THE APPLICATION OF

SET THEORY TO ECOLOGY

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Sets and Membership

There are 32 species of seals (and sea lions) in the world (Bonner, 1999). These species are members of the set of the seals (sea lions) of the world. They are members of this single group, this set. These members inhabit for a part of their lives remote, lonely, predator-free beaches of arctic, tropic, and antarctic regions of the world. The set is supported by these remote regions.

There are six tree squirrel species of North America. These species are members of a set, the set of the tree squirrels of North America (Steele and Kaprowski, 2001). This single group, this set, is supported by the North American continent. Two species occupy the same area in the eastern United States. One extends across Canada. And the others have small areas scattered in the remaining North America. Each area supports each member of the set.

About 900 North American bird species are members of the set of the North American birds, a single taxonomic group (National Geographic, 1999). Each member seems to have a different and unique region where it occurs, that supports it. But the set of all the species is supported by the whole continent.

So one can see that set and membership go hand in hand. Set and membership are illustrated by the North American forest. The set is the single North American forest. The members are 22 associations (Oosting, 1948; Hulburt, 2004). Each association is composed of several principal species. So each association is a set. And the North

American forest is the set of these sets. And further, each association has an area where it occurs and thus supports it.

But the pronghorn (a sort of deer) of the National Bison Range of Montana (U.S.A.) is a single set with a member of one species (Byers, 2003). This set with its single member is found in, is supported by, this one area. The single species is composed of one hundred and twenty animals. So the set in this case is a set with a membership of over a hundred actual animals.

But a set is abstract. So is a species, abstract. We are so impressed by species. They seem so real to us. And they are real, no question about it – so it would seem. But they are abstract. What we see is individual animals, individual pronghorn animals – some of the fastest four-legged animals there are (60 miles per hour). But the species, the pronghorn, we deduce. This species is abstract. Species are abstract. Species are members of sets. These sets are abstract. And so is the set of these sets.

But still what is the point in having sets? What good is a set, one may ask. Perhaps you may see them as artificial ways of putting things into categories. Of course, this is a possible point of view. But not the point of view of this essay – which is that sets and membership are part of the structure of the world. Sets and membership hold nature together.

Nature would seem to be ecological relations between species and the land (or water) where they live. But the locales where they live and support them can be put into sets that match to varying degrees the species that occur in the locales. So the matching

of sets of species and sets of locales will be a prime enterprise. And matching of members of paired sets is a prime enterprise of set theory.

There are different sorts of sets. Sets don't have to be sets of species or sets of organisms. Sets can be sets of attributes, properties. There will be presented a set of five growth rates, division rates, of several algal species. A division rate is a property the algal species possesses. The pumping rate of the oyster is 9.0 liters/hour at 20° (the amount of water made to pass over the oyster's gills). This is an attribute of the oyster. There will be presented a set of five pumping rates, five properties. These properties will be matched one-to-one with the external temperatures that the division rates or pumping rates respond to.

Another sort of property is overwintering; and instances, examples, of this property will include bare limbs of deciduous trees, seeds of annual plants, underground parts of perennial plants, diapause forms of insects, and hibernating mammals. These five collections of instances of the property of overwintering do not match one-to-one the single environmental facet, winter. This set of five collections of instances is very different from the set of five growth rates or set of five pumping rates.

The thing to do, then, it would seem, is to delineate these sets in as all embracing and yet nuanced manner as possible. Set theory, it is hoped one will come to agree, is the ideal way to go about the situation.

A Little Set Theory, Sets and Sets of Sets

A member is shown by a_i. Being a member is shown by E. And a set is shown by A. Assembling these we get:

$$a_i \, E \, A \, ,\qquad\qquad\qquad\qquad 1)$$

which gives us : a_i is a member of A. Also the formula gives us: a_i belongs to A. This is to say that one seal species, a_i, is a member of A. Take A to be the set of 32 seal species of the world (Bonner, 1999).

