Lecture 16. Inertial western boundary currents

I. Inertial western boundary current with two moving layer

The case of one moving layer inertial western boundary current was discussed above. Historically, the study of two-moving layer inertial western boundary current went through quite a struggle. Blandford (1965) found analytical solutions for the case when potential vorticity was constant in both moving layers; however, his solution exhibited an unexpected separation at latitude much lower than that from a single-moving layer model. This puzzle remained unsolved for almost 25 years.

Luyten and Stommel (1985) explained the paradoxical situation in terms of the virtual control first found in the one-dimensional hydraulic problem by Woods (1968). Luyten and Stommel model was based on zero potential vorticity assumption which is not realistic. This problem can be solved for the general case of non-zero potential vorticity.

A. Model formulation:

Basic equations for a two-and-a-half-layer inertial western boundary current are

\[\frac{v_{1x} + f}{h_1} = F_1(\psi_1)\]  \hspace{1cm} (7)

\[\frac{v_{2x} + f}{h_2} = F_2(\psi_2)\]  \hspace{1cm} (8)

and Bernoulli function

\[\frac{v_i^2}{2} + g'(\gamma h_i + h_2) = G_i(\psi_i)\]  \hspace{1cm} (9)

\[\frac{v_i^2}{2} + g'(h_i + h_2) = G_i(\psi_i)\]  \hspace{1cm} (10)

where \(\psi_1\) and \(\psi_2\) are streamfunctions defined as

\(\psi_{ix} = h_i v_i; \psi_{iy} = -h_i u_i, i = 1,2\)  \hspace{1cm} (11)

In addition, there is a simple integral connecting the barotropic streamfunctions with the layer thicknesses

\[(\psi_{1\infty} + \psi_{2\infty}) - (\psi_1 + \psi_2) = \frac{g'}{f} \left[\frac{\gamma (h_{1\infty} - h_{1\infty})}{2} + h_{1\infty} h_{2\infty} - h_1 h_2 + \frac{h_{2\infty} - h_2}{2}\right]\]  \hspace{1cm} (12)
where subscript $\infty$ indicates the interior solution. Using the potential vorticity relations, the downstream momentum equations can be rewritten as a pair of second-order ordinary differential equations for layer thickness $h_1$ and $h_2$

$$\gamma h_{1xx} + h_{2xx} - \frac{f}{g'} F_1(\psi_1) h_1 = \frac{f^2}{g'},$$

$$h_{1xx} + h_{2xx} - \frac{f}{g'} F_2(\psi_2) h_1 = \frac{f^2}{g'},$$

(13)

(14)

In addition, the streamfunctions satisfy

$$\psi_{1x} = h_1 v_1, \psi_{2x} = h_2 v_2$$

(15)

These four equations consist of a high-order system of differential equation. Unfortunately, this is a stiff equation system over the semi-infinite domain $[0, \infty]$. Numerical integration of this system is rather difficult, and it is probably the very reason why Blandford could not find smooth solutions.

B. Streamfunction coordinate transform

Charney’s (1955) paper has been quoted very often; however, the technique used by Charney has seldom been used in other studies. In fact, what Charney used was a streamfunction coordinate transform which can be traced back to von Mises (1927). In the case of one-moving layer model, the streamfunction transformation gives solution in closed analytical form for general potential vorticity profile. For the case of two moving layers, we can introduce the following transformation

$$d\psi_1 = h_1 v_1 dx$$

(16)

In the new streamfunction coordinate $\psi_1$, the problem is reduced to solving two-first-order ordinary differential equations plus three algebraic relations (all in non-dimensional forms)

$$\frac{dv_1^2}{d\psi_1} = 2 \left[ F_1(\psi_1) - \frac{1}{h_1} \right]$$

(17)

$$\frac{d\psi_2}{d\psi_1} = \kappa \frac{h_2 v_2}{h_1 v_1}$$

(18)

$$\frac{\gamma}{2} h_1^2 + \kappa h_2 h_2 + \frac{\kappa^2}{2} h_2^2 = \frac{\gamma}{2} + \kappa + \frac{\kappa^2}{2} - (\psi_{1\infty} + \psi_{2\infty} - \psi_1 - \psi_2)$$

