Lecture 12. Theory of potential vorticity homogenization

1. Introduction
As discussed in previous sections, the most challenging question was how to bring the subsurface layer into motion in a weakly dissipated ocean. This historical puzzle was solved by Rhines and Young (1982a, b) in a series of highly original studies. Based on the quasi-geostrophic theory, they have demonstrated that
1) Closed geostrophic contours appear when forcing is very strong.
2) Potential vorticity is homogenized within the closed contours.
3) There are infinite numbers of possible solutions; however, unique solutions which are stable to small perturbations can be found.
4) Potential vorticity is homogenized toward its value along the northern boundary.
Presentation in this section closely follows their papers.

2. A two-layer model
On a $\beta$-plane the steady circulation of a two-layer QG model can be written as

$$J(\psi_1, q_1) = w_0 - \nabla \cdot \Phi_1$$  
$$J(\psi_2, q_2) = -\nabla \cdot \Phi_2 - D\nabla^2 \psi_2$$

where $J(g, h) = g_x h_y - g_y h_x$ is the Jacobian term (a nonlinear term!), and

$w_0 = \frac{\nabla \times \hat{\tau} \cdot \hat{z}}{\varrho_0 f_0}$ is the Ekman pumping rate

$F = \frac{f_0^2}{g' H} = L_p^2$

where $L_p$ is the Rossby radius of deformation. Note that we have neglected the relative vorticity in these equations because relative vorticity is unimportant for basin-scale motions. The $F(\psi_i - \psi_{i-1})$ terms are the contribution due to interface deformation, also called the stretching term, noting that the interface height is proportional to the streamfunction difference. The $D\nabla^2 \psi_2$ term is the bottom friction, and $D$ is a small parameter.

The interfacial friction is parameterized in terms of $\Phi_i$. As a crude way of mimicking baroclinic instability, the interfacial friction is assumed to be linearly proportional to the velocity shear

$\Phi_1 = R\nabla(\psi_1 - \psi_2)$
$\Phi_2 = R\nabla(\psi_2 - \psi_1)$

where $R$ is a small parameter.

A. Steady, frictionless flow:
Assuming $R$ and $D$ are of the same order and very small, Eqs. (1) are reduced to

$J(\psi_1, \beta y + F(\psi_2)) = w_0 + O(R)$
$J(\psi_2, \beta y + F(\psi_1)) = O(R)$
From these two equations we obtain the barotropic solution
\[ \psi_B = \psi_1 + \psi_2 = -\frac{1}{\beta} \int_{x_e}^{x_f} w_0 dx' \] (7)

where \( x_e(y) \) is the eastern boundary of the model. In this section, we will assume that the Ekman pumping velocity is identically zero outside a circle (Fig. 1). Since we further assume the fluid to be motionless at the infinity, the eastern half of the circle can be chosen as the effective eastern boundary of the model. For given Ekman pumping rate, \( w_0 \), the barotropic streamfunction can be calculated accordingly. Note that the Jacobian term has a very useful property:
\[ J(\psi_1, \psi_1) = J(\psi_2, \psi_2) = 0. \] With the help of the barotropic streamfunction \( \psi_B \), the nonlinear Eqs. (1) can be rewritten as
\[ J(\psi_1, \beta y + F\psi_B) = w_0 + O(R) \] (8a)
\[ J(\psi_2, \beta y + F\psi_B) = O(R) \] (8b)

These equations are linear because \( \psi_B \) is considered to be known. One notices that these equations are first-order partial differential equations in the characteristic form, with a quantity behaves like a "barotropic potential vorticity"
\[ \hat{\beta} \psi_2 = \beta y + F\psi_B \]
as the characteristics. As a result, potential vorticity contours in the second layer are the same as the "barotropic potential vorticity" contours.

It is readily seen that if forcing is weak, the barotropic vorticity is controlled by the planetary term \( \beta y \), so vorticity contours are close to straight lines and there can be no closed vorticity contours. For such cases, the eastern boundary blocks all possible flows in the second layer, as Rooth et al. (1978) pointed out. However, if forcing is strong enough, the second term, \( F\psi_B \), dominates, and there can be closed vorticity contours, as shown in Fig. 1. For such cases, the argument by Rooth et al. (1978) does not apply.