But a set can be shown, also, by curly braces:

$$\{ \, . \, . \, . \, . \, \}. \qquad\qquad\qquad\qquad 2)$$

Now, though, the depiction is much different. The set of seal species, for example, can be given as:

$$\{\text{species}_1, \text{species}_2, \ldots\ldots\text{species}_{32}\} \qquad\qquad 3)$$

which is briefly:

$$\{a_1, a_2, \ldots\ldots a_{32}\} \qquad\qquad\qquad\qquad 4)$$

which might be, using just the subscript numbers:

$$\{1, 2, \ldots.. \, 32\}, \qquad\qquad\qquad\qquad 5)$$

which might be, putting the subscript numbers into left-right order as x's:

$$\{x, x, \ldots\ldots x\},$$

which has all the x's collected as one x before the colon, and each x singled out after the colon:

$$\{x: x \text{ has the property of being a seal species}\}, \qquad\qquad 6)$$

which is: the set of all 32 x's (before the colon) such that each single x (after the colon) has the property of being a seal species. To have this property makes seal species similar and thus makes them the same kind and thus makes them members of the same set, the reality of membership, E, being captured by:

$$\{x: x \text{ E } A\}, \qquad\qquad\qquad 7)$$

$$5.$$

which is: the set of just those 32 seal species such that each is a member of the seal set, A (this is a distillation from Lipschutz, 1998, pp. 1-5; Lightstone, 1964, pp. 40-42; Halmos, 1974; pp. 2-7; Stoll, 1974; pp. 4-9). This is not circular – (see Stoll, p. 8; Halmos, p. 10).

The set of sets has the following formula:

$$\{x: x \text{ E } A \text{ for some } A \text{ E } B\}, \qquad\qquad 8)$$

which is: the set of just those species such that each single species is a member of some set A, which in turn is a member of the set B (Lipschutz, 1998, p. 117; Milewsky, 1939, p. 16; Halmos, 1970, p. 12). Applying this formula to the North American forest, there are a number of important, climax species such that one of them, the beech, is a member in the set, the beech-maple association, which in turn is a member in the set of all these sets, B. Of all the climax species x: one x is picked out to be a member, E, of the beech-maple association, A, which in turn is a member in the set of these associations. There are 22 associations. The set of these is B. This set B is also shown by $\{\ldots\}$. But we have to go further and show this set as follows:

$$\{\{\text{beech, maple}\}\ldots \{\text{white spruce, balsam fir}\}\ldots$$

$$\{\text{piñon pine, juniper}\}\}. \qquad\qquad 9)$$

wherein the sets in single braces are associations, are *A*'s, and these *A*'s are the members of set *B*, in outer braces, which is the set of all the *A*'s.

Straightforward so far, but so far no consideration of the properties that determine sets has been taken up. So the set *B* is replaced by two subsets. One subset, *C*, contains associations specified as follows:

> *A* has the property of being adapted only to moist regions.   10)

> 6.

The other subset, *D*, contains associations specified in the following way:

> *A* has the property of *not* being adapted to *non*-moist regions.   11)

Thus all the 22 associations, all the *A*'s, are included in *C* and *D* by *A* being either in *C* or in *D* (Lipschutz, 1998, p. 5):

> {*A*: *A* E *C* or *A* E *D*}.   12)

This is the principle of the union of sets and replaces 8) here, but in so doing eliminates the species, the *x*'s.

In the portrayal of the set of all associations, 9), the beech-maple association and the white spruce-balsam fir association have the property of being adapted to moist regions and so are members of the subset *C*. On the other hand, in 9) the piñon pine-juniper association has the property of *not* being adapted to *non*-moist regions and is a member of *D*.

Thus the situation is fairly complex now. The symbolization has been minimized. But two features stand out. In 7) each species, shown by a small *x* is a member of some set *A*, and all the species are members of this one *A*. But in 12) all the *A* sets, shown by a

large letter, are members of the set *C* or the set *D*; so not all the *A*'s can be in either *C* or *D*, though *C* and *D* together have all the *A*'s.

A Little More Set Theory, Functions

George Cantor founded set theory in a few decades after 1873. A crucial feature was to match one member of a pair with another member of a pair. For example, each number was matched with each near odd number, each number with its cubed number. Because he was concerned with the collection into sets of these infinitely many numbers,
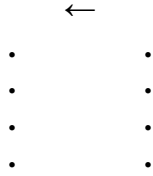
7.

the matched pairs were all equally infinite and the infinities all the same. This was just a first step. Succeeding steps opened up a world of infinitely many infinities (Barrow, 2005, p. 68-76). But it is the first step, the matching of members of pairs that is the concern next.

Two examples of functions show the matching of members of pairs, ordered pairs. They are from (Lipschutz, 1998, pp. 94-95, 98-99). The first is "Consider the function $f(x) = x^3$, i.e., f assigns to each real number its cube." There are two sets, the numbers and the cubed numbers. They are matched in a one-to-one correspondence, each number and each cubed number. The correspondence is reversible as shown by the double arrows, the correspondence is called bijective. The dots show the sets to be infinite.
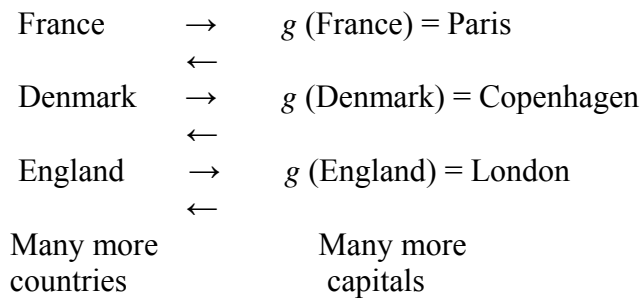
$$x \qquad f(x) = x^3$$

$$
\begin{array}{ccc}
1 & \rightarrow & 1 \\
  & \leftarrow & \\
2 & \rightarrow & 8 \\
  & \leftarrow & \\
3 & \rightarrow & 27 \\
  & \leftarrow & \\
4 & \rightarrow & 64
\end{array}
$$

←

.   .

.   .

.   .

.   .

The second example from Lipschutz is "Let g assign to each country in the world its capital city."

France   → g (France) = Paris
                 ←

Denmark   → g (Denmark) = Copenhagen
                 ←

England   → g (England) = London
                 ←

Many more   Many more
countries   capitals

Thus there are two sets. They are matched in a one-to-one correspondence. Their correspondence is reversible, as shown by the double arrows. So the correspondence is bijective. The sets are finite.

The North American tree squirrels are together a single set. The areas that they inhabit are together a single set. But two squirrel species occupy nearly the same area of the eastern United States. So there is no one-to-one correspondence, and the matching is not reversible. The situation is called surjective. Nevertheless, the other members of the two sets are linked by the function of being adapted of each species to some unique area.