(19)

$$v_1^2 / 2 + \gamma h_1 + \kappa h_2 = G_1(\psi_1)$$

(20)

$$v_2^2 / 2 + h_1 + \kappa h_2 = G_2(\psi_2)$$

(21)

subject to boundary conditions

$$h_1(\psi_{1\infty}) = h_2(\psi_{2\infty}) = 1$$

(22)

$$v_1(\psi_{1\infty}) = v_2(\psi_{2\infty}) = 0$$

(23)

$$\psi_2(0) = 0, \psi_2(\psi_{1\infty}) = \psi_{2\infty}$$

(24)

This system is a well-behave system and can be solved easily.
2. Matching the inertial western boundary current with the mid-ocean thermocline
   
   A. Motivation
   
   We have discussed thermocline structure in the ocean interior and inertial western boundary current with two moving layers. It has been argued that mixing/dissipation within the southern part of the western boundary is negligible, so we would like to match the thermocline solution for the ocean interior with some kind of western boundary currents. This is one more step towards constructing a unified picture for the circulation in a closed basin.

   B. Model formulation
   
   The model ocean consists of three layers of constant density, and the lowest layer is very thick and assumed motionless. The subtropical gyre interior is divided into three domains where the dynamics is slightly different.

   The dynamics of the model is basically the same as the classical ventilated thermocline by Luyten et al. (1983). However, we will assume that potential vorticity in the second layer is uniform for all the subducted water. This is not inconsistent with the observation that potential vorticity in the deep part of the Gulf Stream is practically homogenized. Potential vorticity in the uppermost layer is not uniform because of
turbulent forcing. This assumption gives rise to rather elegant solution for the outcropping line, the western boundary of the shadow zone, and layer thickness.

Fig. 2. Sketch of a model ocean for a subtropical gyre: a) the inertial western boundary current; b) the interior ocean with two-moving layers, where $x_0(y)$ is the outcrop line for the upper layer, and $x_s(y)$ is the boundary of the shadow zone for the second layer; c) a meridional section of the model ocean with three layers.

a. Domain I
North of the outcrop line $x_0 = x_0(y)$, the first layer vanishes, so only the second layer is in motion, and the basic equations are

$$
-f h_2 v_2 = -\gamma_2 h_2 h_{2x} + \tau^x / \rho_0
$$

(25)

$$
f h_2 u_2 = -\gamma_2 h_2 h_{2y}
$$

(26)

$$
(h_2 u_2)_x + (h_2 v_2)_y = 0
$$

(27)

where $\gamma_2 = g (\rho_3 - \rho_2) / \rho_0$. Cross-differentiating gives the Sverdrup relation

$$
\beta h_2 v_2 = -\tau^x / \rho_0
$$

(28)

And the layer thickness satisfies

$$
h_2^2 = h_c^2 + \frac{2 f^2}{\rho_0 \beta^2} \left( \frac{\tau^x}{f} \right)_y (x_c - x)
$$

(29)

where $h_c$ is the constant layer thickness along the eastern boundary of the basin

Since potential vorticity is constant along the outcrop line,

$$
f / h_2 = f_0 / h_c, \text{ along } x_0(y)
$$

(30)

Accordingly, the outcrop line satisfies

$$
x_c - x_0(y) = \rho_0 \beta \gamma_2 \frac{(f / f_0)^2 - 1}{2 f^2 (\tau^x / f)_y} h_c^2
$$

(31)

b. Domain II
South of $x_0(y)$ and north of $x_s(y)$ both layers are in motion, so the basic equations are

$$
-f h_1 v_1 = -h_1 \left[ (\gamma_1 + \gamma_2) h_{1x} + \gamma_2 h_{2x} \right] + \tau^x / \rho_0
$$