Fig. 1. Contours of \( \hat{\beta} \psi_2 \). The dashed circle is \( r = r_1 \), i.e., the bounding contour of the barotropic streamfunction. a) The forcing is weak so \( \hat{\beta} \psi_2 \) is dominated by the planetary vorticity term \( \beta y \), and all the contours are open. b) The forcing is stronger, so there is a region of closed contours. Flows
in this region are shielded from the blocking geostrophic contours started from the “eastern” boundary (the right edge of the forcing domain where $\psi_B = 0$).

Rhines and Young chose a very idealized Ekman pumping function

$$w_0 = -\alpha x, \text{ for } r < r_1; \quad w_0 = 0, \text{ for } r > r_1$$

where $r = \sqrt{x^2 + y^2}$. Therefore, the barotropic streamfunction is

$$\psi_B = \begin{cases} \frac{\alpha}{2\beta}(r_1^2 - x^2 - y^2) & \text{for } r < r_1 \\ 0, & \text{for } r \geq r_1 \end{cases}$$

(10)

The $\hat{q}_2$ contours are circles or arcs of circles if $r \leq r_1$, Fig. 1; outside this circle they are just straight lines

$$\hat{q}_2 = \begin{cases} \frac{\alpha F}{2\beta} \left[ r_1^2 + y_0^2 - x^2 - (y - y_0)^2 \right], & \text{if } r < r_1 \\ \beta y, & \text{if } r \geq r_1 \end{cases}$$

(11)

where $y_0 = \frac{\beta^2}{\alpha F}$. Thus, there will be closed contours if

$$r > y_0, \text{ or equivalently } \frac{\alpha r_1}{\beta} > \frac{\beta^2}{F}$$

(12)

In addition, the closed contours will appear near the northern boundary of the forcing field where the negative meridional gradient of the barotropic streamfunction can cancel the positive meridional gradient of the planetary vorticity. An important technique used in their model is to assume a special Ekman pumping pattern which satisfies

$$\int_{-\infty}^{\infty} w_0(x')dx' = 0$$

(13)

By using such a pattern, they have been able to find nice solutions without worrying too much about what would happen along the western boundary. It is to notice that the same technique was first employed by Goldsborough for the evaporation minus precipitation driven circulation. As we will discuss later, adding the western boundary current is not really an easy exercise at all. IT IS IMPORTANT TO SIMPLIFY THE MODEL AS MUCH AS POSSIBLE TO GAIN THE PHYSICAL INSIGHT, AND TRY TO AVOID UNNECESSARY COMPLICATION IN THE BEGINNING.

When closed geostrophic contours appear in the second layer, the number of possible solutions for ideal fluid is infinite;

$$\psi_2 = A_2(\hat{q}_2)$$

(14)

where $A_2$ is an arbitrary function. In order to find solutions that are physically meaningful, one has to include the next-order terms. Typically, by working on some integrals for the next-order terms, one can find constraints on the lowest-order dynamics, and this will eventually lead to unique solutions.

B. Determination of the flow inside the closed geostrophic contours

For convenience, we assume $D = R$. Integrating (1b) along a closed contour, one obtains

$$R \oint (2\hat{u}_2 - \bar{u}_1) \cdot d\vec{s} = 0,$$

(15)
where \( \bar{u}_1 = \hat{k} \times \nabla \psi_1 \) and \( \bar{u}_2 = \hat{k} \times \nabla \psi_2 \) are velocity in the upper and lower layer, and \( \hat{k} \) is a unit vector in the vertical direction. Note that the Jacobian term vanishes identically! This relation can be rewritten as
\[
\oint \bar{u}_2 \cdot d\bar{s} = \frac{1}{3} \oint \bar{u}_b \cdot d\bar{s}
\]  \( \text{(16)} \)