Species   Area

1

```
&  →  1
2

3  →  3
   ←
4  →  4
   ←
5  →  5
   ←
6  →  6
   ←
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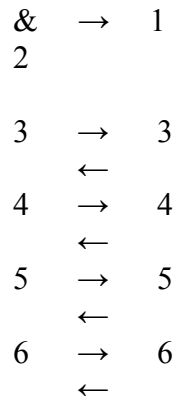
Species 1 has the function of being adapted to area 1, species 2 adapted to area 1, species

3 adapted to area 3, and one-to-one for species 4, 5, 6. The sets are finite if the numbers

9.

attributed to them are not part of them.

Another case is the instances, the exemplifications, of the property of over-

wintering. They compose surjective sets as follows:

1. bare limbs of deciduous trees

2. seeds of annual plants

3. underground parts of perennial plants   winter

4. diapause form of insects

5. hibernating of mammals

The five collections of instances of the property of overwintering are members of the set

to the left. But the set to the right has the single member, winter. The function of being

adapted assigns to each member at the left the environmental feature winter. The bare

limbs of deciduous trees have the function of being adapted to winter. And the other

instances of over-wintering have, likewise, the function of being adapted to winter.

Again the members of the set to the left are finite if the numbers attributed to them are

not part of them. And again the ecological feature that the member to the left is adapted to something, winter, is to be kept in mind – for 'the function of being adapted' is not the whole thing; 'the function of being adapted to winter' is the whole thing.

Further Set Theory, Product Sets

There is a relation between two sets, the product set. For when each member of one set is paired with a member of another set, the result is a relation. The following formulae are from Lipschutz (1998, pp. 64-65). The first is:

$$\{(a, b) : a \text{ E } A \text{ and } b \text{ E } B\} \tag{13}$$

10.

which symbolizes: the set of all ordered pairs, (a, b), is such that (colon) each a is a member of set A and each b is a member of set B. The second formula has two variants.

$$(a, b) \text{ E } R,$$

$$(a, b) \text{ not E } R, \tag{14}$$

which symbolizes: the single pair is, or is not, a member of the relation of being adapted. That is, the members of a pair enter into being related by being adapted. The members of a pair exemplify a relation by entering into it or not entering into (Loux, 2003, p. 30). Then we have,

$$a \text{ } R \text{ } b,$$

$$a \text{ not } R \text{ } b, \tag{15}$$

which are: a is related by being adapted to b, a is not related by being adapted to b.

What can be presented are tabulations of division rates of algal species and pumping rates of oysters and time open of mussels, for it is these properties of these animals at various temperatures that will be members a of the set A. But why would you

want to find out the properties of division rate, etc.? So the temperatures of the

environment, composing set *B*, will be presented so that a one-to-one correspondence

between the members *a* of the set *A* and the members *b* of the set *B* can be shown. For

such a correspondence, such a matching, is of real interest. Thus the division rates of the

diatom species are high from highest to lowest environmental temperatures. But a green

alga has high rates at high environmental temperatures but low or zero rates at low

environmental temperatures. The diatoms grow at Woods Hole (Massachusetts) and the

green alga off New York, portions of the southern New England coast (U.S.A.). The

<div align="right">11.</div>

mussel is like the diatoms and the oyster is like the green alga, and these grow along the

southern New England coast. See Tables at end of article.

<div align="center">

Final Glimpse of Set Theory,
Ordinal Sets

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Ordinals are numbers where each one has a predecessor and a successor, where

each one is bigger than the last and smaller than the next. Thus there is an automatic and

rigid order when numbers are put in counting order. There are a great many ways that

counting numbers can be put into ordered sets. In the presentation next (Halmos, 1974,

p. 44; Copi, 1979, p. 193; Lipschutz, 1998, p. 213; Stoll, 1979, p. 299) what we see are

sets written horizontally. The sets in the curly braces are to the right. Each number to the

left is the number identifying the set and giving the number of members of the set. The

set to the right is composed of the numbered members that precede the numbers to the

left identifying the set; thus $4 = \{0, 1, 2, 3\}$. This crucial precedence feature tells us that

any ordinal is equal to a set, a set of its predecessors, as follows:

0 = Nothing

1 = {0}

2 = {0, 1}

3 = {0, 1, 2}

4 = {0, 1, 2, 3}

.
.
.
.

w = {0, 1, 2, 3, ...}

In order to see what this means, let us take the sea turtles of northern Florida. They have four stages. Each stage has a number. The turtle hatches from its egg in the middle of the night. Before that there is no turtle in the four-legged sense, so this is stage 0. Stage 1 is when the hatchling scrambles up through the sand where it hatched. Stage 2 is when the hatchling scrambles down the beach to the water, toward the brighter ocean horizon (Lohmann 1998, Mrosovsky 1972). Stage 3 is when the hatchling swims at right angles to the incoming east-wind waves out to the Gulf Stream, determined from experiments in cages and wave simulators (Lohmann et al., 1990, 1995). Stage 4 is when the turtle grows within the North Atlantic gyre, maintained there by the earth's magnetic field (Lohmann and Lohmann, 1996, 1998). If the numbers are in the hatchling stages then the numbers go on endlessly but are collected together as omega, w, which is the set of all the infinitely many ordinal numbers, as indicated by the dots after 3 in the last curly braces.

But this is only half of the story. There are no matching pairs yet. So the thing to do is to get something to pair up with the hatchling stages. There is the sand where the hatchling first scrambles up through, stage 1. There is the beach that the hatchling scrambles across, stage 2. There is the continental self water that the hatchling breasts the waves of, stage 3. There is the North Atlantic gyre within which the hatchling grows, stage 4. So all the numbers in the scheme just presented should be doubled up.

Now the odd thing here is that the numbers are doubled in two ways. Where doubling is $1 + 1 = 2$, we get $(1 + 1) w = 2w = w$, because an infinity of doubles, of pairs, is just infinity. But we also get $(1w + 1w) = w + w = w2$, because two infinities are a pair

13.

of infinities.

In order to understand this distinction, we must consider the distribution law $(a + b) c = ac + bc$. The equality does not hold for infinities, because $(a + b) c$, like $(1 + 1) w$, is the infinitization via c of the single pair $(a + b)$ or (a, b) – so we get a whole lot, an infinite lot, an infinite set of (a, b)'s: $\{a_1, b_1, a_2, b_2, a_3, b_3, \ldots\}$ And this looks like just one w, not 2 w's (Kamke, 1950, p. 63).

This is the case for the left side of the distribution. The right side goes as follows: $ac + bc$, like $1w + 1w$, is an infinity of a's and another infinity of b's, a pair of infinities, thus: $\{a_1, a_2, a_3 \ldots; b_1, b_2, b_3 \ldots\}$. And this looks like two separate infinities, $w + w$ or $w2$ (Copi, 1979, p. 206; (Lipschutz, 1998, p. 217; Quine, 1971, p, 158). The difference between $(1 + 1)w = 2w = w$ and $w + w = w2$, between w and w2, is the difference between an integrated and an unintegrated world. The purpose of applied set

theory is to pull the world together, and here is a case of its doing just that – if we go from 2w to w.

It should be pointed out that in the set theorist's presentation the reverse direction of going from $2w = w$ to $w2 = w + w$ is the direction taken. Going from one w to two w's is a big step and it opens up an utterly vast world of unfinitely many infinities (Rucker, 1995, pp. 64-73). This is the great achievement of set theory itself. But here the application of set theory would seem to be to integrate the world.

And so it is clear that the important point is seeing that each number is the set of its predecessors, that each turtle stage and habitat stage is the set of its predecessors, that the world is held together by being in sets, that any ordinal set by overlapping, by

14.

intersecting preceding sets holds the world together too. Surely, this does seem an appealing point of view. This point of view can be put more strongly, as follows.

The delineations of set theory are crucial to seeing the world as it really is. The composition of the world is a vast panorama of sets, interlacing and interconnecting. Everything comes in sets. Sets are everywhere. Sets are to be sought. Sets are to be spot-lighted, high-lighted. For sets are both framework and substance of all that there is.

Table 1

Maximum growth rates in divisions per day of pure cultures of the unicellular diatoms

Asterionella glacialis (*japonica*) and *Thalassionema nitzschioides*, at increasing

temperatures, averaged (from Hulburt, 1982, p. 213). Ocean water temperatures at

Woods Hole, southern Massachusetts (from Schroeder, 1966)[3], where diatoms grow

(Hulburt, 2004).

Set *A*	Set *B*	*A* x *B*	*R*
Division rates, *a*	Ocean temperatures, *b*		
0.59 at 1° a_1	3° b_1	a_1, b_1	$a_1 R_1 b_1$
0.73 at 1° a_2	5° b_2	a_2, b_2	$a_2 R_2 b_2$
1.30 at 10° a_3	10° b_3	a_3, b_3	$a_3 R_3 b_3$
0.80 at 15° a_4	15° b_4	a_4, b_4	$a_4 R_4 b_4$
1.54 at 20° a_5	20° b_5	a_5, b_5	$a_5 R_5 b_5$
1.37 at 25° a_6			

Growth rates in divisions per day of a pure culture of the green alga *Nannochloris atomus*

at increasing temperatures, Set *A* (from Ryther, 1954). Ocean temperatures off New York, Set *B* (from Schroeder, 1966).[3] *N. atomus* grows off New York in summer and fall (Hulburt, 1981).

Set *A*	Set *B*	*A* x *B*	*R*
Division Rates, *a*	Ocean temperatures, *b*		
(0.00 at 1° a_1)[1]	3° b_1[2]	a_1, b_1	a_1 not R_1 b_1
0.00 at 5° a_2	5° b_2	a_2, b_2	a_2 not R_2 b_2
0.14 at 10° a_3	10° b_3	a_3, b_3	a_3 R_3 b_3
0.65 at 15° a_4	15° b_4	a_4, b_4	a_4 R_4 b_4
0.80 at 20° a_5	20° b_5	a_5, b_5	a_5 R_5 b_5
0.71 at 25° a_6			
0.32 at 30° a_7			

_____

[1] It is assumed that because *Nannochloris* grew very poorly at 10°C and did not grow at all at 5°C that it could not grow at 1°C.

[2] b_1 is matched with a_1 arbitrarily (and likewise above)

[3] The temperatures given were gotten by a complex procedure of selection, given in Hulburt (2004).

16.

Table 2

The percentage of time shells of mussels (*Mytilus edulis*) remain open experimentally during 24-hour periods at different temperatures ranging from 1.0° to 20°C (taken from graph in Loosanoff, 1942). The mean rate of water pumping by oysters (*Crassostrea virginica*) at five different temperatures ranging from 1.0° to 24°C (taken from graph in Loosanoff, 1958).

Mussel

Temperature, experimental	1°	5°	10°	15°	20°
Percentage of Time Open	83	93	94	94	86

Oyster

Temperature, experimental	1°	7°	13°	19°	24°
Liters/hour	0	0.5	3.8	9.0	9.0

References

Barrow, J.D., 2005. The Infinite Book. Pantheon Books, New York, 328 pp.

Bonner, N., 1999. Seals and Sea Lions of the World. Blandford A. Cassel, London, 224 pp.

Byers, J.A., 2003. Built for Speed. Harvard University Press, Cambridge, MA, 230 pp.

Copi, I.M., 1979. Symbolic Logic. Macmillan Publishing, New York, 397 pp.

Halmos, P.R., 1970. Naïve Set Theory. Springer-Verlag, New York, 104 pp.

Hulburt, E.M., 1981. An ecological context for classification of phytoplankton and Environmental variables in New York waters. Ocean Sco. Eng., 6: 195-222.

Hulburt, E.M., 1982. The adaptation of marine phytoplankton species to nutrient and Temperature. Ocean Sci. Engin., 7: 187-228.

Hulburt, E.M., 2004. The four principles of adaptation and their set theory foundation. Ecol. Model., 2004; 253-276.

Kamke, E., 1950. Theory of Sets. Dover Publications, Inc., New York, 144 pp.

Lightstone, A.H., 1964. The Axiomatic Method. Prentice-Hall, Inc., Englewood Cliffs, NJ, 246 pp.

Lipschutz, S., 1998. Set Theory and Related Topics. McGraw-Hill, New York, 282 pp.

Lohmann, K.J. and C.M.F. Lohmann, 1996. Detection of magnetic field intensity by sea turtles. Nature, 380: 59-61.

Lohmann, K.J. and C.M.F. Lohmann, 1998. Migratory guidance mechanisms in marine Turtles. J. Avian Bio., 29: 585-596.

Lohmann, K.J., M. Salmon, and J. Wyneken, 1990. Functional autonomy of land and sea orientation systems in sea turtle hatchings. Bio. Bull., 179: 214-218.

Lohmann, K.J., A.W. Swartz, and C.M.F. Lohmann, 1995. Perception of ocean wave direction by sea turtles. Jour. Exp. Bio., 198: 1079-1085.

Loosanoff, V.L., 1942. Shell movements of the edible mussel, *Mytilus edulis* (L.) in Relation to temperature. Ecology 23: 231-234.

18.

Loosanoff, V.L., 1958. Some aspects of behavior of oysters at different temperatures. Biol.Bull. 114: 57-70.

Loux, M.J., 2003. Metaphysics, A Contemporary Introduction. Routledge, New York, 303 pp.

Milewski, E.G., 1989. The Essentials of Set Theory. Research and Education Association, Piscataway, NJ, 71 pp.

Mrosovsky, N., 1972. The water finding ability of sea turtles; behavioral studies and physiological speculations. Brain Behav. Eval., 5: 202-225.

National Geographic, 1999. A Field Guide to the Birds of North America, third edition. National Geographic Society, Washington, DC, 480 pp.

Oosting, H.J., 1948. The Study of Plant Communities. W.H. Freeman and Company, San Francisco, 389 pp.

Quine, W.V., 1971. Set Theory and its Logic. Harvard University Press, Cambridge, MA, 361 pp.

Rucker, R., 1995. Infinity and the Mind. Princeton University Press, Princeton, 342 pp.

Ryther, J.H., 1954. The ecology of phytoplankton blooms in Moriches Bay and Great South Bay, Long Island, New York. Biol. Bull., 106: 198-209.

Schroeder, E.R., 1966. Average surface temperatures of the western North Atlantic. Bull. Mar. Sci., 16: 320-323.

Steele, M.A. and J.L. Koprowski, 2001. North American Tree Squirrels. Smithsonian Books, Washington, 201 pp.

Stoll, R.R., 1979. Set Theory and Logic. Dover Publications, Inc., New York, 474 pp.