(32)
\[ f_h u_1 = -h_1 \left[ (\gamma_1 + \gamma_2) h_{1y} + \gamma_2 h_{2y} \right] \]  
\[ (h, u, v)_x + (h, v)_y = 0 \]  
\[ -f h_2 v_2 = -h_2 \left[ \gamma_2 h_{1x} + \gamma_2 h_{2x} \right] \]  
\[ f_h u_2 = -h_2 \left[ \gamma_2 h_{1y} + \gamma_2 h_{2y} \right] \]  
\[ (h, u_2)_x + (h, v_2)_y = 0 \]  

where \( \gamma_1 = g (\rho_2 - \rho_1) / \rho_0 \). After some manipulations, the layer thicknesses satisfy

\[ \gamma_2 (h_1 + h_2)^2 + \gamma_1 h_1^2 = \gamma_2 h_1^2 + 2 \frac{f h_2}{\rho_0 \beta} \left( \frac{\tau^x}{f} \right)_y (x_e - x) . \]  

Using the potential vorticity homogenization assumption in the second layer, we find the layer thickness

\[ h_1 = \frac{h_e}{\gamma_1 + \gamma_2} \left( \frac{f}{f_0} \Delta^{1/2} \right) \]  

where

\[ \Delta = \left( \frac{\gamma_2 f}{f_0} \right)^2 - (\gamma_1 + \gamma_2) \left\{ \gamma_2 \left[ \left( \frac{f}{f_0} \right)^2 - 1 \right] - \frac{2 f^2}{\rho_0 \beta h_e} \left( \frac{\tau^x}{f} \right)_y (x_e - x) \right\} \]  

\[ \tau^x = -\tau_0 \frac{f}{f_0} \cos \left( \frac{\pi y}{L_y} \right) \]
Where $\tau_0 = 0.75 \text{dyn/cm}^2$. The horizontal dimension of the model basin is $L_x = 6000 \text{km}$, $L_y = 3000 \text{km}$, the stratification parameters are $\gamma_1 = \gamma_2 = 1.5 \text{cm/s}^2$, $h_c = 500 \text{m}$. The solution with continuous western boundary layers is shown in Figs. 3 and 4.

Fig. 3. Structure of the interior solution with external parameters $h_c = 500 \text{m}$, $y_0 = 0.334925$. Heavy curve indicates the outcrop line, and dashed curve for the boundary of the shadow zone.

Fig. 4. Structure of the western boundary currents. A) Meridional section of the inertial western boundary currents. Y is the non-dimensional meridional coordinates, thin curves are layer interfaces at the outer edge of the western boundary, heavy curves are interfaces at the western wall, and the dashed curves are the non-physical solutions. B) Boundary layer structure in the streamfunction coordinates.
However, if the external parameters (such as the location where the outcrop line intercepts the eastern boundary, the lower layer thickness along the eastern boundary, the stratification parameter $\gamma_1$ or the wind stress) are changed, the western boundary layers are interrupted and system behavior can be described in terms of a saddle point in the phase space, Fig. 5.

In the classical theories of wind-driven circulation with single-moving layer, the western boundary layer has been assigned a passive role – closing the mass flux and dissipating the extra vorticity. The interior solution is uniquely determined by the wind stress curl. For a stratified model, the solution also depends on the stratification and surface density distribution, as have been discussed in the previous sections.

It was a surprise when Blandford (1965) failed to find continuous solutions for a two-moving layer inertial western boundary layer model. Here we have shown that continuity of the western boundary layer imposes a constraint over the mid-ocean thermocline. Although there are many freedoms for the interior thermocline, each time we add on a new piece of the circulation, the system loses some freedom, as more and more pieces are added on there will be only few possible solution remain. These implicit constraints reflect the interaction between the interior flow and other parts of the circulation.

![Image of Fig. 5](image-url)

*Fig. 5. Dependence of the boundary layer structure on the lower layer thickness at the eastern wall. The solid line indicates the continuous solution, the long-dashed line for the spurious solution, and the short-dashed lines indicate branches of the solutions.*

### 3. Remark on close the ventilated thermocline with western boundary currents

Since the ventilated thermocline theory has been proposed, and natural question has been whether such beautiful solutions can be closed along the western boundary. It is however very clear that no ideal-fluid model can accomplish such a job because to close the circulation, we need to include friction/dissipation, so that the budget of potential vorticity and energy can be closed for any closed streamlines. Matching an inertial western boundary current with multiple moving layers represents an effort in push this issue to the theoretical limit. Without mixing/dissipation, the circulation cannot be closed. However, including mixing/dissipation will rend the solution analytically rather difficult to solve, and the beauty of analytical solution is hard to achieve.
Reference