Where \( \bar{u}_b = \hat{k} \times \nabla \psi_b = \bar{u}_1 + \bar{u}_2 \) is the barotropic velocity. Using (14), the term on the left-hand side can be rewritten as
\[
\oint \bar{u}_2 \cdot d\bar{s} = \oint A'_2 (\hat{q}_2) \left( \hat{k} \times \nabla \hat{q}_2 \right) \cdot d\bar{s} = A'_2 (\hat{q}_2) \oint (F\bar{u}_b - \beta \hat{x}) \cdot d\bar{s}
\]  \( \text{(17)} \)

Thus, from (16) and (17) we finally obtain a relation
\[
A'_2 = 1/3F
\]  \( \text{(18)} \)

Using (14), the final solution is
\[
\psi_2 = \frac{1}{3F} \hat{q}_2 + \text{const} = \frac{1}{3} \psi_b + \frac{1}{3} \frac{\beta y}{F} + \text{const}
\]  \( \text{(19)} \)

\[
\psi_1 = \psi_b - \psi_2
\]

For the case discussed above, \( \omega_0 = -\alpha x, r \leq r_1 \), so the lower layer flow is
\[
\psi_2 = -\frac{\alpha}{6\beta} (x^2 + (y - y_0)^2) + \text{const. (for closed } \hat{q}_2 \text{)}
\]  \( \text{(19')} \)

\[
\psi_2 = 0, (\text{elsewhere})
\]

The solution is shown in Figure 2.

Fig. 2. The streamfunction maps for the upper (left panel) and lower layer (right panel) when \( y_0 = 0.5r_1 \) (same parameter as in Fig. 1b). The small circle inside is the outmost contour of the potential vorticity homogenization.

The lower layer flow is confined to the region where the \( \hat{q}_2 \) contours close; see Fig. 1b.

In the analysis above, we have assumed \( R = D \). The general case when \( R \neq D \) can be worked out accordingly. One finds
and within the closed geostrophic contours the streamfunction of the second layer is

\[ \psi_2 = \frac{R}{2R + D} \left( \psi_b + \frac{\beta}{F} y \right) + \text{const} \]  

(21)

It is readily seen that potential vorticity in the second layer obeys

\[ q_2 = \left( \frac{D}{2R} + D \right) \hat{q}_2 + \text{const} \]  

(22)

Thus, in the limit of \( D \ll R \), potential vorticity in the second layer becomes homogenized within the closed streamlines. This is an example of the more generalized potential vorticity homogenization theory discussed by Rhines and Young (1982a,b).

C. Important points:
1) Strong forcing induces closed geostrophic contours in the second layer.
2) Non-uniqueness of the flow within the closed contours if dissipation is identically zero.
3) Selection of unique solution by weak dissipation.
4) Using the contour integral to obtain the constraint.
5) Potential vorticity homogenization within the closed contours when the dissipation can be parameterized as lateral potential vorticity diffusion.

3. Three-layer model

We would like to learn more about the vertical structure of the gyre, especially at the middle depth where neither Ekman pumping nor bottom friction can directly affect the circulation. The simplest case is a model with three moving layers and the mean layer depth and density jumps are the same. The basic quasi-geostrophic potential vorticity equations are

\[ J(\psi_1, q_1) = w_0 - \nabla \cdot \Phi_1 \]  

(33a)

\[ J(\psi_2, q_2) = -\nabla \cdot \Phi_2 \]  

(33b)

\[ J(\psi_3, q_3) = -\nabla \cdot \Phi_3 - D \nabla^2 \psi_3 \]  

(33c)

where the Ekman pumping rate is the same as in Eq. (9) and the potential vorticity in each layer is

\[ q_1 = \beta y + F(\psi_2 - \psi_1) \]  

(34a)

\[ q_2 = \beta y + F(\psi_1 - 2\psi_2 + \psi_3) \]  

(34b)

\[ q_3 = \beta y + F(\psi_2 - \psi_3) \]  

(34c)

Note that for the middle layer contributions come from both the upper and lower interfaces. We again assume that the dissipation is due to interfacial friction

\[ \Phi_1 = R \nabla (\psi_1 - \psi_2) \]  

(35a)

\[ \Phi_2 = R \nabla (\psi_1 - 2\psi_2 + \psi_3) \]  

(35b)

\[ \Phi_3 = R \nabla (\psi_3 - \psi_2) \]  

(35c)

Following the same approach in section II, we introduce the barotropic streamfunction

\[ \psi_b = \psi_1 + \psi_2 + \psi_3 = -\frac{1}{\beta} \int_x \psi_{s,(y)} w_0 dx' \]  

(36)

Thus, (33b) becomes
\[ J(\psi_2, \beta y + F\psi_B) = O(R) \]  
\[ (37) \]

Therefore, we obtain an equation in characteristic form again, and the "barotropic" vorticity \( \dot{\psi}_2 = \beta y + F\psi_B \) is the characteristic. Note that the reshaping of the geostrophic contours is done by the barotropic mode \( \psi_B \). Therefore, the general solution is

\[ \psi_2 = A_1(\dot{\psi}_2) + O(R) \]  
\[ (38) \]

\[ \psi_3 = A_2(\dot{\psi}_3) + O(R) \]

where \( \dot{\psi}_3 = \beta y + FA_2(\dot{\psi}_2) \). Integrating (33b) along the closed contours gives

\[ \oint (\dot{u}_2 - \frac{1}{2}(\dot{u}_1 + \dot{u}_3)) \cdot d\tilde{s} = 0 \]  
\[ (39) \]

Using the barotropic streamfunction leads to

\[ \oint \dot{u}_2 \cdot d\tilde{s} = \frac{1}{3} \oint \dot{u}_B \cdot d\tilde{s} \]  
\[ (40) \]

Similarly, the relation for the third layer is

\[ \oint \dot{u}_3 \cdot d\tilde{s} = \frac{1}{2} \oint \dot{u}_2 \cdot d\tilde{s} \]  
\[ (41) \]
Fig. 3. Contours of “barotropic” vorticity in the second ($\hat{q}_2$, left panels) and third ($\hat{q}_3$, right panels) layers for the case of $y_0 = r_1 / 8$ (upper panels) and $y_0 = r_1 / 2$ (lower panels).

Combining (38) and (40) gives $\psi_2$ for the closed contours

$$\psi_2 = \frac{1}{3} \left( \psi_B + \frac{\beta y}{F} \right) + \text{const}.$$  (42)

The solution for the third layer within the closed contours can be obtained in similar way

$$\psi_3 = \frac{1}{6} \psi_B + \frac{2 \beta y}{3 F} + \text{const}$$

**Expulsion of potential vorticity gradient**

By definition

$$q_2 = \hat{q}_2 - 3F \psi_2$$  (44)

Therefore, we have the potential vorticity homogenization within the closed contours

$$q_2 = \beta y + F \psi_B - 3F \left( \frac{1}{3} \psi_B + \frac{\beta y}{3 F} \right) + \text{const} = C$$  (45)

Applying (41) to the most northerly point of the most closed contour with the boundary condition $\psi_2 = 0$

$$\psi_2 = \frac{1}{3} \left( \psi_B + \frac{\beta y}{F} \right) - \frac{1}{3} r_i$$  (46)

Since $\hat{q}_2 = \beta y + F \psi_B = \beta r_i$ on this contour, one obtains

$$q_2 = \hat{q}_2 - 3F \psi_2 = \beta r_i$$  (47)

Thus, potential vorticity within the close geostrophic contours is homogenized toward its value along the northern boundary of the gyre. The potential vorticity homogenization implies that there is a shape potential vorticity front adjacent to the outer most closed geostrophic contours, and this is illustrated in the Fig. 4.

![Fig. 4](image_url)  

Fig. 4. A map of potential vorticity, illustrating the potential vorticity homogenization and expulsion of potential vorticity.

Another important phenomenon related to the three-moving layer model is that the center of wind-driven gyre in each layer is gradually moved northward from the topmost layer to the deep layers,
as shown in Fig. 5. Note that volume flux in the upper layer is reduced over the domain where the lower layers move.

Fig. 5. Streamfunction maps for a three-layer model for the case with $y_0 = r_1/8$. In the middle panel, the smallest dashed circle indicates the outer most closed geostrophic contour in the lower layer, and the middle size dashed circle indicates the outer most closed geostrophic contours in the middle layer, while the outmost dashed circle indicates the domain with non-zero Ekman pumping.

References: