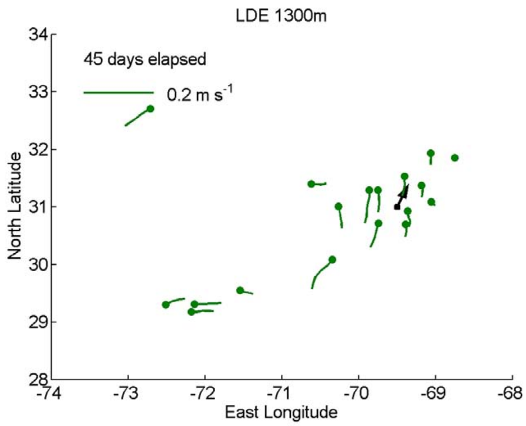
Lagrangian and Eulerian Representations of Fluid Flow:



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Summary: This essay considers the two major ways that the motion of a fluid continuum may be described, either by observing or predicting the trajectories of parcels that are carried about with the flow – which yields a Lagrangian or material representation of the flow — or by observing or predicting the fluid velocity at fixed points in space — which yields an Eulerian or field representation of the flow. Lagrangian methods are often the most efficient way to sample a fluid domain and most of the physical conservation laws begin with a Lagrangian perspective. Nevertheless, almost all of the theory in fluid dynamics is developed in Eulerian or field form. The premise of this essay is that it is helpful to understand both systems, and the transformation between systems is the central theme.

The transformation from a Lagrangian to an Eulerian system requires three key results. 1) The first is dubbed the Fundamental Principle of Kinematics; the velocity at a given position and time (sometimes called the Eulerian velocity) is equal to the velocity of the parcel that occupies that position at that time (often called the Lagrangian velocity). 2) The material or substantial derivative relates the time rate of change observed following a moving parcel to the time rate of change observed at a fixed position; $D(\,)/Dt = \partial(\,)/\partial t \; + \; \mathbf{V}\cdot\nabla(\,)$, where the advective rate of change is in field coordinates. 3) To assert the conservation laws for mass, momentum, etc., within an Eulerian system we need to transform the time derivative of an integral over a moving fluid volume into field coordinates; this leads to or requires the Reynolds Transport Theorem.

In an Eulerian system the process of transport by the fluid flow is represented by the advective rate of change, $\mathbf{V}\cdot\nabla(\,)$. This term is nonlinear (or semi-linear) in that it involves the product of an unknown velocity and the first partial derivative of a field variable. This nonlinearity leads to much of the interesting and most of the challenging dynamics of fluid flows. We can nevertheless put some useful bounds upon what advection alone can do. For variables that can be written in conservation form (e.g., mass and momentum), advection can not be a net source or sink. Advection represents the transport of fluid properties at a definite rate and direction, that of the fluid velocity, so that parcel trajectories are the characteristics of the advection equation. Advection by a nonuniform velocity field may cause a rotation of fluid parcels, which gives fluid parcels angular momentum, and may also cause deformation or straining of fluid parcels.

Contents

Sections marked*: Readers who would prefer a shorter treatment may skip the sections marked ( )* without losing the central theme.

Cover page graphic: If you are using Acrobat 6, click on the figure to start an animation. The green worms are five-day segments of float data from 1300 m depth in the Sargasso Sea. The single black arrow is a current meter measurement. A brief description of this Lagrangian and Eulerian data is in Section 11.2.

1 Kinematics of Fluid Flow.

In this essay we will consider both the observation of fluid flows, and toward the end, the prediction of fluid flows. First we have to define what properties characterize 'fluid flow', a phrase that we will use repeatedly, and then our primary goal will be to erect a coordinate system that will be suitable for analyzing fluid flows. The definition of a coordinate system is always a matter of choice, and the issues to be considered are more in the realm of kinematics, i.e., the description of motion, rather than dynamics, the prediction of motion.

1.1 What makes fluid dynamics challenging?

The physics of the fluid flows that we will consider can be characterized as straightforward classical physics built upon the familiar conservation laws - conservation of mass, (linear) momentum, angular momentum and energy - and a few others. In this regard fluid flows are not different from classical, solid particle dynamics. As we will see, the challenges of classical and geophysical fluid dynamics stem less from exotic physics than from the complex, three-dimensional and time-dependent kinematic properties of most fluid flows.

The properties of a fluid and the kinematics of a fluid flow are likely to be rather different from that of a solid and solid mechanics in four key respects. A fluid, which could be either a gas or a liquid, will be viewed as a **continuum**, within which the fluid velocity, temperature, etc., are defined at every point in space occupied by the fluid, what we will term the domain.[1,2] The molecular properties of the fluid are ignored altogether, and so the physical properties of the fluid (it's viscosity, density, etc.) must be provided from outside of this theory. A differential volume of fluid is termed a parcel, the fluid equivalent of a particle from classical mechanics. They differ from particles in that fluid parcels are imagined to have a finite area and volume so that normal and tangential forces can get a grip. A parcel in a fluid flow is literally pushed and pulled by its surroundings, other fluid parcels or boundaries, and may also be strongly deformed. This important (one is tempted to say intimate) **local interaction** between a fluid parcel and its surroundings is a distinguishing characteristic of fluid flow. A consequence is that we can not predict the motion of a given parcel in isolation from its surroundings; we have to predict the motion of the surrounding parcels as well. How extensive are the surroundings? It depends upon how far we may care to go backward or forward in time and how rapidly signals (taken broadly) are propagated within the fluid. But if we follow a parcel long enough, or if we need to know it's history in detail, then every parcel may well have a **global dependence** over the domain occupied by the fluid. Fluid flows are very often chaotic or turbulent, even when large scale external conditions are held completely steady, and will thus have **variance on all accessible spatial scales**, from the scale of the domain down to a scale set by viscous or diffusive properties of the fluid (a few millimeters in the oceans, say). A consequence is that fluid flows often have many degrees of freedom, with

[1] Footnotes provide references, extensions or qualifications of material discussed in the main text, along with a few homework assignments. They may be skipped on first reading.

[2] Some day your fluid domain will be something grand and important — the Earth's atmosphere or perhaps an ocean basin. For now you can follow along with this discussion by observing the fluid flow in a domain that is small and accessible — the fluid flow in a teacup, for example, as the fundamentals of kinematics are the same for flows big and small. What fundamental physical principles, e.g., conservation of momentum, second law of thermodynamics, can you infer? If your goal was to define completely the fluid flow in the teacup, how many observation points do you estimate would be required? Does the number depend upon the state of the flow, i.e., whether it is weakly or strongly stirred?

the number far exceeding our ability to observe or calculate in complete detail. The phrase 'fluid flow' is then meant to conjure up the mental image of an entire fluid volume in motion, rather than a single parcel in isolation, and with an appreciation that the flow will likely vary over a wide range of spatial and temporal scales.

The third and fourth properties noted above have very important practical consequences in that realistic models of fluid flow are likely to be either (or both) very demanding mathematically or very extensive computationally. The first two of the fluid flow properties noted above — that a fluid is (or must be) idealized as a continuum and that fluid parcels are subject to important local interactions — set the most demanding requirements so far as the choice of a coordinate system is concerned.

1.2 The fundamental principle of kinematics

To set a concrete task, let's suppose that our task is to observe (or measure) the fluid flow within some three-dimensional domain that we will denote by R^3. There are two ways to approach the task of observing a fluid flow, either by tracking individual parcels that are carried about with the flow, or by observing the fluid velocity at locations that are fixed in space. These two sampling strategies are commonly termed the Lagrangian and Eulerian systems, and both of these are widely used in the analysis of continuum dynamics including of the atmosphere and ocean. The Lagrangian perspective is natural for many observational techniques and for the statement of the fundamental conservation theorems. On the other hand, almost all of the theory in fluid dynamics has been developed from the Eulerian perspective. It is for this reason that we will consider both systems in this essay, at first on an equal footing, though for purposes of developing models of fluid flow we will favor the Eulerian or field representation.

Perhaps the most natural way to observe the motion of a fluid is observe the motion of individual fluid parcels that can be identified by dye concentration or by some other marker that does not interfere with their motion. Because fluid motion may vary on what can be very small spatial scales, we will have to consider the motion of correspondingly small parcels. The initial or starting time of the observations will be denoted by t_0 and the corresponding initial position of a parcel denoted by $\boldsymbol{\xi_0}$. Thus the initial position serves to define the identity of a parcel. We somewhat blithely assume that we can determine the position of that specific parcel at all later times, t, to form the parcel trajectory, or pathline,

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{\xi}_0; t). \tag{1}$$

The initial position $\boldsymbol{\xi_0}$ is held fixed for any given trajectory, and hence is a parameter in Eq. (1), in which the only independent variable is the time. The velocity of a parcel, often termed the 'Lagrangian' velocity, $\mathbf{V}_L$, is just the time rate change of it's position,

$$\mathbf{V}_L(\boldsymbol{\xi}_0; t) = \frac{d\boldsymbol{\xi}}{dt}, \tag{2}$$

where this d/dt is an ordinary time derivative (and since we are holding $\boldsymbol{\xi}_0$ fixed, the position of a specific parcel depends only upon the time). To specify *where* this velocity was observed we will have to carry along the parcel trajectory Eq. (1) as well. This Lagrangian velocity of a fluid parcel is exactly the same thing we would mean by the particle velocity in classical dynamics.

If tracking fluid parcels is impractical, say because the fluid is opaque, then we might choose to observe the fluid velocity by means of current meters that we could implant at fixed positions, say $\mathbf{x}$. The essential component of every current meter is a transducer that converts fluid motion into a readily measured signal - e.g., the rotary motion of a propeller or the Doppler shift of a sound pulse. The velocity sampled in this way, often termed the 'Eulerian' velocity, $\mathbf{V}_E$, is intended to be the velocity of the fluid parcels that move through the (fixed) control volume sampled by the transducer, i.e.,

$$\boxed{\mathbf{V}_E(\mathbf{x}, t) = \frac{d\boldsymbol{\xi}}{dt}\,|_{\boldsymbol{\xi}=\mathbf{x}}} \tag{3}$$

The crucial difference between this Eulerian velocity and the Lagrangian velocity $\mathbf{V}_L$ is that the position $\mathbf{x}$ in Eq. (3) is the arbitrary (our choice), fixed position of the current meter, while the position $\boldsymbol{\xi}$ in the Lagrangian velocity of Eq. (2) is the position of a moving parcel. The latter position is, of course, a result of the fluid flow rather than our choice (aside from the initial or starting position). As time runs, the position of any specific parcel will change (barring that the flow is static); meanwhile the velocity observed at the current meter position, $\mathbf{x}$, will be the velocity of the sequence of parcels that move through the current meter position at later times.

Our usage Eulerian and Lagrangian velocity is standard (if no such tag is appended, then Eulerian is almost always understood to be the default), but it may be unfortunate in two respects: it is evidently wrong as historical attribution (Lamb[3] credits Leonard Euler with developing both representations); more importantly for our purpose, it could be misleading if we were led to infer that there are two kinds of fluid velocity. There is one unique fluid velocity that can be sampled in two quite different ways, by tracking moving parcels (Lagrangian) or by observing the motion of fluid parcels past a fixed site (Eulerian). The formal statement of this, Eq. (3), is rather unimpressive, and to emphasize its importance we will give it an imposing title — *the Fundamental Principle of Kinematics* — or FPK for short.[4] Nearly everything we have to say in this essay follows from this principle combined with the familiar conservation laws of classical physics.

Now that we have learned (or imagined) how to observe a fluid velocity, we can begin to think about surveying the entire domain in order to construct a representation of the complete fluid flow. This will require an important decision regarding the sampling strategy; should we make these additional observations by tracking a large number of fluid parcels as they wander throughout the domain, or, should we deploy additional current meters and observe the fluid velocity at many additional sites? In principle, either approach could suffice to define the flow.[5] Nevertheless, the observations themselves and the analysis needed to understand these observations would be quite different, as we will see in examples below. And of

[3]An authoritative and highly recommended text is by H. Lamb, *Hydrodynamics, 6th ed.*, (Cambridge Univ. Press, 1937), the classic tome on fluid dynamics before the age of numerical calculation.

[4]There are about a dozen boxed equations in this essay, beginning with Eq. (3), that you will encounter over and over again in a study of fluid dynamics. These boxed equations are sufficiently important that they should be memorized, and you should be able to explain in detail what each term and each symbol means. The meaning of the symbols is the entire content of this particular equation, which we could have written as an identity.

[5]Examples of fluid flow sampled by Eulerian and Lagrangian methods are in Section 11.1 and 11.2. An excellent web page that develops the theme of Lagrangian and Eulerian observations of a natural system is http://www.sfports.wr.usgs.gov/ rtcheng/Grc.pdf and http://sfbay.wr.usgs.gov/access/flow/drifterstudies/ Another is at http://jerry.ucsd.edu/PDF/SyncL&Emeas.pdf

course in practice our choice of a sampling method will depend at least as much upon purely practical matters - the availability of floats or current meters. Thus it very commonly happens that we may be required to make observations in one system, and then apply theory or diagnostic analysis in the other. A similar kind of duality arises in the development of models and theories; the (Lagrangian) parcels of a fluid flow follow conservation laws that are identical with those followed by the particles of classical dynamics; nevertheless the models and theories commonly applied to fluid flows are almost exclusively Eulerian. The broad goal of this essay is to develop an understanding of Eulerian and Lagrangian representations of fluid flow and specifically to understand how Lagrangian and Eulerian concepts are woven together to implement the observation and analysis of fluid flows.

1.3 About this essay*

This essay is pedagogical in aim and in style. It has been written for students who are somewhere near the beginning of a quantitative study of fluid dynamics and who have at least a modest background of classical mechanics and applied mathematics. In most introductory courses on fluid dynamics the distinction between Eulerian and Lagrangian systems is revealed gradually, as an aside to the main theme - Eulerian theory. The premise of this essay is that the central ideas of fluid kinematics (a rather 'dry' topic) can be developed and organized around an explicit treatment of Eulerian and Lagrangian systems and specifically the transformation between these two systems.

The plan is to describe further the Lagrangian and Eulerian systems in Section 2 and 3, respectively, and then to consider how both systems are invoked to make (visual) descriptions of fluid flows in Section 4. It commonly happens that Eulerian velocity field data, say from a numerical model, have to be transformed into Lagrangian properties, e.g., trajectories, a problem considered in Section 5. The reverse transformation underlies the development of most theories: the conservation laws begin with a Lagrangian perspective, and then for most purposes of theory have to be transformed into an Eulerian system, the topic of Sections 6 and 7. In an Eulerian system the process of transport is represented by advection, the inherently difficult part of most fluid models or theories considered in Section 8. Section 9 completes the development of the ideal fluid model in Eulerian and Lagrangian form. Section 10 is a very brief summary.

The material presented here is not new in any significant way and indeed most of it comes from the foundation of fluid dynamics. There is no attempt to cite original sources since they have long since disappeared from circulation. One notable modern source for kinematics is by Aris (1962) [6] and another, less modern in appearance but no less valuable, is the classic by Lamb (1937). [3] Most comprehensive fluid dynamics texts used for introductory courses include at least some discussion of Lagrangian and Eulerian representations, but not as a central theme. This essay is most appropriately used as a follow-on to a comprehensive text. [7]

[6] A rather advanced and mathematical source for fluid kinematics is Chapter 4 of Aris, R., *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, (Dover Pub., New York, 1962). A particularly good discussion of the Reynolds Transport Theorem (discussed here in Section 7.1) is by C. C. Lin and L. A. Segel, *Mathematics Applied to Deterministic Problems in the Natural Sciences* (MacMillan Pub., 1974).

[7] Modern examples include excellent texts by P. K. Kundu and I. C. Cohen, *Fluid Mechanics* (Academic Press, 2001), by B. R. Munson, D. F. Young, and T. H. Okiishi, *Fundamentals of Fluid Mechanics, 3rd ed.* (John Wiley and Sons, NY, 1998) and by D. C. Wilcox, *Basic Fluid Mechanics* (DCW Industries, La Canada, CA, 2000). Still another superb text that emphasizes experiment and

2 The Lagrangian (or Material) Coordinate System.

One helpful way to think of a fluid flow is that it carries or maps parcels from one position to the next, i.e., from a starting position $\boldsymbol{\xi_0}$ into the positions $\boldsymbol{\xi}$ at later times. We will assume that this mapping is unique in that adjacent parcels will never be split apart, and neither will one parcel be forced to occupy the same position as another parcel. Thus given a $\boldsymbol{\xi_0}$ and a time, we will presume that there is a unique $\boldsymbol{\xi}$. We must assume that a fluid parcel can be taken to be as small as is necessary to meet these requirements, and that a fluid is, in effect, a continuum, rather than made up of discrete molecules. With these conventional assumptions in place, the mapping of points from initial to subsequent positions (i.e., the trajectory) can be inverted, at least in principle. Thus, given a $\boldsymbol{\xi}$ and a time, we can assume that there is a corresponding unique $\boldsymbol{\xi_0}$.

Each trajectory that we observe or construct must be tagged with it's unique $\boldsymbol{\xi_0}$ and thus for a given trajectory $\boldsymbol{\xi_0}$ is a constant. Though $\boldsymbol{\xi_0}$ is constant for a given parcel, we have to keep in mind that our coordinate system is meant to describe a continuum defined over some domain, and that $\boldsymbol{\xi_0}$ varies continuously over the entire initial domain of the fluid. Thus when we need to consider the domain as a whole, $\boldsymbol{\xi_0}$ has the role of being the independent, spatial coordinate; we will see the consequences of this in Section 9.4. This kind of coordinate system in which parcel position is the fundamental dependent spatial variable is often referred to as a Lagrangian coordinate system, and also and perhaps more aptly as a 'material' coordinate system.

We have noted already in Section 1 and Eq. (2) that the Lagrangian velocity is just the ordinary time derivative of the position, and the acceleration of a fluid parcel is just the second (ordinary) time derivative,

$$\frac{d\mathbf{V}_L(\boldsymbol{\xi_0};t)}{dt} = \frac{d^2\boldsymbol{\xi}}{dt^2} \tag{4}$$

From the fact that we are differentiating $\boldsymbol{\xi}$ it should (and must, really) be understood that we are asking for the velocity of a specific parcel, and that we are holding $\boldsymbol{\xi_0}$ fixed during this differentiation. Given that we have defined and can compute the acceleration of a fluid parcel, we go on to presume that Newton's three laws of classical dynamics apply to a fluid parcel in exactly the form used in classical (solid particle) dynamics, i.e.,

$$\frac{d\mathbf{V}_L}{dt} = \mathbf{F}/\rho, \tag{5}$$

where $\mathbf{F}$ is the force per unit volume imposed upon that parcel, and ρ is the density or mass per unit volume

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fluid phenomenon is by D. J. Tritton, *Physical Fluid Dynamics*, (Oxford Science Pub., 1988).

I suggest that this essay is best seen as a secondary rather than a primary reference because many of the concepts or tools used here, e.g., velocity gradient tensor, Reynolds Transport Theorem, characteristics, etc., are reviewed only rather briefly. The contribution of this essay is (or is intended to be) that it shows how these things may be understood as aspects of the Lagrangian and Eulerian representation of fluid flows.

of the fluid. Thus if we observe that a fluid parcel undergoes an acceleration, then we can infer that there had to have been an applied force on that parcel. It is on this kind of diagnostic problem that the Lagrangian coordinate system is most useful, generally.

Before going much further it may be helpful to consider a very simple but concrete example of a flow represented in the Lagrangian framework. Let's assume that we have been given the trajectories of all the parcels in a one-dimensional domain R^1 with spatial coordinate y by way of the explicit formula[8]

$$\xi(\xi_0; t) = \xi_0(1 + 2t)^{1/2} \tag{6}$$

where $t_0 = 0$. Once we specify the starting position of a parcel, $\xi_0 = \xi(t = 0)$, this handy little formula tells us the y position of that specific parcel at any later time. It is most unusual to have so much information presented in such a convenient way, and in fact, this particular 'flow' has been concocted to have just enough complexity to be interesting for our purpose here, but is without physical significance. The velocity of a parcel is then

$$V_L(\xi_0; t) = \xi_0(1 + 2t)^{-1/2} \tag{7}$$

and the acceleration (and also the force per unit volume) is just

$$\frac{d^2\xi}{dt^2} = -\xi_0(1 + 2t)^{-3/2}. \tag{8}$$

Given the initial positions of four parcels, let's say ξ_0 = [0.1 0.3 0.5 0.7] we can readily compute the trajectories and velocities from Eqs. (6) and (7), Fig. 1b and 1c. Note that the velocity depends upon the parcel initial position, ξ_0. If V did not depend upon ξ_0, then the flow would necessarily be spatially uniform, i.e., all the fluid parcels in the domain would have exactly the same velocity. The flow shown here has the following form: all parcels shown (and we could say all of the fluid in $y > 0$) are moving in the direction of positive y; parcels that are at larger y move faster; all of the parcels in $y > 0$ are decelerating in the sense that their speed decreases with time, and the magnitude of the deceleration increases with y.

If, as presumed in this example, we are able to track parcels at will, then we can sample as much of the domain and any part of the domain that we may care to investigate. In a real, physical experiment (Section 11.2) this could be a bit problematic; we could not be assured that any specific portion of the domain will be sampled unless we launched a parcel there. Even then, the parcels may spend most of their time in regions we are not particularly interested in sampling, a hazard of Lagrangian observation.

Consider the information that this Lagrangian representation provides; in the most straightforward way it shows *where* fluid parcels released into a flow at a given time and position will be found at some later time. If our goal was to observe how a fluid flow carried a pollutant, say, from a source (the initial position) into the rest of the domain, then this Lagrangian representation would be ideal. We could simply release (or tag) parcels over and over again at the source position, and then observe where the parcels were subsequently carried by the flow. By releasing a cluster of parcels we could observe how the flow deformed or strained the fluid (e.g., the float cluster shown on the cover page and in Section 11.2; and taken up in detail in Section 8.3). Similarly, if our goal was to measure the force applied to the fluid, then by tracking parcels and

[8]When we separate a list of parameters and variables by semicolons or commas as $\xi(\xi_0; t)$ on the left hand side, we mean to emphasize that ξ is a function of ξ_0, a parameter, and t, an independent variable. When variables are separated by operators, as $\xi_0(1 + 2t)$ on the right hand side, we mean that the variable ξ_0 is to be multiplied by the sum $(1 + 2t)$.

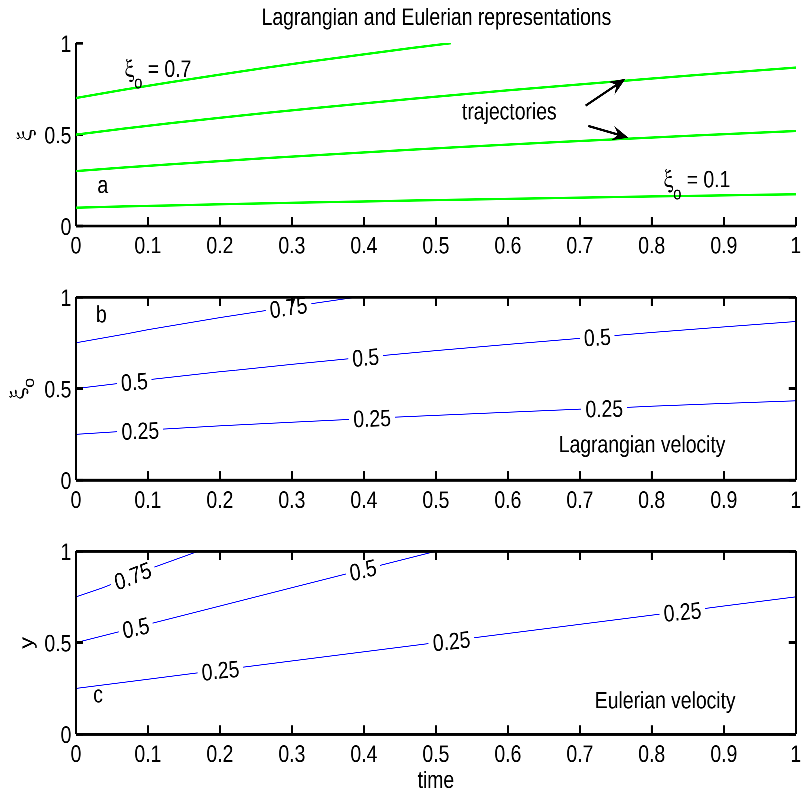

Figure 1: Lagrangian and Eulerian representations of the one-dimensional, time-dependent flow defined by Eq. (6). (a) The solid lines are the trajectories $\xi(\xi_0; t)$ of four parcels whose initial positions were $\xi_0 = 0.1$, 0.3, 0.5, 0.7. (b) The Lagrangian velocity as a function of time and initial position. The lines plotted here are contours of constant velocity, *not trajectories*. (c) The Eulerian velocity field computed in Section 3 by solving for $V(y, t)$, and again the lines are of constant velocity.

observing their accelerations we could estimate the force directly via the Lagrangian equation of motion, Eq. (4) (it is very hard to envision a force attached to parcels in just the way indicated by Eq. (8)). These are important and common uses of the Lagrangian coordinate system but note that they are all related in one way or another to the observation of fluid flow rather than to the calculation of fluid flow.

If our goal was to carry out a forward calculation, i.e., to compute rather than observe parcel trajectories, then in order to use the Lagrangian equation of motion, Eq. (5), we would have to be able specify the sum of the forces, $\mathbf{F}$, acting on the fluid parcel. As was hinted before, this can be very awkward in a Lagrangian system. The reason is that $\mathbf{F}$ on a parcel will very likely depend upon the spatial gradient, $\partial/\partial x$, of the pressure and the velocity in the vicinity of the parcel. The spatial gradient is not readily calculated in a Lagrangian system since the independent (spatial) variable is the initial parcel position and not the usual spatial coordinates, which are dependent variables in the Lagrangian system. Although the Lagrangian equation of motion is simple and familiar, nevertheless the specification of $\mathbf{F}$ in a multi-dimensional *continuum* is usually extremely difficult (shown in Section 9.4) and the upshot is that Lagrangian models are well-suited to forward calculation only in somewhat special circumstances.[9] Though

[9]To be sure, some of these circumstances are both important and inherently interesting. In this course we will examine a (La-

we will not try to solve more than one or two Lagrangian systems, it is often highly desirable to compute Lagrangian diagnostics — the trajectories, accelerations, etc. of fluid parcels given the velocity field of an Eulerian model. This will be considered in the following Section 4.

3 The Eulerian (or Field) Coordinate System.

We have been keen on extolling the virtues of Lagrangian observation, but we should admit to some inherent problems, as well. The spatial sampling of Lagrangian data is more or less uncontrolled since the parcels will go wherever the flow takes them, and that may not be where we our interest lies! And so, for example, if our goal was to observe the long term average flow through a channel, we might prefer to moor a current meter directly in the channel rather than chase floats or drifters in and out. If high temporal resolution was desirable then again we might find that it was preferable to install a rapidly sampling current meter rather than resample with Lagrangian methods. Examples in which the flow conditions at a specific site are the desired goal abound and so do the need for Eulerian observations.

An important and rapidly developing observational technique involves the generation of the Eulerian velocity field from Lagrangian data by the analysis procedure of interpolating or mapping irregularly sampled Lagrangian data $V_L(\xi_0; t)$ on to a spatial grid. To know where to assign the velocity we will also have to know the position, $y = \xi(\xi_0; t)$. This kind of procedure, which is a direct application of the FPK, is widely used to make maps of entire flow fields at once. One such method is known as Particle Imaging Velocimetry, or PIV. On the laboratory scale, the PIV technique uses successive photographic images of neutrally buoyant particles that might be illuminated by a pulsed laser source. Provided that the particles (or the pattern that they form) can be recognized from one image to the next, then it is fairly simple to differentiate parcel position with respect to time and then form a map, sometimes in three-dimensions, of the flow throughout the domain. These kinds of data can provide excellent spatial resolution of flow features that is often essential for the diagnosis of fluid dynamics.[10]

In the contrived example of a Lagrangian flow considered in Section 2 we have the huge advantage of knowing all the parcel trajectories via Eq. (6) and so we can so make the transformation from the Lagrangian to the Eulerian system explicitly. Formally, the task is to eliminate all reference to the parcel initial position, ξ_0, in favor of the position $y = \xi$. This is readily accomplished since we can invert the trajectory Eq. (6) to

grangian) model of interacting free vortices. To the approximation of potential flow theory these vortices interact by advecting one another about. Some important microscale aspects of a dilute gas, e.g., diffusivity or viscosity or reaction dynamics, can be modeled using kinetic theory that takes the individual gas molecules to be hard spheres that interaction via two-particle collisions (an accessible and highly recommended introduction to gas theory is by A. J. Garcia, 'Numerical Methods for Physics' (Prentice Hall, New Jersey, 2000; see Ch. 11)). A collection of stars may be modeled as a dilute gas with interactions via gravitational mass attraction. Finally, the method of characteristics considered in Section 8.2 takes a Lagrangian perspective on a continuum fluid model, but is usually applicable only in rather special (but very interesting) circumstances. So, while we may claim that Lagrangian models are not widely applicable, investigators who work in some other disciplines would not agree!

[10] Links to PIV web pages are found at http://www.efluids.com/efluids/pages/products/piv.html A very nice illustration of the use of PIV to diagnose a Taylor-Couette flow can be found at http://www/widget.ecn.purdue.edu/∼swereley/piv.html The tracers used in the PIV technique may be naturally occurring features as well, see http://www.irisa.fr/vista/Themes/Demos/MouvementFluide/infra.html for the use of satellite imagery to infer the motion of the underlying atmosphere. There are float systems (satellite-tracked) that allow high density observations of ocean currents at the sea surface and at depth, and allow construction of quite detailed maps of the time-mean Eulerian circulation, see http://iprc.soest.hawaii.edu/research/slides/theme2/04-spring-t2-1.pdf

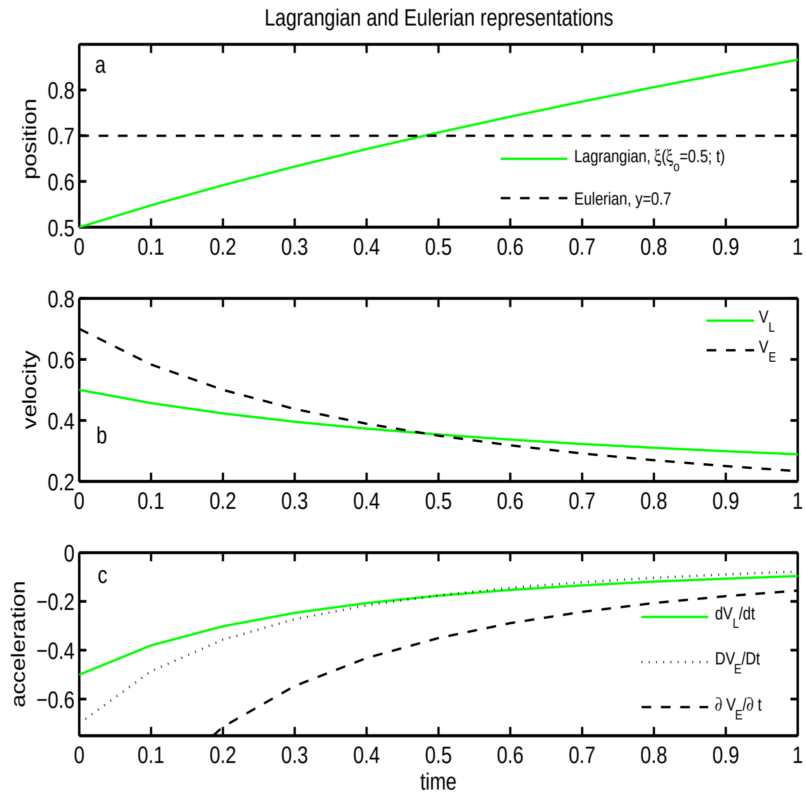

Figure 2: Lagrangian and Eulerian representations of the one-dimensional, time-dependent flow defined by Eq. (6). (a) Positions of the trajectory $\xi(\xi_0 = 0.5, t)$ (green, solid line) and the Eulerian observation site $y = 0.7$ (dashed line), which is, of course, fixed. (b) The Lagrangian and Eulerian velocity at $\xi_0 = 0.5$ and $y = 0.7$, respectively. Note that the parcel identified by $\xi_0 = 0.5$ crosses the Eulerian observation position $y = 0.7$ at time $t = 0.48$, computed from Eq. (7). At that time these specific Lagrangian and Eulerian velocities are equal, but not otherwise. That this equality holds is at once trivial - a non-equality could only mean an error in the calculation - but also consistent with and illustrative of the FPK, Eq. (3). (c) Accelerations for the parcel and the fixed position used just above. There are two ways to compute a time rate change of velocity at a fixed point; one of them, DV_E/Dt, is the counterpart of the Lagrangian acceleration, in the sense that at the time the parcel crosses the Eulerian observation site, $dV_L/dt = DV_E/Dt$, a crucially important point discussed in detail in Section 6.

find ξ_0,

$$\xi_0 = y(1+2t)^{-1/2}. \tag{9}$$

where we have already substituted y for ξ. Substitution into (7) and a little rearrangement gives the velocity field for this flow

$$V_E(y,t) = y(1+2t)^{-1} \tag{10}$$

which is plotted in Fig. (1c). Admittedly, this is not an interesting velocity field, but rather a very simple one, and partly as a consequence, this (Eulerian) velocity field looks a lot like the Lagrangian velocity of moving parcels (cf, Eq. 6 and Fig. 1b). However, the spatial coordinates in Figs. (1b, 1c) are different - in the b its ξ_0, while in c its an ordinary spatial coordinate, y - and so to compare the Eulerian and the Lagrangian velocities over some region is a bit like comparing apples and oranges; they are qualitatively different things. Though different generally, nevertheless there are times and places where the two velocities are exactly the same, as evinced by the Fundamental Principle of Kinematics or FPK. By tracking a parcel around in this flow and by observing velocity at a fixed site (in Fig. 2 we have arbitrarily chosen the parcel tagged by $\xi_0 = 0.5$ and the observation site $y = 0.7$), we can verify that the Eulerian and the Lagrangian velocities are equal at a common y and time consistent with the FPK, Eq. (3) (Fig. 2b). Indeed, there should be an *exact* equality since there has been no need for approximation in this transformation Lagrangian $\rightarrow$ Eulerian.[11]

The acceleration (Fig. 2c) is a little more involved. For now, suffice it to say that the derivative with respect to time of velocity at a fixed position, $\partial V_E/\partial t$, is generally *not* equal to the Lagrangian acceleration at the same time and place, but another kind of time derivative, denoted by DV_E/Dt, is equal to the Lagrangian acceleration, a very important matter taken up in Section 6.

4 Depictions of Fluid Flow.

It may be apparent from the previous discussion that simply showing what a fluid flow looks like will be a significant task in cases where the domain is multi-dimensional and the flow is time-dependent. A variety of methods are commonly used to show the flow dependence upon one or more of the independent variables, and it is important to understand what specific aspects of a flow can be revealed by one means or another. This topic of flow representation can be seen as a continuation of the Lagrangian-Eulerian transformation problem considered above. In this section we will assume that we know the Eulerian velocity field, and that we may need to compute certain Lagrangian properties of the flow. [12]

[11] Here's one for you: assume Lagrangian trajectories $\xi = a(e^t + 1)$ with a a constant. What is the position of parcels at $t = 0$? Compute and compare the Lagrangian velocity $V_L(\xi_o, t)$ and the Eulerian velocity field $V_e(y,t)$. Suppose that two parcels have initial positions $\xi_o = 2a$ and $2a(1+\delta)$ with $\delta \ll 1$; how will the distance between these parcels change with time? How is the rate of change of this distance related to V_E? (Hint: consider the divergence of the velocity field, $\partial V_E/\partial y$.) Suppose the trajectories are instead $\xi = a(e^t - 1)$.

[12] An excellent web page that shows the practical reasons for tracking air parcels (and the things that go with them) using Eulerian data from large scale numerical models of the atmosphere is at http://www.arl.noaa.gov/slides/ready/conc/conc2.html

4.1 Pathlines, or trajectories

One important example is the parcel trajectories, often called pathlines. In this section we will consider position and velocity in a two-dimensional space, R^2, and $\mathbf{x}$ and $\mathbf{V}$ indicate vector position and velocity. From here on out we are going to drop the subscripts L and E that have been used to emphasize Lagrangian and Eulerian velocity. The kind of velocity should be clear from the context, or from the list of independent variables.

We can compute parcel trajectories from the Eulerian velocity field via

$$\boxed{\frac{d\mathbf{x}}{dt} = \mathbf{V}(\mathbf{x}(t), t)} \tag{11}$$

provided we recognize that $\mathbf{x}$ on the right side is the moving (time-dependent) parcel position. The appropriate initial condition is just

$$\mathbf{x}(t = 0) = \xi_0. \tag{12}$$

Note that (11) is in the form of the FPK, or Eq. (3). In component form this may be written out

$$\frac{dx}{dt} = U(x, y, t); \quad \frac{dy}{dt} = V(x, y, t) \tag{13}$$

and with the initial conditions (ICs)

$$x(t = t_0) = \xi_{xo}; \quad y(t = t_0) = \xi_{yo} \tag{14}$$

which makes clear that we have two first order ODEs. On first sight these trajectory equations (13) could be deceptive; as here written they are quite general and applicable to any fluid motion in R^2. Thus it should not be surprising if on most occasions they prove intractable by elementary methods. If U depends upon y or V, or if V depends upon x or U, then these are coupled equations that have to be solved simultaneously; if U or V are nonlinear then they are nonlinear equations. Either way their solution may have to be sought with numerical techniques. What is surprising about Eq. (13), even after several encounters, is that what can seem to be very simple velocity fields can yield complex and interesting trajectories (one example is in Section 5.2).

We can best illustrate these diagnostic quantities with a two-dimensional velocity field,

$$\mathbf{V} = x\mathbf{e}_x + \frac{y}{1 + 2t}\mathbf{e}_y, \tag{15}$$

which is plotted for two times in Fig. 3. The component equations are then

$$\frac{dx}{dt} = x; \quad \frac{dy}{dt} = \frac{y}{1 + 2t}, \tag{16}$$

and with ICs as above. The dependent variables are uncoupled, and moreover, within each component equation the independent variables can be readily separated,

$$\frac{dx}{x} = dt; \quad \frac{dy}{y} = \frac{dt}{(1 + 2t)}.$$

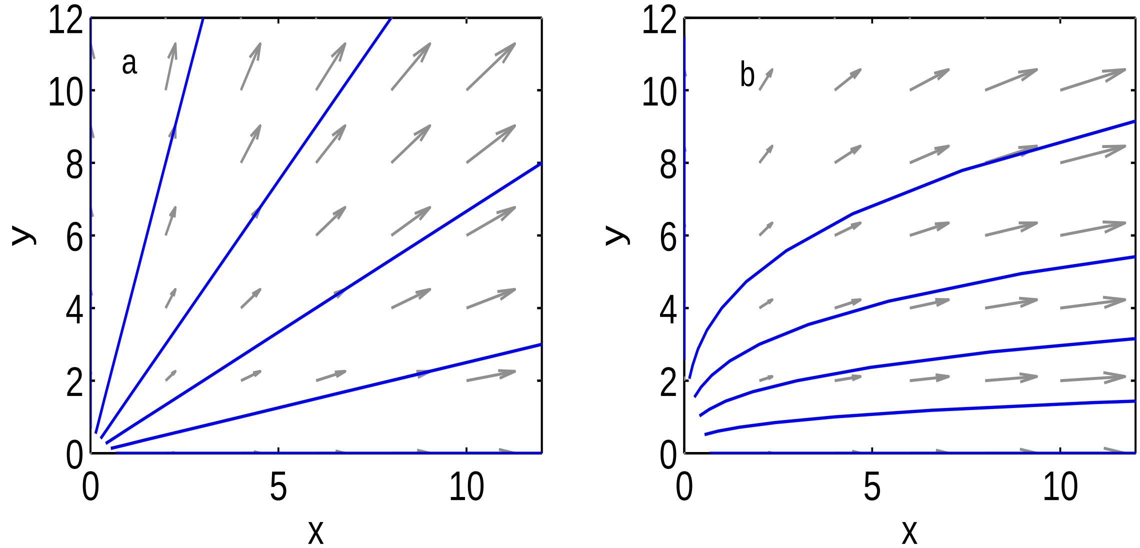

Figure 3: (a) Velocity field and streamlines (the family of solid lines) for Eq. (16) at $t = 0$. (b) At $t = 1$. Notice that the velocity at a given point turns clockwise with time as the y-component of the velocity decreases with time.

These can then be integrated over the limits ξ_{xo} to x (ξ_{yo} to y) and t_0 to t to yield the trajectory

$$x(\xi_{xo}, t_0; t) = \xi_{xo} exp(t - t_0); \quad y(\xi_{yo}, t_0; t) = \xi_{yo} \frac{(1 + 2t)^{1/2}}{(1 + 2t_0)^{1/2}}. \tag{17}$$

Notice that the y-component is just as before, Eq. (6), except that we have retained the initial time as a parameter (we will need it below). Trajectories staring from a few different ξ_0 are in Fig. 4. In this case the trajectories are roughly in the direction of the flow as seen in Fig. 3, and appear to bend over in time, consistent with the temporally-decreasing y-component of the velocity field. There is nothing surprising in this case, but in a later section, 5.2, we will see an example where the trajectories could not have been anticipated in advance of an integration, and the Eulerian and Lagrangian mean flows are qualitatively different.

4.2 Streaklines*

Another useful characterization of the history of parcel positions is the so-called streakline, which shows the positions, at a fixed time, of all of the parcels which at some earlier time passed through a given point. An example of this would be the plume of smoke coming from a point source located at $\mathbf{x}_p$ and recorded, say by a photograph taken at a time, t_p. The information needed to construct a streakline is contained within the trajectory, Eq. (17). To see this we will construct a streakline by releasing parcels one after the other from a fixed source. The first parcel is released at time $t_0 = 0$, and we let the trajectory run until $t = t_p$, the time we make the photograph. The only data point we retain from this trajectory is the position at time $t = t_p$, i.e., we record $\mathbf{x}(t_p; \mathbf{x}_p, t_0 = 0)$. A second parcel is released a little later, say at $t_0 = \frac{1}{4}$, and again we let the

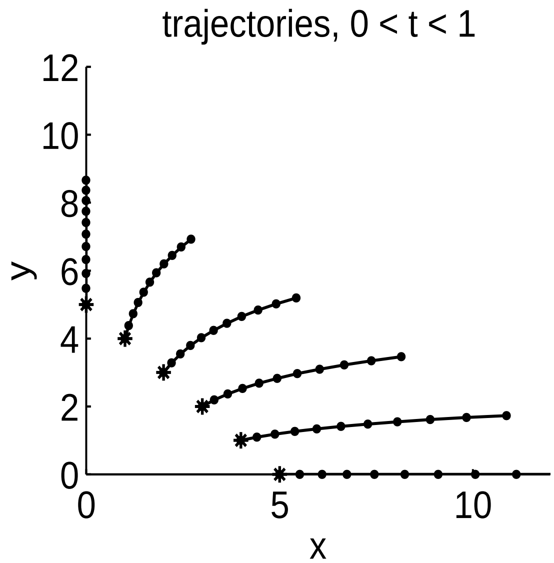

Figure 4: Trajectories of six parcels that were released into the flow given by Eq. (16) at the same time, $t_0 = 0$, and tracked until $t = 1$. The sources are shown by asterisks. Dots along the trajectories are at time intervals of 0.1

trajectory run until $t = t_p$, where we retain only the last point, $\mathbf{x}(t_p; \mathbf{x}_p, t_0 = \frac{1}{4})$. A third parcel is released at $t_0 = \frac{2}{4}$, and again we record it's position at $t = t_p$, $\mathbf{x}(t_p; \mathbf{x}_p, t_0 = \frac{2}{4})$. It appears, then, that a recipe for making streakline from a trajectory is that we treat the initial time, t_0, as a variable, while holding t constant at t_p, and also the initial position at $\mathbf{x} = \mathbf{x_p}$. Several streaklines are in Fig. 5. Notice that in this time-dependent flow, trajectories and streaklines are not parallel.

4.3 Streamlines

Still another useful method is to draw streamlines, a family of lines that are everywhere parallel to the velocity. Time is fixed, say at $t = t_f$, and thus streamlines portray the direction field of a velocity field, with no reference to parcels or trajectories or time-dependence of any sort. There is more than one way to construct a set of streamlines, but a method that lends itself to generalization is to solve for the parametric representation of a curve, $\mathbf{X}(s)$ that is everywhere parallel to the velocity;

$$\boxed{\frac{d\mathbf{X}}{ds} = \mathbf{V}(t_f; x, y)} \tag{18}$$

or in components;

$$\frac{dX}{ds} = U(t_f; x, y); \quad \frac{dY}{ds} = V(t_f; x, y). \tag{19}$$

A suitable 'initial' condition is $X(s_0) = X_0$, etc. Notice that s is here a dummy variable; we could just as well have used any other symbol but s is conventional.[13] $\mathbf{X}$ is the position of a point on a line, where just

[13] In fact, s could be regarded as time, provided we make certain not to confuse this use of time with the time-dependence of the velocity field (which is suppressed while we draw a given map of streamlines). This helps make clear that streamlines are parallel

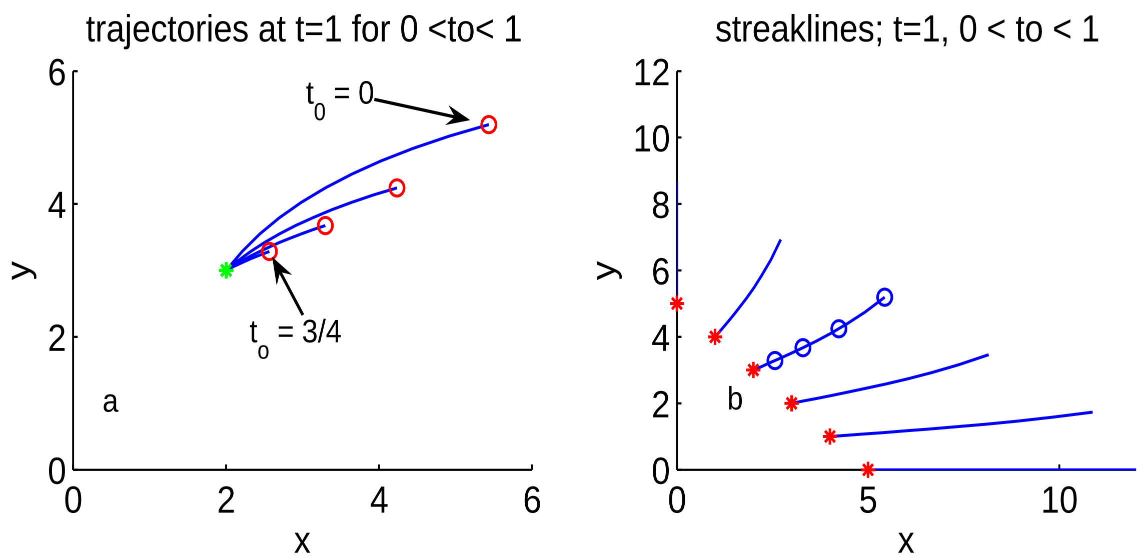

Figure 5: (a) Trajectories of five parcels that were released from a common source, (x,y) = (2,3), and tracked until $t = 1$. The parcels were released at different initial times, $t_0 = 0,\ 1/4,\ 2/4,\ 3/4,$ and 1. The latter trajectory has zero length. The end points of the trajectories are the open circles, the locus of which forms a streakline. (b) Streaklines from several different sources. These streaklines start at $t_0 = 0$ and the 'photograph' was taken at $t_p = 1$. Notice that these streaklines end at the endpoint of the trajectories of Fig. 4 (they have that one point in common) but that streaklines generally have a different shape (different curvature) from the trajectories made over the same time range. In this figure and in previous ones (Fig. 2b) the two quantitites being compared were only slightly different and one might well wonder if, for example, streaklines are some kind of approximation to trajectories. The answer is 'no', in general, they are qualitatively different.

above $\mathbf{x}$ meant the position of a parcel. This reuse of symbols is certainly a risky practice, but it's also almost unavoidable. Given the velocity components Eq. (16), these equations are also readily integrated to yield a family of streamlines:

$$X = X_0 exp(s - s_0); \quad Y = Y_0 exp\left(\frac{s - s_0}{1 + 2t_f}\right), \tag{20}$$

and recall that t_f is the fixed time that we draw the streamlines. We are free to choose the integration constants so that a given streamline will pass through a position that we specify. There is no rule for choosing these positions; in Fig. 3 we arbitrarily picked five positions and then let s vary over sufficient range to sweep through the domain. Other streamlines could be added if needed to help fill out the picture. No particular value is attached to a given streamline. In the future we will consider the streamline's sophisticated cousin, the streamfunction, which has isolines that are also parallel to velocity, but which assigns values that are related to the speed of the flow.

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to parcel trajectories in steady flows. If we marked off equal increments of time along a streamline we could depict the speed of the flow.

5 Eulerian to Lagrangian Transformation by Approximate Methods.

The previous sections emphasized the purely formal steps required to transform from one reference frame to the other. An understanding of the formal steps is important, of course, but the ease with which we could make the transformation in those cases could be positively misleading. In actual practice, an explicit and invertible specification of trajectories over an entire domain is highly unlikely, and even in the case that a complete field specification is available, it probably can not be integrated by elementary methods. In this section we will consider an approximate method based upon an expansion of the velocity field in Taylor series. This yields results that are interesting and important of themselves, and introduces some new tools, e.g., the velocity gradient tensor, that are widely useful.

5.1 Tracking parcels around a steady vortex

The power and the limitations of the Taylor series method can be appreciated by analysis of parcel motion in a steady, irrotational vortex in R^2. The radial and azimuthal velocity components are given by

$$\mathbf{V} = (U_{rad},\ U_{azi}) = (0,\ C/2\pi r), \tag{21}$$

where r is the distance from the vortex center. The $1/r$ dependence of azimuthal speed is the distinguishing feature of an irrotational vortex. C is a constant, termed the circulation,

$$C = \oint \mathbf{V} \cdot d\mathbf{s},$$

where $d\mathbf{s}$ is the vector line segment along a path that encloses the vortex center and that is traversed in an anti-clockwise direction. C measures the vortex strength, and without loss of generality we can set $C = -2\pi$ to define a vortex that rotates clockwise, Fig. (6). An irrotational vortex is an idealization of the vortex flow produced by the convergent flow into a drain, for example, and has several interesting properties that we will consider in later sections (including why it it is said to be irrotational). For now it makes a convenient flow into which we can insert floats and current meters to investigate kinematics. It is apparent that parcel trajectories in this steady vortex will be circular, and that a parcel will make a complete orbit in time $T = (2\pi r)^2/C$. The Cartesian velocity components are

$$\begin{pmatrix} U \\ V \end{pmatrix} = \frac{C}{2\pi r} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \tag{22}$$

where $r = (x^2 + y^2)^{1/2}$ and the angle $\theta = \mathrm{atan}(y/x)$ is measured counter-clockwise from the x-axis.

For the sake of this development let's suppose that the only thing we know is the velocity observed at one fixed site, say $(x_s,\ y_s) = (0,\ 1)$.[14] The velocity observed at this fixed site, $\mathbf{V}_0 = \mathbf{V}(x_s,\ y_s)$, an Eulerian velocity, would then be a steady, uni-directional flow having Cartesian components

[14] When we write the Cartesian components of a vector within the body of the text as here, they will be written as if they were in a row matrix, i.e., $(x,\ y)$; when they are written as a separate equation they will be written as a column matrix, Eq. (22), which is the form in which they actually appear in tensor equations.

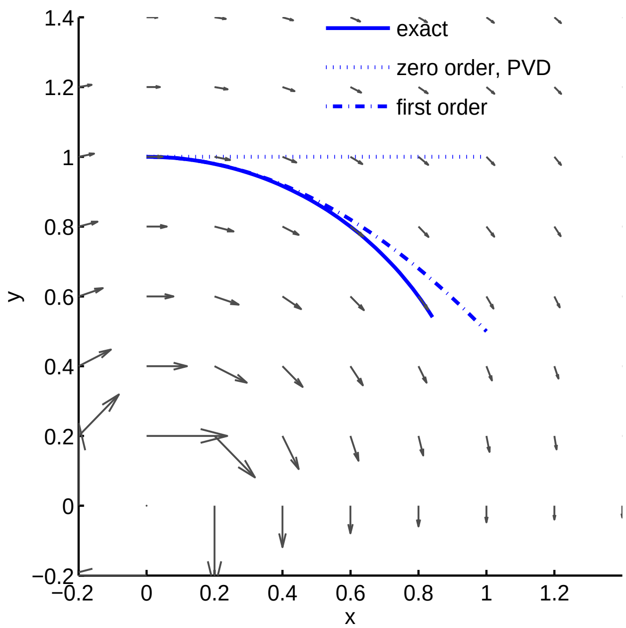

Figure 6: Parcel tracking around a clockwise-rotating, irrotational (!) vortex. Parcels were released at $(x, y) = (0, 1)$ and then followed for one unit of time. The exact trajectory (solid line) was easily computed in cylindrical coordinates. The zero order approximation, Eq. (23), or PVD, (dotted line) starts off well enough, but begins to accumulate a noticeable error when the displacement becomes even a small fraction of the radius. The first order trajectory, Eq. (27), (dash-dot line) accounts approximately for the spatial variation of the velocity and is considerably better than the PVD, but it too begins to fail when the displacement becomes comparable to the radius.

$(U_0,\ V_0) = -(C/2\pi r,\ 0) = (1,\ 0)$. What, if anything, can be inferred about parcel trajectories from this limited data? If no other information was available, then we might make a first attempt at estimating the displacement of the parcel by integrating the Eulerian velocity in time as if it were the Lagrangian velocity, i.e.,

$$\delta \mathbf{X}_0 = \int \mathbf{V}_0 dt = \begin{pmatrix} U_0 t \\ 0 \end{pmatrix}. \tag{23}$$

It is essential to understand that such a procedure is wrong, formally. But it is also fruitful to see the result, termed a progressive vector diagram, or PVD, as a lowest (zero) order approximation of the trajectory.[15] The PVD in this case indicates a linear (pseudo-)trajectory, consistent with the uni-directional velocity observed at the fixed site, Fig. 6. A PVD is a useful way to visualize a current meter record or a wind record insofar as it gives a direct measure of how much fluid has gone past the observation site. But the question here is to what extent does a PVD show *where* fluid parcels will go after they pass through the observation site? That's a hard question to answer generally, but a related question - is there any flow condition under which we could we interpret a PVD as if it were a parcel trajectory? - leads to a useful analysis. A PVD would represent a true trajectory if the Eulerian velocity field was spatially uniform. Observations made at any position would then be equal to observations made anywhere else, including at the moving position of a parcel. A spatially uniform flow is a degenerate case of little interest, but this helps us to see that the issue insofar as this Eulerian to Lagrangian transformation is concerned is the spatial variation of the flow.

If we consider the example of a steady vortex flow, then it would appear that the PVD is an acceptable trajectory estimate only for (pseudo-)displacements that are much less than the horizontal distance over

[15]The notation $\delta\mathbf{X}$ would usually mean a displacement vector that is small in some sense, e.g., compared to the radius of convergence of a power series. Here we are going to integrate long enough for the displacement to be substantial, and then we will call $\delta\mathbf{X}$ the trajectory. This abuse of δ is intentional, because we want to see the consequences of violating the small displacement restriction.

which the flow changes significantly. By inspection, the horizontal scale of this vortex is estimated to be the radius (at a given point), and so this condition could be written $\delta \mathbf{X}_0 << r$. When the displacement is greater than this, the velocity at the position of the parcel (the Lagrangian velocity that we should be integrating) will begin to differ significantly from the velocity observed back at the fixed site (the Eulerian that we are integrating in this PVD-approximation). Once this discrepancy is evident, the PVD will soon fail to make a good approximation to the actual trajectory.

We could improve on this first attempt at computing a trajectory from Eulerian data if we could take some account of the spatial variation of the velocity. To do this we can represent the velocity field in the vicinity of the observation point by expanding in a Taylor series, here for each component separately,

$$\begin{aligned} U(x,y) &= U_0 + \frac{\partial U}{\partial x}\delta X + \frac{\partial U}{\partial y}\delta Y + HOT, \\ V(x,y) &= V_0 + \frac{\partial V}{\partial x}\delta X + \frac{\partial V}{\partial y}\delta Y + HOT, \end{aligned} \tag{24}$$

where (U_0, V_0) is the velocity observed at the observation site, the partial derivatives are evaluated at the observation site, and HOT is the sum of all the higher order terms that are proportional to $\delta x^2, \delta x^3$, etc. In effect, we are now allowing that we know not only the velocity but also the four partial derivatives, though at one position only. It is very convenient to use a vector and tensor notation to write equations like these in a format

$$\mathbf{V}(x,y) = \mathbf{V}_0 + \mathbb{G}\cdot\delta\mathbf{X} + HOT, \tag{25}$$

where $\mathbb{G}$ is the velocity gradient tensor,

$$\boxed{\mathbb{G} = \begin{pmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{pmatrix}} \tag{26}$$

$\mathbb{G}$ recurs in the study of kinematics and we will encounter it again in Section 8.3. For now $\mathbb{G}$ is a device to streamline notation; when we matrix multiply[16] $\mathbb{G}$ into a displacement vector (written as a column vector), we get the velocity difference that corresponds to that displacement vector. It is easy to see that if we doubled or halved the length of the displacement vector we would get twice or half the velocity difference. Thus, multiplication by the velocity gradient tensor serves to make a linear transformation on a displacement vector. In general the result will be a velocity difference vector having a different direction from that of the displacement vector, and of course it has a different amplitude and different dimensions as well. The velocity gradient tensor evaluated at the observation site $(x_s,\ y_s) = (1,\ 0)$ has a simple form

$$\mathbb{G} = \frac{C}{2\pi}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

The PVD approximation amounts to an integration of the first term only of the Taylor series, while

[16] $\mathbf{G}$ is a Cartesian tensor that can be manipulated as if it were a matrix. $\mathbf{G}$ is a tensor insofar as its elements will transform with a rotation of coordinate axes in a way that leaves tensor equations invariant to any (time-independent) coordinate rotation. No such transformation properties are implied for the elements of a matrix.

ignoring the spatial variation of the velocity altogther. Since this approximation omits all terms that are first and higher power in the displacement, the PVD is termed the zeroth order displacement or trajectory. The first order trajectory, by which we mean the first correction to the trajectory, can then be computed by dropping the HOT from Eq. (25), taking $\delta\mathbf{X} = \delta\mathbf{X}_0 = \int \mathbf{V}_0 dt$ and integrating in time,

$$\delta\mathbf{X}_1 = \delta\mathbf{X}_0 + \mathbb{G}\cdot\mathbf{V}_0 t^2/2. \tag{27}$$

This first order trajectory is a considerable improvement upon the PVD (Fig. 6). Nevertheless, after sufficient time has passed and the displacement becomes comparable to the length scale of the flow, the radius, then this first order trajectory also accumulates a noticeable error. Adding an evaluation of the next term of the HOT would delay the failure, in general, but in any steady vortex flow the displacement will eventually carry a parcel long distances from its origin. Approximation methods built around a Taylor series expansion are not uniformly valid in time when applied to a steady vortex flow.

5.2 Parcel tracking in gravity waves; Stokes drift*

This method has somewhat better success when applied to a wavelike motion in which the parcel displacements over a single wave passage are small compared to the wavelength. In this flow there are then two length scales (where in the vortex flow above there was only one, the radius). As an example, we will analyze the parcel motion associated with a surface gravity wave having a surface displacement

$$\eta(x,t) = A\cos(kx - \omega t), \tag{28}$$

where a is the amplitude of the surface displacement, $k = 2\pi/\lambda$ is the wavenumber given the wavelength λ and ω is the wave angular frequency (it is assumed that ω and $k > 0$). The argument of the trigonometric function shows that this surface displacement moves rightward as a progressive wave having a phase speed $c = \omega/k$. The two-dimensional and time-dependent velocity field associated with this wave

$$\mathbf{V}(x,z,t) = Ue^{kz}\begin{pmatrix}\cos(kx-\omega t)\\ \sin(kx-\omega t)\end{pmatrix}, \tag{29}$$

where z is the depth, positive upwards from the surface. The amplitude or speed at the surface is $U = a\omega$ and decays with depth on an e-folding scale $1/k$. This exponential decay with depth is appropriate for a wave whose wavelength is less than the water depth, a so-called deep water wave. If the wavelength is much greater than the water depth, a shallow water wave, the x component of the velocity is independent of depth and the z component is linear with depth and vanishes at the (flat) bottom.[17]

The (Eulerian) velocity observed at a fixed point is a rotary current, often called the orbital velocity, of amplitude $A\omega e^{kz}$ that turns clockwise with time at the angular frequency ω. Given the known velocity we

[17] A very clear reference on Stokes drift is Longuet-Higgins, M. S., 1969. 'On the transport of mass by time-varying ocean currents'. *Deep-Sea Res.*, **16**, 431-447. Still another very interesting and accessible discussion of Eulerian and Lagrangian methods applied to the measurement of gravity waves is Longuet-Higgins, M. S., 1986. 'Eulerian and Lagrangian aspects of surface waves', *J. Fluid Mech.*, **173**, 683-707.

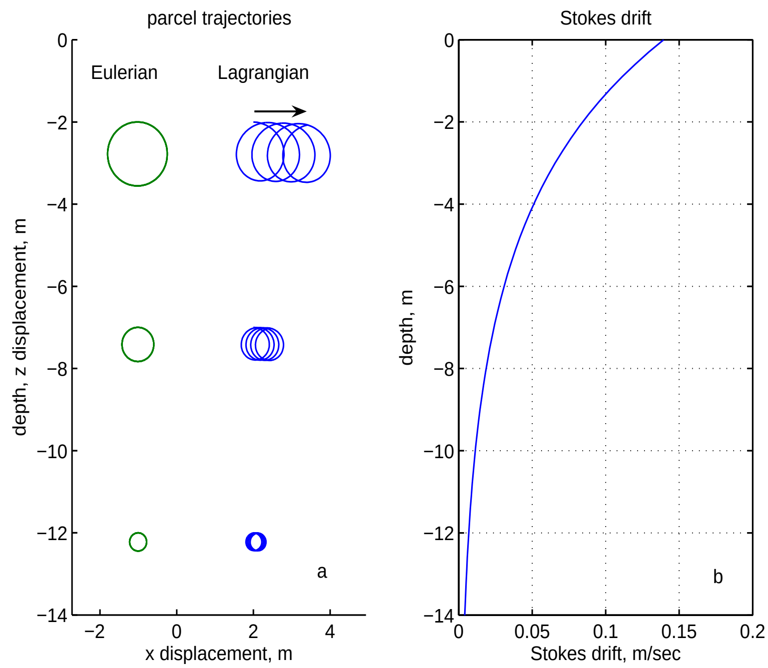

Figure 7: (a) Parcel trajectories underneath a deep-water, surface gravity wave that is presumed to be propagating from left to right. The amplitude of the surface displacement was taken to be $A = 1$ m and the wavelength λ = 50 m. The Eulerian trajectories (or PVDs) are closed circular orbits around which the parcels move clockwise. The Lagrangian trajectories were computed by integrating numerically for four periods (= 22.6 sec) and are open loops indicating there is a net drift of fluid parcels from left to right (shown as an arrow). (b) Stokes drift for this wave computed from Eq. (31). An animation of a gravity wave and parcel trajectories is in Section 11.3.

can readily calculate the PVD-like parcel displacements by integrating $\mathbf{V}(x, z, t)$ with respect to time while holding x and z constant,

$$\delta \mathbf{X}_0 = Ae^{kz} \begin{pmatrix} -\sin(kx - \omega t) \\ \cos(kx - \omega t) \end{pmatrix}. \tag{30}$$

The PVD indicates that parcels move in a closed rotary motion with each wave passage and that the net motion is zero, consistent with the wave-average of the Eulerian velocity.

From the analysis of motion around a vortex we might have developed the insight that this PVD for a gravity wave would probably give a fairly accurate prediction for the actual parcel displacements provided that the parcel displacements were very much less than the scale over which the wave orbital velocity varies. In this case the scale is k^{-1} in either direction, so that this condition is equivalent to requiring that the wave steepness, $Ak = 2\pi A/\lambda$ must be much less than 1. This is also the condition under which the linear solution gives an accurate waveform of the surface displacement, a pure sinusoid, which we have assumed with Eq. (28).

If we are dubious of this zeroth order approximation of trajectories then we may want to calculate the first order velocity via Eq. (25), and neglecting the HOT. The velocity gradient tensor for this wave is just

$$\mathbb{G} = A\omega k e^{kz} \begin{pmatrix} -\sin(kx-\omega t) & \cos(kx-\omega t) \\ \cos(kx-\omega t) & \sin(kx-\omega t) \end{pmatrix}$$

and matrix-multiplying into the zeroth order displacement given by Eq. (30) gives the first order velocity,

$$\begin{aligned} \mathbf{V}_1(z) &= A^2\omega k e^{2kz} \begin{pmatrix} \sin^2(kx-\omega t) + \cos^2(kx-\omega t) \\ 0 \end{pmatrix} \\ &= UAke^{2kz} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned} \tag{31}$$

where recall $U = A\omega$ is the amplitude of the orbital velocity at the surface. The coefficient $UAk = A^2k^2c$ is the wave steepness squared times the phase speed. This velocity is independent of time and x, and so the first order displacement is easily computed,

$$\delta\mathbf{X}_1(z) = \mathbf{V}_1(z)t, \tag{32}$$

where there is no particular initial position. Notice that this formula relates the displacement of a parcel to a depth, so that it gives a field representation of a Lagrangian property.

Quite unlike the PVD, Eq. (32) indicates that fluid parcels have a substantial net motion in the direction of the wave propagation, often called Stokes drift or mass transport velocity, that is a fraction $2\pi\frac{A}{\lambda}e^{kz}$ of the orbital motion amplitude. For example, for a wave having an amplitude of A = 1 m and wavelength of $\lambda = 50$ m, the orbital motion at the surface is about 1.11 m s^{-1} where the Stokes drift is about 0.14 m s^{-1}. The Stokes drift decreases very rapidly with depth; twice as rapidly as does the wave orbital velocity. The vertically integrated Stokes drift is the mass transport per unit length,

$$M = \rho\int_{-\infty}^{0} \mathbf{V}_1(z)dz = \begin{pmatrix} \rho UA/2 \\ 0 \end{pmatrix},$$

and also the momentum per unit area associated with the wave. The kinetic energy per unit area of the wave motion is

$$K = \rho\int_{-\infty}^{0} \mathbf{V}(z)^2 dz = \rho UcA/4.$$

Thus the momentum in the direction of gravity wave propagation and the kinetic energy of gravity waves are related by the particle-like relation

$$M = \frac{2K}{c},$$

which holds for other kinds of waves, e.g., electromagnetic waves, that have genuine momentum.[18] Thus, the

[18]The physics of wave momentum is remarkably intricate: waves that exist on a physical medium may *not* have a genuine momentum of this sort, though they will have a momentum flux, a point made famously in this context by M. E. McIntyre, 'On the wave momentum myth', *J. Fluid Mech.*, **106**, 331-347, 1981. An excellent recent reference on the topic of wave momentum is by D. Rowland, 'Comments on "What happens to energy and momentum when two oppositely-moving wave pulses overlap?", *Am. J. Phys.*, **72**(11), 1425-1429, 2004. Surface gravity waves have momentum by virtue of the displaced free surface. Here's a small problem for you: show that the Eulerian mean momentum over the water column is equal to the mass transport associated with Stokes drift by making the same kind of (bilinear) approximation for transport that we made for Stokes drift, i.e., assume that the wave velocity $\mathbf{V}$ under the displaced surface is given by Eq. (26) evaluated at $z = 0$, and compute the wave mean of $\eta\mathbf{V}$.

Stokes drift turns out to be much more than a residual effect of switching from an Eulerian to a Lagrangian coordinate system and indeed it is one of the most important means by which surface gravity waves interact with other scales of motion.[19] It is notable that all of the information needed to calculate the Stokes drift was present in the Eulerian velocity field, Eq. (29). However, to reveal this important phenomenon, we had to carry out an analysis that was explicitly Lagrangian, i.e., that tracked parcels over a significant duration.[20]

6 Transforming a Time Derivative.

We noted in the first section that the application of Newton's laws of motion to a fluid is simplest in the Lagrangian coordinate system wherein it has the form of (solid) particle dynamics. The reason, worth repeating, is that in the Lagrangian system we are tracking specific parcels and momentum balance, for example, applies to a specific fluid parcel and not to positions in space. In the Eulerian system the partial derivative with respect to time represents the rate of change (of velocity, say) at a point fixed in space, and will not equal the Lagrangian time rate change except in the somewhat degenerate case that there are no spatial variations of the flow. However, we have also emphasized that the velocity, temperature etc., of a fluid parcel is exactly the same thing as the velocity, temperature, etc. observed in an Eulerian frame at that same time and at the parcel position (if this statement sounds circular when we say it this way, then it is!). This suggests that we should be able to write the time rate of change following a parcel in terms of Eulerian (or field) variables, and this will turn out to be the key step in deriving equations suitable for modelling most fluid flows.

[19]Stokes drift will transport fluid properties as well as any material, i.e., plankton, carried by the fluid. The role of Stokes drift in ocean circulation and ocean ecology is an active topic in ocean science. A good entry to some of this research can be found at http://www.aslo.org/lo/toc/vol_49/issue_4/1214.pdf. Other, more advanced studies are by Leibovich, S., 'The form and dynamics of Langmuir circulation.' *Ann. Rev. Fluid Mech.*, **15**, 391-427, 1983 and by McWilliams, J.C. and J.M Restrepo, 'The wave-driven ocean circulation.' *J. Phys. Oceanogr.* **29**, 2523-2540, 1999.

[20]Stokes drift is a very robust phenomenon that can be produced and observed with simple means: fill a flat container with water to a depth of about 2-4 cm. A bath tub works well, but even a large cake pan will suffice. To make gravity waves use a cylinder having a diameter of roughly the water depth and a length that is about half the width of the tank. Oscillate the wave maker up and down with a frequency that makes gravity waves and observe the motion of more or less neutrally buoyant particles; some that float and others that sink to the bottom. You can easily vary three things, the amplitude of the waves, the depth of the water, and the width of the wave maker. Are the waves in your tank shallow water or deep water waves? Describe the mean flow (if any) set up by the oscillating wave maker, and how or whether it varies with the configuration of the tank and wave maker.

Suppose that the one-dimensional velocity in a progressive wave is given as $Ucos(kx - \omega t)$. Calculate the Stokes drift approximation of the mean parcel motion in such a wave, and compare the result to the numerical integration of the full trajectory, i.e., $dx/dt = Ucos(kx(t) - \omega t)$. For what range of wave steepness does the Stokes drift estimate give an accurate estimate of the mean flow? Why do parcels have a Stokes drift in this wave?, i.e., explain why parcels in this wave velocity field have a net motion. What happens at very large steepness? How does this compare with your observations from the bath tub? How does this compare with the Stokes drift of a deep water gravity wave?

6.1 The material derivative in field form

To accomplish this transformation of a time derivative we have to write the time rate of change of a fluid variable, say temperature, T, in terms of $y(t)$, and as before eliminate ξ_0. Thus

$$\frac{dT(\xi_0;t)}{dt} = \frac{dT(y(t),t)}{dt}, \tag{33}$$

where we are assuming that the parcel trajectory Eq. (1) can be used to go from ξ_0 and t to a unique y position, $y(t) = \xi(\xi_0;t)$. To compute the time derivative we apply the chain rule and find that

$$\frac{dT(y(t),t)}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial t}. \tag{34}$$

If the time rate of change of position, y, on the right hand side is evaluated as the position of a moving parcel, then this can be rewritten in terms of the fluid velocity, $V = \frac{\partial y}{\partial t}$ and so

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + V\frac{\partial T}{\partial y}. \tag{35}$$

In case the time derivative could be misinterpreted it is common to use the operator D/Dt and to write equations of this sort in the form

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + V\frac{\partial(\ )}{\partial y} \tag{36}$$

that recurs in fluid dynamics. The operator D/Dt gives, entirely in field coordinates (just y in this case), the time rate of change that would be observed at a parcel moving through the point y at time t. D/Dt is the second key piece in the transformation of Lagrangian dynamics into Eulerian or field form, the FPK being the first.[21] In Fig. (2c) we showed an example, the time rate change of a velocity component following a parcel, $d^2\xi(\xi_o;t)/dt^2$, along with the field equivalent, DV/Dt defined above. These two accelerations are equal at a common position and time in exactly the way that Lagrangian and Eulerian velocities are equal at a common position and time. In the next section, 7, we will find that the acceleration DV/Dt is equal to the sum of the forces applied to the fluid at that spatial position. Thus the forces can be specified in field coordinates, say Cartesian, x, y, z, which was the goal from the outset.

The operator D/Dt goes by a number of different names, including the convective derivative, the substantive (or substantial) derivative, the Stokes derivative and the material derivative (our choice), the profusion of names giving a clue to its importance. It is very often said to be the time derivative 'following the flow', in the sense that $\frac{\partial y}{\partial t}$ in Eq. (34) is the fluid velocity. This identification is central to this development, but it does not imply parcel tracking in the Lagrangian sense. We do not keep track of individual parcels in a field description of fluid flow; we have to do a deliberate Lagrangian analysis of the flow, as in the Stokes drift analysis of Section 5.2, in order to infer Lagrangian properties. In circumstances where confusion with the ordinary time derivative is not likely, then the material derivative is often written as d/dt (and we will revert to this form once we are finished with kinematics and ensconced in an Eulerian frame).

[21] Can you show that $D\xi/Dt$ leads directly to the FPK, and in that sense is more general than the FPK.

In the example above we have assumed a one-dimensional domain to minimize the algebra. The same relations hold in a three-dimensional flow having Cartesian coordinates (x,y,z) and velocity (U,V,W), the material derivative then being

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} + W\frac{\partial}{\partial z}, \tag{37}$$

or using more compact vector notation,

$$\boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}} \tag{38}$$

The gradient operator expanded in Cartesian coordinates is

$$\boldsymbol{\nabla} = \boldsymbol{e}_x\frac{\partial}{\partial x} + \boldsymbol{e}_y\frac{\partial}{\partial y} + \boldsymbol{e}_z\frac{\partial}{\partial z},$$

with $\boldsymbol{e}$ the unit vector. It is important to notice that the advective derivative term $\boldsymbol{V}\cdot\boldsymbol{\nabla}$ of (38) is, in effect, a scalar that is multiplied into the variable being differentiated. To emphasize this property, the advective derivative is sometimes written $(\boldsymbol{V}\cdot\boldsymbol{\nabla})$.

Care has to be taken when D/Dt is applied to velocity (or any vector) and then expanded in other than Cartesian coordinates. In the Cartesian example we find that the D/Dt produces four terms for each component of the vector being differentiated, one term being the partial with respect to time, and three terms arising from the advective term. However, when D/Dt is expanded in cylindrical polar coordinates we get five terms, with an additional advective term arising from the θ dependence of the r and θ unit vectors (in the Cartesian system the unit vectors are constant). In spherical polar coordinates the D/Dt operator expands to six terms. It is sometimes convenient to represent the advective derivative of velocity by the following vector identity

$$(\mathbf{V}\cdot\nabla)\mathbf{V} = \frac{1}{2}\nabla(\mathbf{V}^2) \ + \ (\nabla\times\mathbf{V})\times\mathbf{V}, \tag{39}$$

which is less likely to be misinterpreted. This form is especially useful in the idealized 'perfect fluid', wherein frictional effects vanish and so too does the curl of the velocity, also called the vorticity (Section 8.3).

6.2 Modes of an advection equation

In Section 7 we will derive the full, formal conservation law for a continuum, and so complete the first stage of the kinematic analysis. However, while the material derivative is still fresh and in front of us we will take the opportunity to note the possible balances among the terms in a two-dimensional heat budget,

$$\begin{array}{ccccc} \frac{\partial T}{\partial t} & + & \mathbf{V}\cdot\nabla T & = & Q, \\ \mathrm{time\ rate\ change} & + & \mathrm{advection} & = & \mathrm{source} \end{array} \tag{40}$$

where T is the temperature,

$$\mathbf{V} \cdot \nabla T = U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y}$$

and Q is the divergence of a kinematic heat flux, i.e., it is a heat flux divided by some depth times density times heat capacity (this is a preliminary heat budget; important details will follow beginning in Section 7).[22]

Often the balance will involve all three terms (taking advection as one term). But it may also happen that at some times, or averaged over some time scales, the balance will be dominated by just two terms, which we will term a 'mode'. There are three modes inherent in a three term equation like this one, and as part of building a vocabulary for describing fluid dynamics it is useful to note each of the possibilities:

Steady balance. The term $\partial/\partial t$ is the rate of change observed at a fixed position. A steady flow is one that has $\partial/\partial t = 0$ for all relevant properties throughout the domain. A steady flow may nevertheless be subject to significant external forcing in which the advection term (nearly) balances the source,

$$V \cdot \nabla T = Q.$$

For example, if the source was positive or heating and yet we observed that there was little or no local time rate of change (i.,e that the flow was steady) then we would infer that the advection term was acting to cool the observation site.

Such a steady balance may occur instantaneously, or it may be very useful to think of the balance over a significant time scale, say averaging over a year. The temperature at a given site in the lower atmosphere is likely to go through a nearly closed annual cycle, so that the annual average of the local rate of change would nearly vanish. Nevertheless the local source term Q may be significant when averaged over the same period. In that case we can infer that the advection term averaged over the annual cycle must also have been significant. For example, the large scale meridional flux of heat associated with the ocean's overturning circulation transports warmer waters and thus heat to middle and higher latitudes, where it is given up to the atmosphere so that the ocean is cooled by Q. More or less the reverse happens at lower latitudes; the source term Q is positive due to excess solar radiation, and the overturning circulation acts to bring in cooler waters, in part by vertical advection (so that the advection term is then three-dimensional).[23] This advection or transport of heat from lower to higher latitudes, which occurs in both the atmosphere and ocean, makes a very significant contribution to the moderation of Earth's meridional temperature gradient.

Local balance. The advective term $V \cdot \nabla T$ will vanish if the fluid is at rest, if the temperature is

[22]For example, the temperature T could be the temperature of the atmosphere observed near ground level, i.e., your local climate. In that case Q would be due mainly to solar insolation and radiative cooling, and the advection term would be associated mainly with the advection of differing air masses to the observation site. Over the next week or two, take notice of your local climate and how it varies on a diurnal to weekly basis. What causes the local temperature to change, advection or the local source? Assuming that the latter is mainly radiative, then it can be inferred, very roughly, from the diurnal variation due to radiative fluxes. To evaluate advection requires that you monitor the mesoscale temperature in your region and the local wind; the necessary data are available from good weather maps in the newspaper or better, from a weather forecasting center such as FNMOC. We are not looking for precise quantitative estimates so much as a qualitative discussion of the two-dimensional balance sorted upon time scale.

[23]The vertical advection term is just $W\partial T/\partial z$. In natural flows of the atmosphere and ocean the vertical velocity W is usually much smaller than the horizontal velocity, but then the vertical gradient of most properties is also much, much larger than the horizontal gradient. The result is that the vertical advection term is sufficiently different from the horizontal advection terms that it would be reasonable in this context to treat it as a separate, fourth term in a three-dimensional heat budget. We won't do that, however.

spatially uniform so that ∇T vanished, or if the fluid velocity is parallel to isolines of temperature. In that case the local time rate of change would reflect the source term, Q, in what is called a local balance,

$$\frac{\partial T}{\partial t} = Q,$$

where 'local' means at one place, 'nonlocal' meaning that spatial gradients are important. For example, the rapid rise of surface air temperature during a daily cycle that includes significant solar radiation is indicative of a local, source-driven heat balance. Much the same thing happens on a seasonal cycle, especially at middle and higher latitudes.

Frozen field. Finally, we note that a third important mode inherent to Eq. (40) is that

$$\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T = 0,$$

while both terms are considerably larger than the source, Q. Said a little differently, the local rate of change of temperature may be due mainly to advection rather than to a local (heat) source term. This kind of balance is sometimes referred to as a 'frozen field', though the thing imagined to be frozen is the spatial structure of temperature that is embedded within the moving fluid (and not frozen in space). When this spatial structure is carried past fixed points by the fluid flow there is then a local rate of change of the property. For example, the passage of a frontal boundary between warm and cold air may cause a rapid and significant change in the locally observed air temperature. Thus, the observation of a changing temperature at a fixed site is not by itself sufficient to infer the presence of a local heat source unless the advection of temperature (heat, really) was known to be negligibly small from some additional information.[24]

7 Transformation of Integrals and the Conservation Laws.

We have noted that the conservation principles of classical physics can be applied to a specific parcel or volume of fluid, but we have also indicated that the development of models of fluid flow is generally best done in an Eulerian or field coordinate system (the demonstration of this is in Section 9). Thus it will prove to be essential to transform integrals taken over fluid volumes into their equivalent field form. This requires or leads to the third key piece of the transformation of dynamics from Lagrangian to Eulerian form, the Reynolds Transport Theorem.

7.1 Reynolds Transport Theorem

Consider the integral in R^1 of a scalar property, let's say temperature, T, over a moving 'volume' of fluid (the results are easily extended to R^3),

$$H(t) = \int_{\xi_1}^{\xi_2} T d\xi. \tag{41}$$

[24]This description of modes has been for the Eulerian balance. Go back and describe the Lagrangian balance for each of these modes. Consider whether the temperature can or must vary with ξ_0.

The limits on this integral are the positions of moving parcels, e.g., $\xi_1(\xi_{o1}, t)$ is the y-position of the parcel tagged by the initial location $y = \xi_{o1}$. This integral is proportional to the heat content (internal energy) of the fluid (linear) volume encompassed in the integral, and it is entirely sensible to ask for its time derivative, dH/dt, which we assume will be proportional to an applied heat source. It is far preferable to specify heat sources in field coordinates rather than material coordinates, and thus the need for the present development.

This integral looks a little exotic because the independent coordinates are material coordinates, but nevertheless it is no more than the sum of an integrand, in this case the temperature, T, multiplied by a length, $d\xi$, that happens to be embedded in a moving fluid. Thus the length could change with time. To transform this integral to field variables we have to transform the time derivative of T, which we have already done (Section 6), and the time derivative of a material length,

$$L(t) = \int_{\xi_1}^{\xi_2} d\xi = \xi_1 - \xi_2 \tag{42}$$

which is just the distance between the two moving parcels. The time derivative of this length is then just the velocity difference at the location of the two parcels that mark it's endpoints, i.e.,

$$\frac{dL}{dt} = \frac{d}{dt}(\xi_2 - \xi_1) = V_2 - V_1, \tag{43}$$

where V_1 is the fluid velocity at $y = \xi_1$. The velocity difference on the right hand side can be written

$$V_2 - V_1 = \frac{\partial V}{\partial y} L$$

as L is made infinitesimal (the quantity $\frac{1}{L}\frac{dL}{dt}$ is called the linear strain rate, something we will return to in Section 8.3). Thus the time derivative of an infinitesimal material length transforms to field coordinates as

$$\frac{d}{dt} dy_{material} = \frac{\partial V}{\partial y} dy_{field}, \tag{44}$$

where the derivative $\frac{\partial V}{\partial y}$ is in field coordinates and dy_{field} = $dy_{material}$ at the time that the transformation is made. This accounts for the possible change in the length of a material line. If instead of a line segment we had transformed a differential volume, then the corresponding result would be

$$\boxed{\frac{1}{dVol}\frac{d}{dt}(dVol) = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = \nabla \cdot \boldsymbol{V}} \tag{45}$$

$\nabla \cdot \boldsymbol{V}$ is the divergence of the fluid velocity and is the volumetric (or normalized) time rate change of the volume (notice that the units are 1/time). We will return to the divergence again in Section 8.3;

With these results in hand, we are ready to state the Reynolds Transport Theorem (or RTT) which relates the time derivative over a material volume to the equivalent field quantities. For this one-dimensional case:

$$\frac{d}{dt}\int_{\xi_1}^{\xi_2} T d\xi = \int_{y_1}^{y_2} (\frac{DT}{Dt} + T\frac{\partial V}{\partial y}) dy, \tag{46}$$

where $y_1 = \xi_1$, etc. at the time the transformation is made. Notice that the integral on the left side of the equation is over material coordinates, while the integral on the right side is over field coordinates (in case there could be any ambiguity the type of coordinate will be written out). In three dimensions the RTT applied to a scalar T that has a body source Q is just

$$\boxed{\frac{d}{dt}\iiint_{material} T\, dVol = \iiint_{field} (\frac{DT}{Dt} + T\nabla\cdot\boldsymbol{V})\, dVol = \iiint_{field} Q\, dVol} \tag{47}$$

The RTT is a purely *kinematic* relationship that holds for any fluid property, momentum, angular momentum, energy etc. that is subject to a conservation law.[25] The appropriate source terms Q will be different for each of these quantities. The specification of the appropriate Q is a nontrivial *physical* problem in as much as it depends upon the physical properties of the fluid and upon the boundary conditions. This is obviously a very important matter that is, however, also outside the scope of this essay.

7.2 Mass conservation

An important application of the RTT is to the mass of a moving, three-dimensional volume of fluid,

$$M = \iiint_{material} \rho\, dVol. \tag{48}$$

There is no body source of mass and thus we can assert that M must remain exactly constant, $dM/dt = 0$, with no approximation for any flow condition (since we are ignoring relativistic effects). It is crucial to understand that we could make no such general assertion for a volume that was fixed in space. By means of the Reynolds Transport Theorem we can write this in field variables as

$$\frac{dM}{dt} = 0 = \iiint_{field} (\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{V}) dVol, \tag{49}$$

where the triple integral is over the spatial position of the volume. If this integral relation holds at all times and for all positions within a domain, and if the integrand is smooth (no discontinuities), then the integrand must vanish at all times and positions in that domain[26] yielding the differential form of the mass conservation relation,

$$\boxed{\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{V} = 0} \tag{50}$$

If we expand the vector velocity and gradient operator in Cartesian components the result is

$$\frac{\partial\rho}{\partial t} + U\frac{\partial\rho}{\partial x} + V\frac{\partial\rho}{\partial y} + W\frac{\partial\rho}{\partial z} + \rho(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}) = 0.$$

[25] And of course, most quantities are *not* subject to a conservation law. For example, in a two particle collision, the kinetic energy will be conserved only in the special case that the collision is 'elastic', so that no energy is lost to deformation, acoustic modes, etc. On the other hand, if the particles stick together after the collision, then the kinetic energy will decrease by an amount that depends upon the particular conditions of mass and initial velocity. Thus the kinetic energy and higher moments of the velocity are not conserved in most collisions or during mixing events in a fluid.

[26] We have an integral of the form $\int_{x_1}^{x_2} \Psi(x)dx = 0$ for any x_1, x_2 and Ψ is smooth. Suppose that $\Psi(x_a) > 0$; we can show that this leads to a contradiction. If Ψ is smooth, then there will be a neighborhood around x_a where $\Psi(x) > 0$; choose x_1 and x_2 to lie within this neighborhood, and apply the mean value theorem to the integral to find that $\Psi(\bar{x})(x_2 - x_1) \geq 0$ because $\Psi(\bar{x}) \geq 0$. However, this contradicts what we know about this integral and hence $\Psi(x)$, the integrand, must be zero at every point.

What we have gained or learned from the RTT (compared with the informal advection equation of Section 6.2) is that the mass conservation equation (50) has a divergence term $\rho\nabla \cdot \boldsymbol{V}$. Thus if the density of a fluid parcel changes in time, say it increases, $\frac{D\rho}{Dt} > 0$, then we can conclude that there must have been a decrease in the volume of the parcel and thus a convergence, $\nabla \cdot \boldsymbol{V} < 0$, of the fluid velocity. This holds regardless of the cause of the density change, whether due to a pressure variation or to heat exchange with the surroundings.

An example may help to clarify this: Consider a mass of gas enclosed in a cylinder that is capped by a frictionless piston. The walls of the cylinder may allow the passage of heat but not material so that the chemical composition and the mass of the enclosed gas must remain constant. In steady state the weight of the piston is supported by the pressure within the gas. Now imagine that the gas absorbs heat through the walls of the cylinder. Before we can calculate the new pressure and density of the gas we have to specify whether the piston is free to move or is clamped. Suppose that the piston is clamped so that volume of the gas is held fixed during the heating process. The pressure of the gas will increase as heat is absorbed in an isovolume process, but there will be no change in the volume of the gas, no divergence of the fluid velocity, and the density will remain constant, consistent with Eq. (50).

The atmosphere and ocean are not enclosed fluids so that heating and cooling processes are effectively isobaric, i.e., the piston is free to move. The gas will expand just enough so that the pressure force on the piston continues to balance the weight of the piston. In such a constant pressure or isobaric heating process, the volume occupied by the gas will increase at a rate Adh/dt, where A is the area of the piston and h is the height of the gas column and the fluid velocity will necessarily be divergent, $\nabla \cdot \mathbf{V} = h^{-1}dh/dt$. The density of the gas will necessarily decrease, consistent with Eq. (50).

The divergence term is necessary to make a fully consistent mass conservation equation, and yet for most phenomenon of the atmosphere or ocean, the fluid velocity associated with this divergence can be safely ignored compared to other fluid velocities.[27] Under this so-called incompressibility assumption, the velocity is assumed to follow

$$\nabla \cdot \mathbf{V} = 0, \tag{51}$$

which in effect says that the volume of fluid parcels is exactly constant. Density may nevertheless change, and indeed density changes may be of primary importance however density is computed from an equation of state given the pressure, temperature, salinity, etc., with no reference made to the divergence of the fluid velocity. That we can ignore fluid divergence is a very important mathematical approximation that is contingent upon the physical phenomenon under consideration. One important class of phenomena, acoustic waves, owe their entire existence to velocity divergence and associated pressure changes. Once a few more pieces are in place, we will be able to appreciate that the incompressibility assumption for the velocity field, Eq. (51), can be made with negligible error provided that the fluid velocity is much less than the speed of sound, which holds well for most natural flows of the atmosphere and ocean.

[27] A question for you: Suppose that a volume of air, 10 km on a side by 1 km thick, is heated at constant pressure by 5 C in a period of an hour (as might occur in a vigorous sea breeze circulation). What is the magnitude of the associated (divergent) velocity, assuming that it appears on one vertical face only of the volume?

7.3 Momentum balance

A second important application of the Reynolds Transport Theorem arises on consideration of momentum balance. The momentum of a moving, three-dimensional volume of fluid can be written

$$\boldsymbol{B} = \iiint_{material} \rho \boldsymbol{V}\; dVol. \tag{52}$$

Because this is a material volume we can assert Newton's Second Law,

$$\frac{d\boldsymbol{B}}{dt} = \iiint_{material} \boldsymbol{F}\; dVol = \iiint_{field} \boldsymbol{F}\; dVol, \tag{53}$$

where $\boldsymbol{F}$ is the sum of all forces acting on the volume. (The second equality holds because the integral over the field volume is equal to the integral over the material volume assuming, as in all of this development, that the volumes coincide.) These forces could include body forces, such as gravity, that act throughout the volume, and forces that can be imagined to act on the surface of the volume, such as normal stress (pressure) and shear stress. By means of the Reynolds Transport Theorem and the mass conservation relation we can write the left side of Eq. (53) in field coordinates as

$$\iiint_{field} \rho \frac{D\boldsymbol{V}}{Dt}\; dVol = \iiint_{field} \boldsymbol{F}\; dVol \tag{54}$$

and the volume integrals are performed over the volume occupied instantaneously by the moving fluid. The volume considered here is arbitrary, and so the differential form of the momentum balance for a fluid continuum is

$$\boxed{\frac{D\boldsymbol{V}}{Dt} = \frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V} = \boldsymbol{F}/\rho} \tag{55}$$

A term that might have been expected, $\frac{D\rho}{Dt}\boldsymbol{V}$, has dropped out by application of the mass conservation requirement (you should be sure to check that this is correct). Thus the momentum of a fluid parcel (or marked fluid volume) can change only because of a change of velocity.

When the advective term is expanded in Cartesian components the result to this point is

$$\begin{aligned}
\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} &= \frac{F_x}{\rho} \\
\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z} &= \frac{F_y}{\rho} \\
\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z} &= \frac{F_z}{\rho},
\end{aligned} \tag{56}$$

a truly formidable set of coupled, nonlinear partial differential equations. The advective terms are nonlinear in that they are the product of an unknown velocity component and the first derivative of an unknown velocity component (or other flow variable). And we are not finished; to make Eq. (55) useful for prediction we have to specify the relevant force, $\mathbf{F}$, as well as suitable boundary and initial conditions, tasks that we will take up further in Section 9.

8 Aspects of Advection.

From a physical perspective, it is the process of advection that endows many fluid flows with rich spatial structure and complexity. From a mathematical perspective, it is the nonlinearity of the advective terms that stymie most of the familiar PDE solution techniques that require superposition of solutions. Thus when the advection terms are retained in model systems, feasible solution methods involve numerical analysis, generally. Nevertheless, in this section we will begin to consider ways to set some bounds upon what advection alone can do in a fluid flow, and just as important, to understand what advection can not do. There are three topics in this section; the first and third, Fluxes in space (8.1) and The Cauchy-Stokes Theorem (8.3), are essential elements of fluid kinematics. The second topic, The method of characteristics (8.2), is a little less so, but is (almost) irresistible given the Lagrangian/Eulerian theme of this essay.

8.1 Fluxes in space

The mass conservation equation Eq. (50) can be written in a 'flux' form by expanding the material derivative and collecting terms under the divergence operator,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0, \tag{57}$$

or in Cartesian components,

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho U)}{\partial x} + \frac{\partial (\rho V)}{\partial y} + \frac{\partial (\rho W)}{\partial z} = 0.$$

Budget equations that can, like this one, be written in the specific form

$$\boxed{\frac{\partial f}{\partial t} + \nabla \cdot \mathbf{g} = 0} \tag{58}$$

in which the local time rate of change is due solely to the divergence of a flux are said to be 'conservation equations' and have the following important property. We can assume without loss of generality that the variable f and the flux $\mathbf{g}$ are vanishingly small as x and y go to infinity, i.e., that the flow is bounded in space. Denote the areal integral of f as

$$F = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f \, dxdy,$$

which is thus the total amount of f. Then by integrating both sides of the conservation law from $-\infty < x, y < +\infty$, and use of the (Gauss') divergence theorem on the right hand side we find that

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\partial f}{\partial t} dxdy &= -\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \nabla \cdot \mathbf{g} \, dxdy \\ &= \oint \mathbf{g} \cdot \mathbf{n} \, ds \\ \frac{dF}{dt} &= 0. \end{aligned} \tag{59}$$

In going from the second to the third line we have used that $\mathbf{g}$ vanishes at infinity and thus along the path of the line integral. Thus the total amount of f is constant, even though the flux $\mathbf{g}$ may act to redistribute f within the domain so that f at a given point could be highly time-dependent, as in the frozen field balance of Section 6.2. Nevertheless, the conservation law assures us the the total amouunt of f will remain constant. The same thing would result if $\mathbf{g}$ vanished on the boundary of a domain, e.g., if the flux were due to fluid motion and the boundary was closed. If the flux across the boundary (or at infinity) vanishes, then we can conclude that the total amount of f, indicated by F, must be constant in time. We would expect that to be true on purely physical grounds when f is fluid density and F is thus the net mass within the domain, there being no sources or sinks for mass in the classical physics that we presume holds. This same conservation relation will hold for other properties, such as momentum and kinetic energy, provided that the momentum and energy budgets can be written in the form of a conservation law, Eq. (58). It is important to understand that in many and perhaps most circumstances a conservation law will not obtain, for example if source terms are present in the budget equation for f. In that common event F need not be constant, and in that case we should probably call the corresponding governing equation an f 'balance' or 'budget' rather than a 'conservation equation', though this distinction is sometimes ignored.

If the volume or areal integrals of Eq. (60) are taken within a portion of a domain, and assuming a body source Q, then another important interpretation of the RTT is that

$$\begin{aligned} \int_a^b \int_c^d \frac{\partial f}{\partial t} dxdy &= \int_a^b \int_c^d \nabla \cdot \mathbf{g}\, dxdy + \int_a^b \int_c^d Q dxdy \\ \frac{dF}{dt} &= \oint \mathbf{g} \cdot \mathbf{n}\, ds + \int_a^b \int_c^d Q dxdy \end{aligned} \tag{60}$$

where the time derivative can be moved inside the spatial integral over what are now presumed to be the fixed limits, a, b, etc., of a control volume. The total amount of f in this control volume, F, can thus change due to either a flux divergence within the control volume or due to a body source within the volume. [28] Notice that only the flux divergence appears in these budget equations; the flux *per se* is not in evidence and the magnitude of the flux may have no physical significance.

8.2 The method of characteristics*

Advection transports, and in the simplest case, translates, fluid fields. To begin to examine this, consider the case of a constant (spatially and temporally uniform) velocity, $U \geq 0$, and some scalar property of the fluid, say T, in R^1. Assuming that there is no body force for T, then the Eulerian (field) equation for the evolution of T reads

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + U \frac{\partial T}{\partial x} = 0. \tag{61}$$

[28] This latter result can and often is turned around and made the basis of the differential, Eulerian balance equations. We need merely assert that the amount of some property f can change only if there is a flux of f through the sidewalls of a fixed control volume due to transport by the fluid flow, in which case $\mathbf{g} = \rho \mathbf{V} f$, or by diffusion, $\mathbf{g} = -K \nabla f$, if a Fourier diffusion law is appropriate, or if there is a body source of the property f within the control volume. As the volume is made very small, out pops the differential conservation laws, e.g., Eqs. (50) or (55). The only slightly disagreeable part of this procedure is that the forcing terms have to be applied to fluid at points in space rather than to specific fluid parcels, but we have to admit that the RTT comes to the same thing.

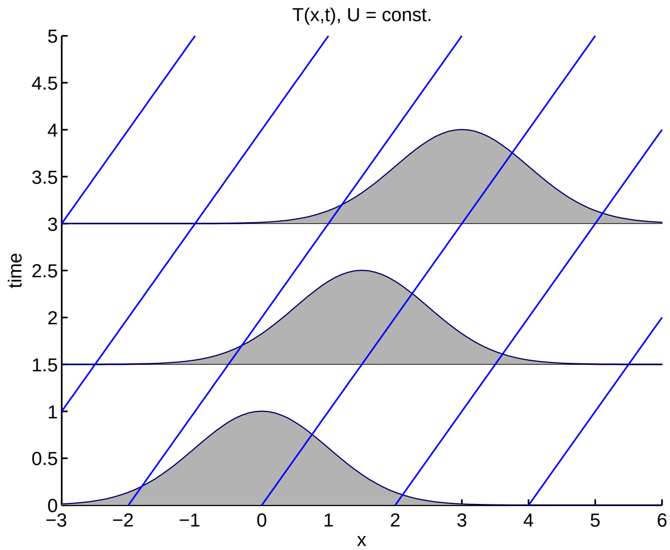

Figure 8: A scalar field, $T(x)$, in a uniform flow $U = 1$; the field T is translated to the right at the rate U. The straight solid lines are the characteristics of the flow, each of which has a slope $dt/dx = 1/U$. Notice that along these lines, T is constant.

In Lagrangian (material) terms this states that T is constant in time on each parcel, though T could very well be a different constant on each parcel. If we could specify T for each parcel at some point along its trajectory, and if in addition we could predict the trajectories of all parcels, we would have a complete solution.

For the problem at hand we will assume that $T(x)$ is known along the line $(x, t = 0)$,

$$T(x, t = 0) = H(x) = A \exp(-x^2/2d^2), \tag{62}$$

where $H(x)$ is a Gaussian hump, Fig. (8) of amplitude $A = 1$ and width $d = 1$. Advection by the uniform velocity U will act to shift each parcel to the right at the rate U and according to our governing equation (61), the parcel retains its initial T. Since all parcels move with the same speed, the inital shape or profile of $T(x)$ is unchanged in width or amplitude. This advection by a uniform velocity amounts to a simple translation of the field T.

Notice that the field $T(x, t)$ is constant in the x, t plane along lines given by $x - Ut = constant$ that are said to be the *characteristics* of the PDE, Eq. (61). If we set

$$T(x, t) = f(x - Ut), \tag{63}$$

where $f(x, t)$ is any differentiable function, this $T(x, t)$ satisfies the PDE, Eq. (61). If we further set $f = H$ we can satisfy the initial condition, and thus $T(x, t) = H(x - Ut)$ is the solution to the problem posed (as you should verify by substitution into the governing equation, Eq. (61)).

This is a nearly trivial example, of course, but it illustrates the basis of a very powerful solution technique for first order PDEs, including some nonlinear forms, called the method of characteristics. [29] The appeal of this method for our present purpose is that it solves the advection equation by recognizing that

[29] The method of characteristics is described well in many textbooks on partial differential equation. An excellent text that emphasizes numerical solution methods built upon the idea of characteristics is by R. J. LeVeque, *Numerical Methods for Conservation Laws* (Birkhauser Verlag, Basel, 1992). A clear and concise online source is at http://www.scottsarra.org/shock/shick.html

advection alone does not change the scalar T of a given parcel (though, of course, advection can change T at a fixed point in space). The mathematical strategy is to convert the governing PDE into a set of (usually coupled) ODEs; one of the ODEs will describe the rate of change of the property T along the trajectory of a parcel (zero if there is no forcing, as in Eq. (61)), and the other ODEs will serve to define the trajectory of each parcel. When combined, these will give the full solution.

To start we will seek the path in (x, t) along which the governing equation reduces to an ordinary differential. To do this we will seek the parametric form of a curve $(x(s), t(s))$ where s is distance along the curve. Assuming that there is such a curve, then we can write $T(x(s), t(s))$, and the (directional) derivative along s is just

$$\frac{dT}{ds} = \frac{\partial T}{\partial t}\frac{dt}{ds} + \frac{\partial T}{\partial x}\frac{dx}{ds}. \tag{64}$$

Notice that we can write the differentials of x and t with respect to s as ordinary differentials. Comparing this with Eq. (61), it appears that we can set

$$\frac{dT}{ds} = 0, \text{ and thus } T = constant \tag{65}$$

along this path, provided that the parametric representation of the path satisfies

$$\frac{dt}{ds} = 1 \tag{66}$$

and

$$\frac{dx}{ds} = U. \tag{67}$$

These latter two ODEs define the family of lines along which Eq. (65) holds. The first of these can be immediately integrated to yield

$$t = s, \tag{68}$$

where the integration constant can be set to zero, and the second condition integrates to

$$x = Us + b, \tag{69}$$

where b is the value of x when $t = 0$. Using Eq. (68) this last can be written

$$x = Ut + b. \tag{70}$$

Thus the family of lines along which T =constant are given parametrically by Eqs. (68) and (69) or by Eq. (70). These lines are called the *characteristics* of the governing PDE, Eq. (61). In this extremely simple problem all of the characteristics have the same slope, $dt/dx = 1/U$, Fig. (8), since we have presumed that U is constant.

Along each characteristic line T remains constant, according to Eq. (65), and to find what constant value holds on a given charateristic we need initial data on each characteristic; here we have Eq. (62), or

$$T(x, t) = H(b).$$

We can then eliminate b using Eqs. (68) and (69), and find the explicit solution,

$$T(x,t) = Aexp(-(x-Ut)^2/2d^2). \tag{71}$$

Thus the field $T(x,t)$ is translated to the right at the constant rate U, as we had surmised already from very basic considerations.

The method of characteristics can be applied to real advantage to many first order PDEs, including some that are not linear. For example, consider a problem in which the field being advected is the current U itself,

$$\frac{DU}{Dt} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = 0, \tag{72}$$

so that U is now the dependent variable. The initial condition is again presumed to be a Gaussian hump, Fig. (9),

$$U(x,t=0) = A\exp{(-x^2/2d^2)}. \tag{73}$$

This governing equation, often called the inviscid Burgers' equation, is not linear because the advection term is the product of two unknowns and so the wide array of methods that are available to solve linear PDEs will not be applicable. It is linear in the partial derivatives, however, and the method of characteristics is well-suited for such quasi-linear problems.[30] We again seek solutions of

$$\frac{dU}{ds} = 0$$

and as above find that the characteristics $x(s)$ and $t(s)$ are (straight) lines, Figs. (9) and (10). A significant difference is that the slope of the characteristic lines now varies from characteristics to characteristic since U varies with x. Far from the origin the characteristic lines are nearly vertical in the x,t plane (large slope corresponds to small U) while close to the origin the characteristics have a minimum slope and thus a maximum dx/dt and $1/U$. The solution for this problem can be written,

$$U(x,t) = A\exp{(-(x-U(x,t)t)^2/2d^2)}, \tag{74}$$

which is not separable into an explicit solution (in which U is on the left side and the right side is a function of x and t). It can be readily graphed, however, Fig. (9), and interpretation of the solution is made clear by the characteristics, Fig. (10). The most rapidly moving part of the hump that begins near the origin starts to overtake the slower moving part that starts at larger x; after some time U evolves to have an infinite derivative and then becomes triple-valued. In the case that U is the current speed, a multi-valued solution makes no physical sense and has to be rejected. It isn't that the method of characteristics has failed, but rather that the governing equation (72) has omitted some physical process(es), e.g., diffusion or viscosity, that will become important when the derivative becomes very large. Even with diffusion present, the derivative may, nevertheless, become very large, and the flow said to form a shock wave, across which the current speed is nearly discontinuous. The conservation of momentum holds regardless of the details of the field, and the subsequent motion of a shock wave can be determined using fundamental principles. [31]

[30]It should be noted that this is not a complete model of any fluid flow in that we have not considered the conservation of volume

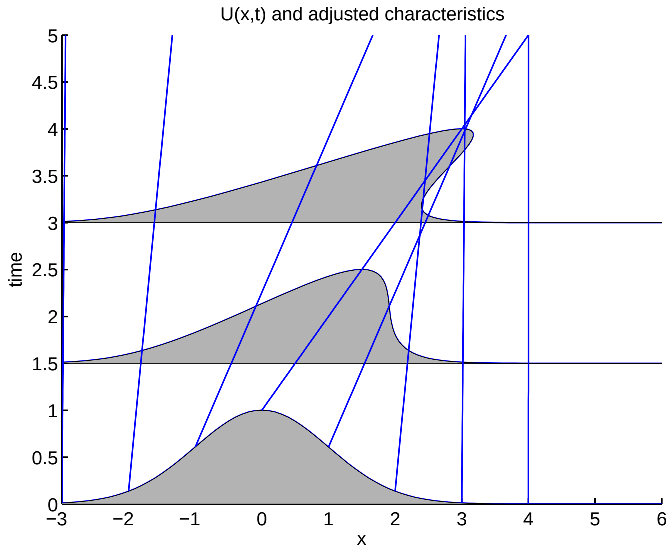

Figure 9: In this case the field being advected is the current itself. Because the initial field is nonuniform (Gaussian), the subsequent advection distorts the pattern and at about $t = 1.7$ the field predicted by advection alone becomes multivalued; at that time we could say that the wave has broken. Subsequent evolution has to account for the formation of a shock. Notice that the abscissa in this plot is both time and current amplitude. To show where points of constant current amplitude move, the lines plotted here are characteristics that have been adjusted upward by the amount $U(x, t = 0)$. The next figure shows the characteristics without this adjustment.

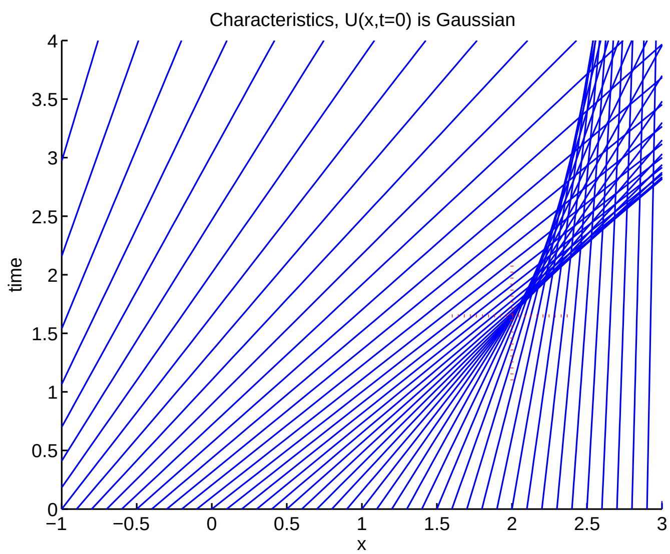

Figure 10: The characteristics of the flow shown in Fig. (9). The short dashed lines denote $x = 2, t = exp(1/2)$ where characteristics are expected to cross first. Since only a few characteristic lines are shown here, the first crossing that actually occurs in this figure is at a slightly later time and place.

The method of characteristics can be readily extended to higher dimensions and to problems in which there is some forcing. Suppose that we have a model

$$\frac{\partial T}{\partial t} + U(x,y)\frac{\partial T}{\partial x} + V(x,y)\frac{\partial T}{\partial y} = Q(x,y). \tag{75}$$

The governing equation can be reduced to an ODE:

$$\frac{dT}{ds} = Q \tag{76}$$

provided that

$$\frac{dt}{ds} = 1,$$

and

$$\frac{dx}{ds} = U \quad \text{and} \quad \frac{dy}{ds} = V. \tag{77}$$

Notice that Eq. (77) is exactly Eq. (19) for streamlines that we considered in Section 4.3. Thus streamlines and characteristics are one and the same when the velocity is steady.[32]

8.3 Rotation rate and strain rate; the Cauchy-Stokes Theorem

You have probably noticed that advection does much more than simply translate fields from place to place; in the flow in a teacup, presuming it has been stirred somewhat, advection will also act to draw out parcels into long, thin streaks. These streaks will have a much greater surface area than did the initial parcel, and as a direct consequence, diffusion can then act to homogenize the tracer property much more quickly than would diffusion alone. This second aspect of advection, that it may change the shape of fluid parcels, is an important part of kinematics that we consider in this section.

To pose a definite problem, we will exploit the last result of Section 8.2 to calculate the motion of a parcel (identified with a colored tracer) that is embedded in a steady, clockwise rotating, vortical (circular) flow; either an irrotational vortex, Fig. (11, left), discussed in Section 5.1 and defined by Eq. (21), or a solid body rotation, Fig. (11, right) in which the azimuthal speed increases with radius as

$$\mathbf{V} = (U_{rad},\ \ U_{azi}) = (0,\ \ \Omega r), \tag{78}$$

(or mass) nor the possibility of a pressure gradient. These will be considered in Section 9.

[31] To calculate when characteristics will first cross, consider the following problem, the governing equation is the inviscid Burgers' equation, and the initial data is piece-wise linear: $U = U_1$ for $-\infty < x \leq 0$; $U = U_1 - x/L$ for $0 < x \leq L$ and $U = 0$ for $x > L$. Assume that $U_1 > 0$. Sketch the characteristics and the solution at several times, and show that the characteristics starting from $x = 0$ and $x = L$ will cross at $t_c = L/U_1$. In the limit that L is small this can be written $t_c = -1/(\partial U/\partial x)$, and for a continuous intial $U(x)$ it is plausible that the first crossing will be due to the largest (negative) value of $(\partial U/\partial x)$. For the Gaussian of Eq. (73) this is $\sim \exp(-1/2)$. Where in space does the first crossing occur? What would happen in the piece-wise linear case if $U_1 < 0$?

[32] Model PDE systems in which the fields are propagated at a definite, finite rate are said to be *hyperbolic*; in the advective equation (75) the speed is simply the fluid velocity. The elementary wave equation is also hyperbolic, since fields are propagated at the wave phase speed. A system that includes diffusion is said to be *parabolic*; at a given point the field will be influenced by the entire domain at all previous times. The identification of a model system as hyperbolic or parabolic is a key step in the design and implementation of efficient numerical schemes (e.g., LeVeque[29]).

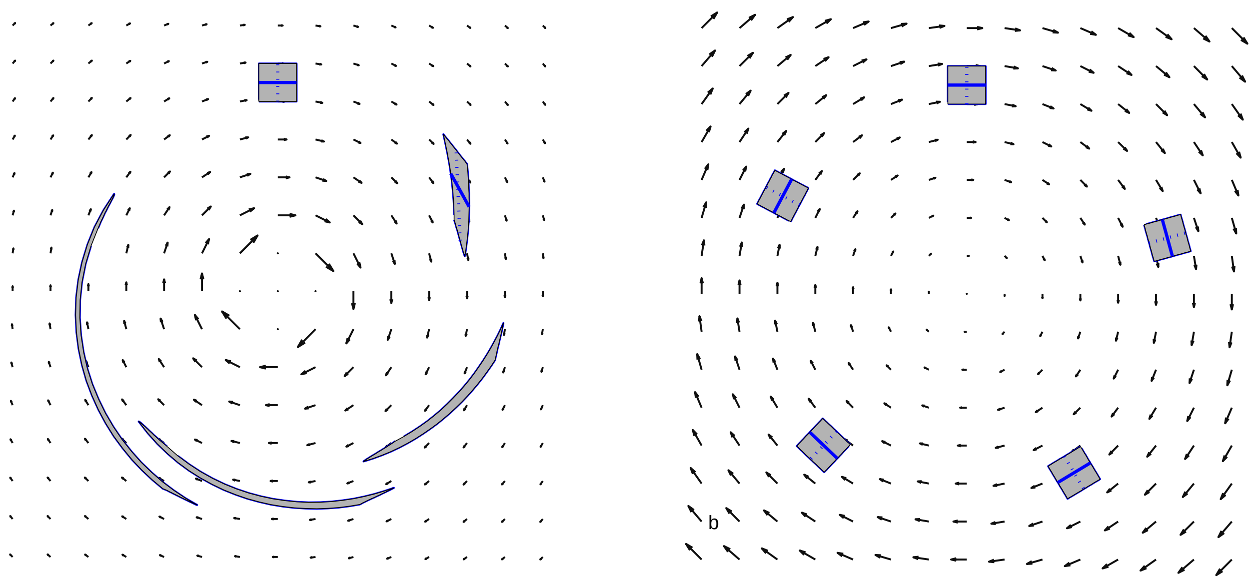

Figure 11: (a) A small patch of colored tracer, or parcel, has been set into a steady, clockwise rotating irrotational vortex and advected through most of one revolution. The largest velocity vectors near the center of the vortex were not plotted. The parcel was square, initially (at 12 o'clock) and then rather severely deformed by this flow. Nevertheless, the area of the parcel was conserved, as was its average orientation (though the orientation is obscured by the very large deformation). (b) In this experiment the vortex flow was a solid body rotation. As the name implies, this motion could just as well be that of a solid, rotating object. The orientation of the parcel changes in time, but the area and the shape are conserved, i.e., there is no deformation.

where Ω is the uniform rotation rate. It is assumed that $C < 0$ and $\Omega < 0$ so that both vortical flows are clockwise. These are idealizations, of course, and yet with a little effort (imagination?) something akin to both kinds of vortices can be observed in the flow in a teacup: more or less irrotational vortices are observed to spill off the edges of a spoon that is pushed through the fluid, and at longer times the azimuthal motion that fills the teacup will often approximate a solid body rotation (except very near the edges).[33]

The first thing to note is that the parcels are transported clockwise with the clockwise flow in either vortex; in these flows there is nothing quite as exciting as Stokes drift (Section 5.2) that can make the Lagrangian mean flow (i.e., what we see as the displacement of the parcel) qualitatively different from the Eulerian mean flow (what you would expect given the field of vector velocity). Aside from that, the effects of advection are remarkably different — the irrotational vortex produces a very strong deformation of the parcel, while the solid body rotation leaves the shape of the parcel unchanged. The irrotational vortex leaves the (average) orientation of the parcel unchanged, though this is impossible to verify in Fig. (11) given the very large deformation, while the solid body rotation changes the orientation at the rate Ω that characterizes the rotation rate of the vortex. The area of the parcel is unchanged in either case.

[33]The irrotational vortex has a singularity in U_{azi} at $r = 0$, while in a solid body rotation U_{azi} grows linearly and without bound with r. A hybrid made from these two idealizations — solid body rotation near the center of a vortex, Eq. (78), matched to an irrotational profile $U_{azi}(r)$ from Eq. (21), that continues on for larger r — avoids both problems and can make a convenient, useful approximation to a real vortex, e.g., a hurricane. This kind of hybrid is called a Rankine vortex.

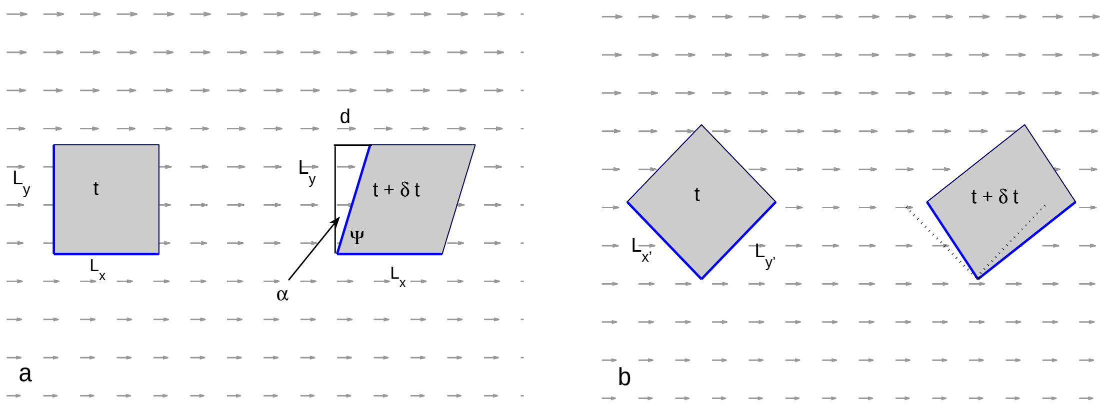

Figure 12: (a) A parcel (marked with colored tracer), has been set into the shear flow defined by Eq. (80) and advected for a time interval, δt. The sides on the lower and left edges of the parcel were initially orthogonal, and of length L_x and L_y. After a time interval δt, the upper left corner has been displaced a distance $d = \delta t L_y \partial U/\partial y$ with respect to the lower left corner and so the left edge of the parcel has rotated clockwise through an angle $\alpha \approx \tan\alpha = -d/L_y$. The angle α thus changes at the rate $d\alpha/dt \approx \alpha/\delta t = -\partial U/\partial y$, as long as α is small. (b) The same flow and the same parcel, but compared to the example at left, the initial orientation of the parcel was rotated by 45 deg. The original lower left and lower right sides are shown at $t + \delta t$ as the dotted lines. In this case the angle defined by the lower left and lower right edges remains 90 deg, while the length of these sides is compressed or stretched. Evidently this particular shear flow has both a shear strain rate, emphasized at left, and a linear strain rate, emphasized at right (strain rate will be defined in the text below). An orthogonal axes pair, e.g., the lower and left sides of this parcel, will thus sample one or the other (or a little of both) of these strain rates depending upon their orientation with respect to the flow.

These changes in the orientation and shape of a fluid parcel are caused solely by advection and are thus a consequence of the velocity field. Our goal in this section is to find out what specific properties of the velocity field are relevant. Once again we are asking for what amounts to Lagrangian properties — the size, shape, etc. of a fluid parcel — in terms of the Eulerian velocity field, $V(x, y, z, t)$. To make the analysis tractable we are going to consider flows that are two-dimensional and we will follow the parcel only for short times (unlike the examples of Fig. 11) so that the velocity field can be assumed steady. Given these restrictions, the Eulerian velocity field around a given point, x_0, y_0 can be calculated with sufficient accuracy by the first terms of a Taylor expansion (as in Section 5.1 and repeated here),

$$\mathbf{V}(x, y) = \mathbf{V}(x_0, y_0) + \mathbb{G} \cdot \delta\mathbf{X} + HOT, \tag{79}$$

where $\delta\mathbf{X}$ is a small displacement and

$$\mathbb{G} = \begin{pmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{pmatrix}$$

is the velocity gradient tensor, which represents all that we presume to know regarding the velocity field. These restrictions to small displacement and steady flow may seem severe, but in the end we come to results that can be applied to the differential (Eulerian) conservation equations, which is just what we need.

The velocity gradient tensor is the center of attention for now, and we'd like to know what it looks like.

We can not make a diagram of a tensor *per se*, but we can show what $\mathbb{G}$ does when it operates on a displacement vector, and that is what counts. We will illustrate this with a very simple shear flow [34]

$$\mathbf{V}(x,y) = \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} U_0 + \gamma y \\ 0 \end{pmatrix}, \tag{80}$$

where the spatial mean flow is $U_0 = 0.5$ and the shear is $\partial U/\partial y = \gamma = 0.7$ and constant. A fluid parcel embedded in this flow evolves as shown in Fig. (12). The velocity gradient tensor evaluated at any point in this flow is just

$$\mathbb{G} = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0.7 \\ 0 & 0 \end{pmatrix}. \tag{81}$$

To see what $\mathbb{G}$ does, we can multiply $\mathbb{G}$ into a set of unit vectors, $\mathbf{e}$, that span 360 degrees in direction, and plot the resulting velocity difference, $\delta\mathbf{V} = \mathbb{G}\mathbf{e}$, at the end of each unit vector, Fig. (13a). The velocity difference plotted in this way looks a lot like the velocity field itself since the shear is presumed spatially uniform. However, these diagrams show properties of the velocity field at the specific point where $\mathbb{G}$ has been evaluated, and are not a map of the velocity field *per se*.

Rotation rate: Consider a spatial variation of the velocity that could cause the sides of the parcel to change orientation: in the example of Eq. (80) and Fig. (12a), the U component of velocity increases with increasing y, and this causes the left and right sides of the parcel to rotate clockwise. If we denote the angle of the left side of the parcel with respect to the y-axis by α, then by simple geometry we can see that

$$\frac{d\alpha}{dt} = -\frac{\partial U}{\partial y}.$$

Similarly, if we denote the angle between the lower side of the parcel and the x-axis by θ, then the lower side of the parcel would rotate counter-clockwise if the y-component of velocity increased with x, i.e.,

$$\frac{d\theta}{dt} = \frac{\partial V}{\partial x}$$

(the angle $\theta = 0$ in this figure, and so you should make a sketch to verify this, keeping in mind that the angle θ can be assumed to be very small). The angles θ and α may change independently. A sensible measure of the average rotation rate of the parcel, ω, also called the physical rotation rate, is the average of these angular rates,

$$\omega = \frac{1}{2}(\frac{d\theta}{dt} + \frac{d\alpha}{dt}) = \frac{1}{2}(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}). \tag{82}$$

A small, rigid orthogonal vane embedded in the flow would rotate at this rate (assuming that the force on the vanes is linear in the velocity). For many purposes, e.g., Eq. (39), it is convenient to use a slightly different measure of the rotation rate called the vorticity, $\xi = 2\omega$,

$$\boxed{\xi = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = \nabla \times \mathbf{V}} \tag{83}$$

[34] The word *shear* has an origin in the Middle English *scheren*, which means to cut with a pair of sliding blades, for example sheep shears. The fluid mechanics usage emphasizes the idea of sliding planes; a shear stress is a force (per unit area) that is parallel to a specified surface, and a velocity shear is a spatial variation of the velocity in a direction that is perpendicular to the velocity vector, e.g., in the shear flow Eq. (79), the x component of the velocity varies in the y direction.

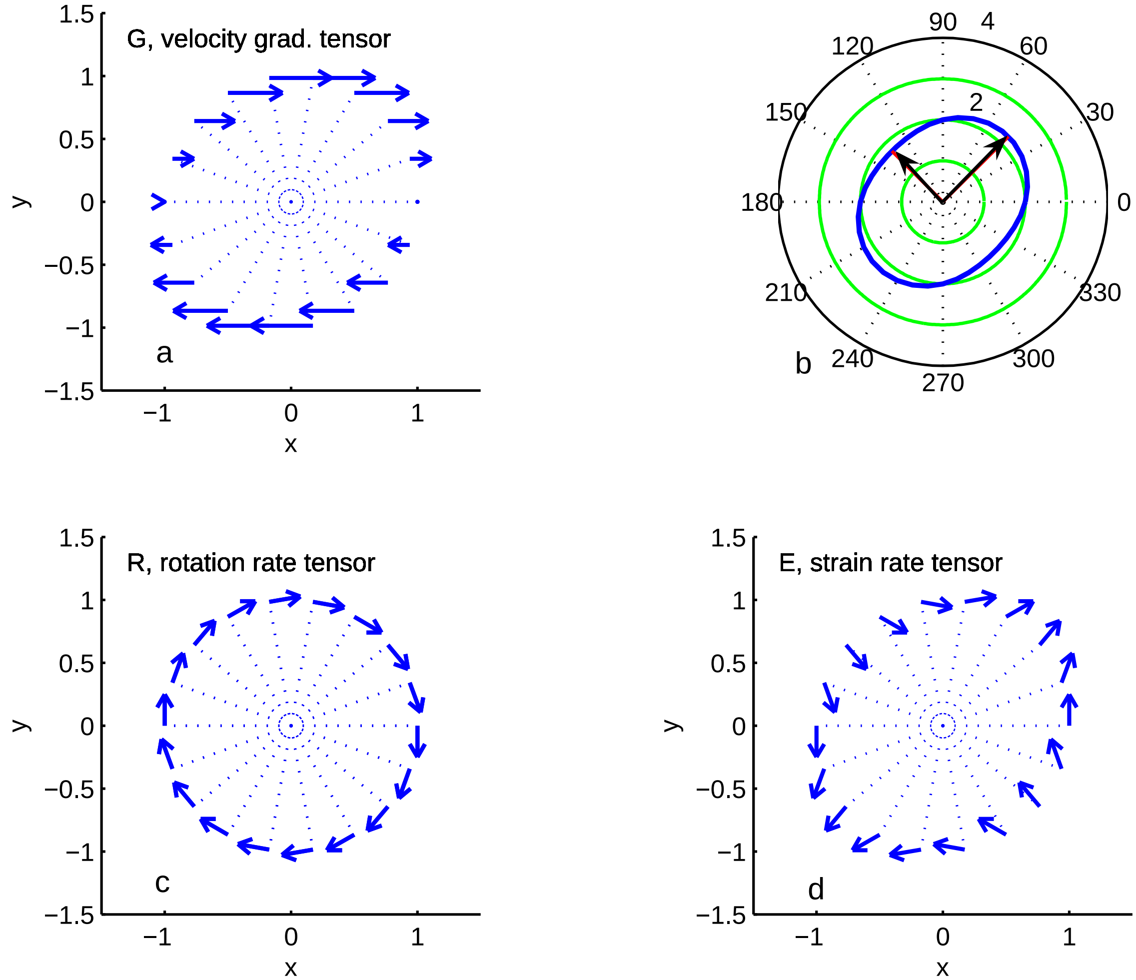

Figure 13: (a) The velocity gradient tensor $\mathbb{G}$, Eq. (81), has been multiplied into a sequence of unit vectors with varying directions (the dashed lines) and the resulting velocity plotted at the end of the unit vectors. (b) The eigenvectors of the strain rate tensor are shown as the vectors, and the linear strain rate + 1 is the peanut-shaped ellipse (ovals of Cassini). The maximum and minimum values of the linear strain rate are aligned with the eigenvectors. (c) The rotation rate tensor $\mathbb{R}$ has been multiplied into a sequence of unit vectors. (d) The strain rate tensor $\mathbb{E}$ multiplied into the unit vectors. The resulting velocity vectors $\delta\mathbf{V}$ have a variable direction (and magnitude) with respect to the unit vector. Notice how $\mathbf{e}\cdot\delta\mathbf{V}$ compares with the linear strain rate shown in the upper right panel.

The vorticity is invariant to the orientation of the parcel with respect to the flow, and hence is invariant to a rotation of the coordinate system. In R^2 vorticity is effectively a scalar, i.e, a single number; in R^3 the vorticty $\nabla \times \mathbf{V}$ is a vector with three components. The vorticity of a fluid is analogous in many respects to the angular momentum of a rotating solid. Vorticity follows a particularly simple conservation law that makes it an invaluable diagnostic quantity.

Strain rate: In the case that the two angles θ and α change at different rates, then our parcel will necessarily change shape or deform. One of several plausible measures of the shape of the parcel is the angle made by the lower and left sides, ψ, and evidently $\psi = \pi/2 + \alpha - \theta$, Fig. (12a). If α and θ change by the same amount, then $\psi = constant$, and the parcel will simply rotate without deforming; this is what we could call a solid body rotation, considered just above. But if the angles change at a different rate, then the shape of the parcel will necessarily change, and for the angle ψ,

$$\frac{d\psi}{dt} = \frac{d\theta}{dt} - \frac{d\alpha}{dt} = \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}. \tag{84}$$

If the angle α alone changes while the lengths of the parcel sides remain constant, then this kind of time-varying deformation is called a *shear* strain rate. This is the case in Fig. (12a) when the angle α is very small, i.e., for very small times after the parcel is released in the flow. This particular shear strain rate is in the plane parallel to the x-axis.

But suppose that we rotate the parcel by 45 degrees before we release it into the same shear flow. The result of advection then appears to be quite different if we continue to emphasize an orthogonal axes pair (Fig. 12b); the sides of the parcel remain orthogonal, but now the lengths of the sides are compressed (lower left side, $d(L_{x'})/dt < 0$) or stretched (lower right side, $d(L_{y'})/dt > 0$). A strain rate that causes a change in the length of a material line is termed a *linear* strain rate, e.g., in the x'-direction or y'-direction, rewriting Eq. (44),

$$\frac{dL_{x'}}{dt} = L_{x'}\frac{\partial U}{\partial x'} \quad \text{and} \quad \frac{dL_{y'}}{dt} = L_{y'}\frac{\partial V}{\partial y'}. \tag{85}$$

Divergence, another very important quantity that we have encountered already in Section 7.1, is the sum of the linear strain rates measured in any two orthogonal directions and gives the normalized rate of change of the area, $A = L_x L_y$, of the parcel, or rewriting Eq. (45),

$$\frac{1}{dA}\frac{d}{dt}(dA) = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = \nabla \cdot \boldsymbol{V}.$$

Unlike the vorticity and the divergence, the strain rate can not be written as a vector operator on the velocity, and from Fig.(12) it seems that the strain rate that we diagnose with a given orthogonal axes pair (the sides of the parcel), is entirely dependent upon the orientation of these axes with respect to the flow. It is highly unlikely that a quantity that depends entirely upon the orientation of the coordinate system can have any fundamental role, and this in turn implies that while Eqs. (84) and (85) are useful, nevertheless there is more to say about the strain rate than Eqs. (84) and (85) alone.[35]

The Cauchy-Stokes Theorem: The strain rate may change the shape and size of a parcel, while rotation

[35] It was noted above that the rotation rate defined by Eqs. (82) or (83) is independent of the orientation of the axes with respect to the flow. Can you verify this (semi-quantitatively) from Fig. (12)?

alone does not. This suggests that there is something fundamentally different in these quantities, and that it may be useful to separate the rotational part of the velocity gradient tensor from all the rest. This turns out be straightforward because the rotation is associated with the anti-symmetric part of the velocity gradient tensor, and any tensor can be factored into symmetric and anti-symmetric component tensors as follows. Let $\mathbb{G}'$ be the transpose of $\mathbb{G}$,

$$\mathbb{G}' = \begin{pmatrix} \frac{\partial U}{\partial x} & \frac{\partial V}{\partial x} \\ \\ \frac{\partial U}{\partial y} & \frac{\partial V}{\partial y} \end{pmatrix}.$$

We can always subtract and add $\mathbb{G}'$ from $\mathbb{G}$

$$\begin{aligned} \mathbb{G} &= \tfrac{1}{2}(\mathbb{G} - \mathbb{G}') + \tfrac{1}{2}(\mathbb{G} + \mathbb{G}') \\ &= \mathbb{R} + \mathbb{E}, \end{aligned} \tag{86}$$

and thereby define two new tensors

$$\mathbb{R} = \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial U}{\partial y} - \frac{\partial V}{\partial x} \\ \\ \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} & 0 \end{pmatrix} \tag{87}$$

and

$$\mathbb{E} = \begin{pmatrix} \frac{\partial U}{\partial x} & \frac{1}{2}(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}) \\ \\ \frac{1}{2}(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}) & \frac{\partial V}{\partial y} \end{pmatrix}. \tag{88}$$

$\mathbb{R}$ is called the rotation rate tensor and is anti-symmetric, $R_{12} = -R_{21}$; $\mathbb{E}$ is called the strain rate tensor and is symmetric, $E_{12} = E_{21}$. in R^2, $\mathbb{R}$ has only one unique component that is proportional to the rotation rate ω. When $\mathbb{R}$ is multiplied onto a set of unit vectors the result is a velocity difference $\delta\mathbf{V} = \mathbb{R}\mathbf{e}$ that is normal to the unit vector and has the same amplitude (same speed) for all directions of the unit vectors. The rotation associated with $\mathbb{R}\mathbf{e}$ is apparent, Fig. (13, lower left), and the magnitude is just $\omega = -\frac{1}{2}\partial U/\partial y = -0.35$; the vorticity is -0.7.

The stain rate tensor $\mathbb{E}$ has three independent components (in general) and is a little more involved. The resulting velocity difference, $\delta\mathbf{V} = \mathbb{E}\mathbf{e}$, varies in direction and amplitude depending upon the direction of the unit vector (though in the specific case shown in Fig. (13d) the amplitude happens to be constant). The linear strain rate in a given direction is given by the component of $\delta\mathbf{V}$ that is parallel or anit-parallel to the unit vector in that direction. There are two special directions in which the linear strain rate is either a minimum or a maximum. Given the specific $\mathbb{G}$ of Eq. (81), the minimum linear strain rate is -0.35 and is found when the unit vector makes an angle of 135 (or 315) degrees with respect to the x axis (Fig. 13b); the maximum linear strain rate is 0.35 and is at 45 (or 225) degrees. Thus a parcel will be compressed along a line 135 degrees (with respect to the x axis) and will be stretched along a line normal to this, 45 degrees. Notice too that when the unit vector is pointing in these special directions the velocity difference and the unit vector are either anti-parallel or parallel, and the relationship among $\mathbb{E}$, $\mathbf{e}$, and $\delta\mathbf{V}$ may be written

$$\mathbb{E}\mathbf{e} = \delta\mathbf{V} = \lambda\mathbf{e},$$

where λ is a real number. These directions are thus the directions of the *eigenvectors* of the symmetric tensor $\mathbb{E}$, and the amplitude of the linear strain rate in those directions is given by the *eigenvalues*, λ, of $\mathbb{E}$; $\lambda = \pm 0.35$ for this particular $\mathbb{E}$.[36] The divergence, $\nabla \cdot \mathbf{V} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}$, is the sum of the eigenvalues, and also the trace of $\mathbb{E}$; in the specific case of the shear flow Eq.(80), the divergence happens to be zero though the linear strain rate in most directions is not. Like the rotation rate, the divergence is invariant to rotation of the coordinate system. In the discussion above we have sketched out

> ***The Cauchy-Stokes Decomposition Theorem:*** *an instantaneous fluid motion may, at each point, be resolved into three components;*
>
> *1) a translation* — the velocity $\mathbf{V}(x_0, y_0)$ of Eq. (79),
>
> *2) a rigid body rotation* — the amplitude of which is given by the off- diagonal elements of the rotation rate tensor, $\mathbb{R}$, and,
>
> *3) a linear strain rate along two mutually perpendicular directions* — the directions and the amplitudes of the maximum and minimum linear strain rate are given by the eigenvectors and eigenvalues of the strain rate tensor, $\mathbb{E}$.

In most real fluid flows the rotation rate and the strain rate will vary spatially and in time, right along with the velocity itself, and each of them have important roles in Eulerian theories that seek to predict the evolution of fluid flows: the rotation rate (vorticity) is analogous to angular momentum, as we have already noted; the strain rate is proportional to the rate at which adjacent fluid parcels are sliding past a given point, and the viscous stress in a fluid is often assumed to be proportional to the strain rate tensor times the fluid viscosity, a scalar that is considered to be a thermodynamic property of a fluid. We will see these things again.[37]

---

[36] Eigenvectors and eigenvalues are a central theme of linear algebra and will not be reviewed here. Excellent references are by J. Pettofrezzo *Matrices and Transformations* (Dover Pub., New York, 1966) and for undergraduate-level applied mathematics generally, M. L. Boas, *Mathematical Methods in the Physical Sciences, 2nd edition* (John Wiley and Sons, 1983). If you have access to Matlab, a search on 'eigenvector' will return several useful, concise tutorials.

[37] So soon? 1) Go back and take another look at the (stirred) fluid flow in a teacup. Do you see evidence of divergence, rotation or strain rate? 2) Compute the velocity gradient tensor, and the associated divergence, rotation, and strain rates for the case of an irrotational vortex, and for the case of a solid body rotation. You may compute the derivatives either analytically or numerically, and use Matlab to calculate the eigenvalues and vectors. Choose several points at which to do the calculation and interpret your results in conjunction with Fig. (11). 3) The Cauchy-Stokes Theorem is useful as a systematic characterization and interpretation of the velocity expansion formula, Eq. (79), and especially of the velocity gradient tensor. A characterization of the strain rate tensor by its eigenvectors/values is surely the most natural, but suppose that we have a special interest in the shear strain rate, or, perhaps we just want to be perverse — can you state an equivalent theorem in terms of the shear strain rate? 4) Now that we have the hang of these tensors we decide to make yet another one: move the trace (the divergence) of the strain rate tensor into what we might call the divergence tensor, D, i.e., $D(1,1) = D(2,2) = \frac{1}{2}(\nabla \cdot V), D(1,2) = D(2,1) = 0$. What properties does this new tensor D have? What simplification results so far as the eigenvectors/values of the strain rate tensor are concerned? 5) In a similar vein, the velocity gradient tensor in R^2 has four independent elements, $\partial U/\partial x, \partial U/\partial y, \partial V/\partial x$ and $\partial V/\partial y$. We have seen that three of the four possible combinations of these terms have real importance, the divergence, $\partial U/\partial x + \partial V/\partial y$, the rotation rate or vorticity, $\xi = \partial V/\partial x - \partial U/\partial y$, and the shear strain rate, $\partial V/\partial x + \partial U/\partial y$ of Eq. (84). Can you interpret the fourth possible combination, $\partial U/\partial x - \partial V/\partial y$? (Don't expect anything of cosmic significance.)

9 The Euler Fluid

In this section we will consider a comparatively simple but important idealization of a fluid, the 'Euler fluid', that recognizes only two forces on a fluid parcel, a pressure force and gravitational acceleration. Our goals in this are two-fold: 1) To complete the development of a fluid model in Sections 9.1 and 9.2 (since we have studied the difficult, formal part, the kinematics) and 2) To compare the Eulerian and Lagrangian equations of motion in Sections 9.3, and specifically to show why the Eulerian or field system is strongly preferred over the Lagrangian system for most problems.

9.1 Pressure and inertial forces

Imagine a parcel immersed in a volume of fluid; at each point on the surface of the parcel there will be a compressive force given by $-P\mathbf{n}d\sigma$ where P is a scalar called the pressure, $\mathbf{n}$ is the unit normal that points outward from the parcel into the surrounding fluid, and $d\sigma$ is the differential area. The vector $-P\mathbf{n}$ is thus a normal force per unit area (or normal stress) exerted on the parcel by the surrounding fluid. The *cause* of pressure[38] is varied, but for now we assume that pressure exists, and look for the immediate consequences so far as the momentum balance is concerned. The net pressure and gravitational force on the parcel is given by the integral over the entire surface of the parcel,

$$- \oint \mathbf{n}P \, d\sigma = - \oint \nabla P \, dVol, \tag{89}$$

where the right hand side follows on application of Gauss' (or Green's) theorem. We will also allow that a gravitational acceleration of magnitude g may be present, and will act upon a fluid parcel exactly as it would a solid particle. The net force on a parcel of Euler fluid is then just

$$\oint \mathbf{F} \, dVol = - \oint (\nabla P + \rho g \mathbf{e_z}\,) \, dVol. \tag{90}$$

The unit vector $\mathbf{e_z}$ is defined to be anti-parallel to the gravitational acceleration vector field (as measured by a plumb line, say). Note that gravitational acceleration is a body force; other inertial forces, e.g., the Coriolis force, will appear as additional body forces. In the limit that the parcel is made very small, as in going from Eq. (54) to (55), the pressure and gravitational force per unit volume is given by

$$\mathbf{F} = -\nabla P - \rho g \mathbf{e_z}. \tag{91}$$

Gravitational acceleration can usually be regarded as a known quantity, independent of the fluid. Pressure is in general an *unknown* quantity that has to be solved alongside velocity and density. [39] Pressure *per se* is of great significance so far as thermodynamics properties of a fluid are concerned, but notice that it is only the pressure *gradient* that appears in the momentum balance of a parcel.

[38] This P is the absolute pressure, $P \geq 0$, and is the pressure that appears in thermodynamic relations. There are other kinds of pressure, for example the pressure value that is read from an air gauge used to check the inflation of a tire is a 'gauge pressure' and is measured relative to the ambient atmospheric pressure. Gauge pressure can be less than zero (and then your tire is very flat).

[39] That it can be helpful, indeed necessary, to introduce an unknown pressure into the momentum balance is by no means an obvious step, and in fact pressure first appeared in the historical development of fluid mechanics long after other seemingly more subtle concepts were well understood. An account of the development of fluid mechanics is by G. A. Tokaty, *'A History and Philosophy of Fluid Mechanics'*, (Dover Pub., New York, 1971)

9.2 The Euler fluid in Eulerian form

The Eulerian momentum equation Eq. (55) that includes these two terms as the only force is just

$$\boxed{\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = \frac{-1}{\rho}\nabla \mathbf{P} - g\mathbf{e_z}} \tag{92}$$

and the mass conservation relation is as before,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0. \tag{93}$$

It is often useful to write these equations in the flux or 'conservation form' described in Section 8.1. The mass conservation equation (93) is already written that way. If we regard the density as variable, then we can not write the velocity of the Euler fluid in flux form directly, we have to use instead the momentum density, i.e., $\rho\mathbf{V}$. By adding the mass conservation equation to the momentum equation and minor rearrangement (that you should be sure to verify) we can write that

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho U \\ \rho V \\ \rho W \\ \rho \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho U^2 + P \\ \rho UV \\ \rho UW \\ \rho U \end{bmatrix} + \frac{\partial}{\partial y}\begin{bmatrix} \rho VU \\ \rho V^2 + P \\ \rho VW \\ \rho V \end{bmatrix} + \frac{\partial}{\partial z}\begin{bmatrix} \rho WU \\ \rho WV \\ \rho W^2 + P + \Phi \\ \rho W \end{bmatrix} = 0 \tag{94}$$

where the gravitational potential $\Phi = \rho g z$.[40]

9.3 Equation of state*

To this point the model equations are rather general. However, we have five unknowns — three velocity components, U, V and W, the density, ρ, and the pressure, P — and only four independent equations. Clearly something more is required. To complete this model we have to specify also the relationship between pressure and density (at least), and this depends upon the physical properties of the fluid, i.e., whether a liquid and almost incompressible, or a gas which is readily compressed, and whether the fluid exchanges heat with its surroundings.

The simplest possibility is to take the density to be a known constant,

$$\rho = \rho_o,$$

which is appropriate for a model of gravity waves on a water-air interface, since the density variations within the water will be much, much smaller than the density contrast across the surface. If the fluid is a gas, then the density likely varies significantly with the pressure, and so it is more appropriate to take density to be some function of the pressure,

$$\rho = \rho(P). \tag{95}$$

[40]In Section 6.1 we made a lot of the fact that advection alone could not be a net source or sink of momentum in the case that a fluid was contained within some bounded domain. We can readily write the Euler fluid momentum equation in flux form, but what happens to the conservation property when pressure and gravitational potential are included? Consider also an integral over a finite domain.

This kind of relationship among thermodynamic variables is often called an equation of state. A fluid described by Eq. (95) is called 'barotropic' in that lines of constant density will be parallel to lines of constant pressure $\nabla\rho = (\partial\rho/\partial P)\nabla P$. [41]

To summarize: the Eulerian momentum and mass conservation equations, Eq. (94), plus the state equation, Eq. (95), make what is in principle a complete set of equations for an Euler fluid. To solve a specific problem we also have to specify appropriate boundary conditions and inital conditions, important topics taken up elsewhere.

9.4 The Euler fluid in Lagrangian form.*

The considerations that lead to Eq. (91) and the equation of state, Eq. (95), can be applied straight away to the Lagrangian system, Eq. (5), which we will consider briefly in this section. But first a warning: the Lagrangian equations of this section are not easily or often used for most practical purposes, including ours, and we will not see them again after this section. Nevertheless, if you are curious to know why the Lagrangian equations of motion are seldom seen, and why nearly all of theoretical fluid dynamics is carried out with the Eulerian equations of motion, then read on (otherwise this can be skipped without loss of continuity).

We are going to make a small change in our Lagrangian notation before going further, and that is to use $\mathbf{X}$ as the position of a fluid parcel, with components x, y, z, and $\mathbf{A}$ as the initial position with components a, b, c for (this avoids the subscript $(\ )_o$.) The Lagrangian equation of motion is then;

$$\frac{d^2\mathbf{X}}{dt^2} = \nabla P/\rho + \mathbf{g}, \tag{96}$$

where the d/dt is an ordinary time derivative. In component form this is

$$\frac{d^2x}{dt^2} = \frac{1}{\rho}\frac{\partial P}{\partial x}, \quad \frac{d^2y}{dt^2} = \frac{1}{\rho}\frac{\partial P}{\partial y} \text{ and } \frac{d^2z}{dt^2} = \frac{1}{\rho}\frac{\partial P}{\partial z} + g. \tag{97}$$

On first sight these look promising compared with Eq. (94) since there is nothing quite like the nonlinear advection terms of the Eulerian differential conservation equations. Ineed, advection does not arise as a separate term in the Lagrangian system. However, on second view there is a serious problem lurking in the pressure gradient (noted toward the end of Section 2). The gradient of the pressure is necessarily taken with respect to the usual spatial coordinates x, y, z, which in this Lagrangian system are *dependent* variables, and the derivative of one unknown with respect to another unknown is extraordinarily difficult. In the Lagrangian system the independent spatial coordinates are the initial position of the parcel, the a, b, c, and so the pressure derivative we would much prefer is

$$\frac{\partial P}{\partial a} = \frac{\partial P}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial a}.$$

[41] Lest this seem hopelessly vague we will note two important classes of barotropic fluids; if the fluid is a mass of ideal gas that can quickly exchange heat with an isothermal heat reservoir, then the gas may remain isothermal under pressure changes and so $\rho = \rho_0\frac{P}{P_0}$. If the fluid is a mass of ideal gas that is effectively insulated from heat exchange, then $\rho = \rho_0(\frac{P}{P_0})^{\frac{1}{\gamma}}$, where $\gamma = C_p/C_v$ is the ratio of specific heats, about 1.4 for air (derived in, e.g., Kundu and Cohen [7], Section 1.9). If the density depends upon a variable temperature and heat is exchanged with the surroundings, then the model equations have to be augmented with an energy balance equation, the First Law of thermodynamics applied to a continuum.

To eliminate the spatial derivative $\partial/\partial x$ in favor of $\partial/\partial a$ we will use the chain rule, and that $x = x(a, b, c, t)$, $y = y(a, b, c, t)$ and so on.[42] Solving this for the pressure gradient $\partial P/\partial x$ and substitution into the x-component of the Lagrangian momentum equation gives

$$\frac{\partial^2 x}{\partial t^2} = -\left(\frac{\partial P}{\partial a} - \frac{\partial P}{\partial y}\frac{\partial y}{\partial a} - \frac{\partial P}{\partial z}\frac{\partial z}{\partial a}\right)/\rho\frac{\partial x}{\partial a},$$

which does not look promising (and note that we picked up $\partial P/\partial y$ and $\partial P/\partial z$). To put this into a form analogous to Eq. (94) in which only a single pressure term appears in each component equation we can multiply the x-component equation of Eq. (97) by $\partial x/\partial a$, the y-component equation by $\partial y/\partial a$ and the z-component by $\partial z/\partial a$ and add the resulting three equations; the procedure is repeated for the other components with the result (Lamb,[3] Article 1.13):

$$\begin{aligned}
\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} + (\frac{\partial^2 z}{\partial t^2} + g)\frac{\partial z}{\partial a} &= \frac{1}{\rho}\frac{\partial P}{\partial a}, \\
\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial b} + (\frac{\partial^2 z}{\partial t^2} + g)\frac{\partial z}{\partial b} &= \frac{1}{\rho}\frac{\partial P}{\partial b}, \\
\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial c} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial c} + (\frac{\partial^2 z}{\partial t^2} + g)\frac{\partial z}{\partial c} &= \frac{1}{\rho}\frac{\partial P}{\partial c}.
\end{aligned} \tag{98}$$

For completeness we will list but not derive the mass conservation relation (see Lamb, loc. cit.)

$$\rho\frac{\partial(x, y, z)}{\partial(a, b, c)} = \rho_o,$$

where the partial differential operator is the Jacobian of the transformation from material to spatial coordinates.

The Lagrangian momentum equations written for a continuum do not retain the simplicity of the corresponding equations for solid particles, and, more to the point, they are less amenable for most purposes of fluid dynamics than are the corresponding Eulerian continuum equations. With no intent to make the situation seem hopeless, we will note two significant hurdles to their use: 1) The Lagrangian equations involve a second derivative with respect to time, where even one integration can be challenging. In effect, the solution of the Lagrangian equations requires solving for parcel trajectories in one fell swoop. In the Eulerian system we can solve for velocity (one integration in time), and then can integrate the velocity solution separately to find parcel trajectories (Section 4.1), if they are required. In many problems, trajectories will not be required, and indeed we may have no interest in knowing the trajectories of specific parcels. To say it a little differently, a Lagrangian solution may tell us much more than we may need to know. That would not be grounds for complaint, except that nothing comes free, and Lagrangian solutions are likely to require a very high price in computational or mathematical effort. 2) While the nonlinear advection terms in an Eulerian momentum equation have a clear-cut physical interpretation (as in the modes of Section 6.2), and are semi-linear (so that the method of characteristics is applicable) the same can not be said for the corresponding nonlinear terms of Eq. (98). And specifically, the derivative of parcel position

[42] One way to think of this is that the differential displacement, δx, depends upon the initial position of the parcel as $\delta x = \frac{\partial x}{\partial a}\delta a + \frac{\partial x}{\partial b}\delta b + \frac{\partial x}{\partial c}\delta c$. Substitution into the partial derivative gives what amounts to the chain rule.

with respect to the initial position, e.g., $\partial x/\partial a$, can be extraordinarily complex in cases like the flow in a stirred tea cup where parcels move a significant distance away from their starting position. As we noted in Section 1, this difficulty stems from the inherent need to represent local (spatial) interactions within a continuum. At another level we might add that some form of nonlinearity is inevitable in a fluid dynamics model, and it happens that the Lagrangian form of it is unfortunately rather awkward.

There are always exceptions and remedies; in wave problems, where particles may not move long distances (Section 5.2), the Lagrangian system may be tractable and in some special problems involving nonlinear surface gravity waves, even advantageous. It is also possible to reinitialize parcel positions from time to time and so avoid the worst of $\partial x/\partial a$. While there are exceptions, and we should keep an eye open for others, it is a fair generalization to say that the Lagrangian momentum equations, Eq. (98), are not as suitable for most theoretical problems that require integrating the equations of motion as is the Eulerian system. It is for that reason that we worked through the development of the Eulerian conservation laws in Section 6 and 7, and will devote nearly all of our effort to the Eulerian system from here on.

10 Concluding Remarks.

The broad goal of this essay has been to introduce some of the central concepts of kinematics applied to fluid flows, and especially to develop an understanding of Eulerian and Lagrangian representations of fluid flow. The starting point is the so-called Fundamental Principle of Kinematics, or FPK, (Eq. (3) and Section 1.1), which asserts that there is one unique fluid velocity. The fluid velocity can be sampled either by tracking fluid parcels or by emplacing current meters at fixed locations. We have seen by way of a simple example (Sections 2 and 3) that is possible to shift back and forth from a Lagrangian to an Eulerian representation provided that we have either (1) a complete knowledge of all parcel trajectories, or, (2) the complete velocity field at all relevant times. In Section 2 we first presumed (1), which happens only in the special world of homework problems. However, in the usual course of a numerical model calculation we probably will satisfy (2), and thus can compute parcel trajectories on demand (Sections 3 and 4). Numerical issues and diffusion (numerical and physical) will complicate the process and to some degree the result. Nevertheless, the procedure is straightforward in principle and is often an important step in the diagnostic study of complex flows computed in an Eulerian frame. Approximate methods may be usefully employed in the analysis of some flows, an important example being the time-mean drift of parcels in a field of surface gravity waves (Section 5.2). The Eulerian mean motion below the wave trough is zero on linear theory, while the Lagrangian mean flow may be substantial, depending upon wave steepness.

The statement of conservation laws for mass, momentum, etc., applies to specific parcels or volumes of fluid, and yet to apply these laws to a continuum it is usually preferable to transform these laws from an essentially Lagrangian perspective into Eulerian or field form (Section 9.3). There are three key pieces in this transformation; the first is the FPK noted above. The second is the material derivative; an ordinary time derivative transformed into the Eulerian system is $D(\,)/D = \partial(\,)/\partial t + \mathbf{V} \cdot \nabla(\,)$, the sum of a local time rate of change and an advective rate of change (Eq. (38) and Section 6.1). The third is that integrals and their time derivatives can be transformed from material to field coordinates by way of the Reynolds Transport Theorem (Eq. (47) and Section 7.1). Important applications of the RTT yield the mass conservation relation and the momentum balance (Sections 7.2 and 7.3), which are the starting point for classical fluid dynamics.

The process of advection contributes much of the interesting and most of the challenging dynamics and kinematics of fluid flows. The advection term is semi-linear in that it involves the product of an unknown (generally) velocity component and the first partial derivative of a field variable. There are some important bounds on the consequences of advection. For variables that can be written in a conservation form (e.g., mass and momentum), advection alone can not be a net (globally integrated) source or sink, though it may cause variations at any given point in the domain (Section 8.1). Advection alone transports fluid properties at a definite rate and direction, that of the fluid velocity. The method of characteristics (Section 8.2) exploits this hyperbolic property of the advection equation to compute solutions of nonlinear PDEs; along a characteristic line (which are streamlines in steady flow) the governing equation is exactly as seen by a parcel. In the instances where we can solve for the characteristics, this leads to insightful solutions. The idea of characteristics is the starting point for the development of efficient numerical advection schemes [43] and for the interpretation of many fluid flows. Besides merely transporting fluid properties, advection by a nonuniform velocity field (which is to say nearly all velocity fields) will also cause a rotation of fluid parcels that is akin to angular momentum. Advection may also cause straining or deformation of fluid parcels that may lead to greatly increased mixing rates in a stirred fluid compared to diffusion alone (Section 8.3).

Once the RTT is in hand, it is easy to write down the Eulerian or field form of the momentum, continuity and state equation for the so-called ideal (or Euler) fluid, in which only pressure gradient and gravitational forces are recognized (Section 9.1). The Lagrangian form of the momentum balance is also easy enough to write down, but the spatial derivatives required for the pressure gradient lead to very difficult nonlinear terms (Section 9.3). It is for that reason that the Eulerian form of the momentum equation is almost always the first choice for problems in which the equations of motion are integrated in time.

10.1 Where next?

The next steps for us are to consider the appropriate boundary and initial conditions for the Euler fluid model (boundary conditions being the defining element in many problems), and to determine whether there are useful first integrals of the motion, the various Bernoulli functions. The Euler fluid model has built in limitations in that it ignores diffusion and dissipation; the inclusion of these physical processes leads to a more general fluid model, the Newtonian fluid, and a new set of boundary conditions, physics and phenomena. The one thread that runs through nearly all of the vast subject of fluid dynamics is the foundation of continuum kinematics that we have begun to lay down here. Not a few of the most important developments in theoretical fluid dynamics begin with an essentially Lagrangian perspective that is implemented in an Eulerian framework. [44]

[43] Staniforth and J. Cote, 'Semi-Lagrangian integration schemes for atmospheric models - a review', *Mon. Wea. Rev.*, **119**, 2206-2223, 1991.

[44] An example of considerable importance in GFD is the generalized Lagrangian mean, Andrews D.G., and M.E. McIntyre, An exact theory of nonlinear waves on a Lagrangian-mean flow. *J. Fluid Mech.*, **89**(4), 609-646, 1978, which be warned, is not an easy read.

10.2 Acknowledgments

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11 The Movie Corner.

The following pages each have a movie clip that may be played by Acrobat Reader 6. Double click on the image frame and after a short delay the movie should begin. The slider that appears at the bottom of the movie frame can be used to replay the movie.

11.1 Eulerian and Lagragian velocity in an ocean circulation model

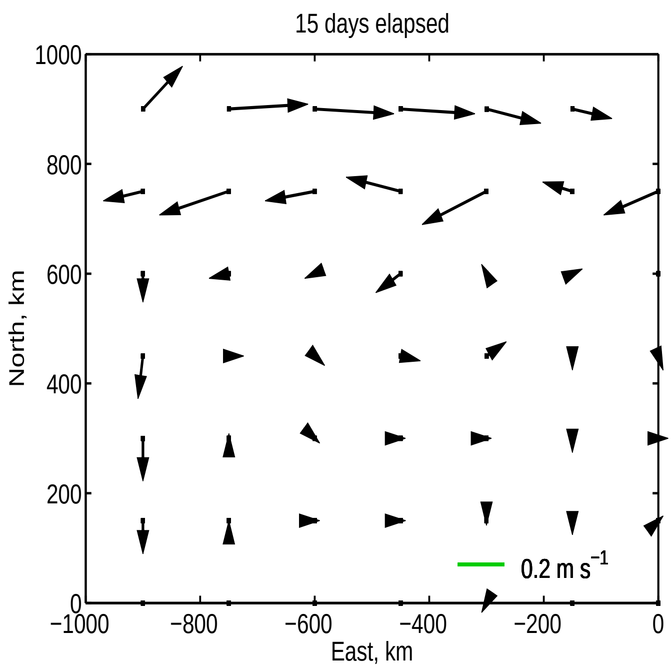

Figure 14: An ocean circulation model sampled with Eulerian and Lagrangian methods (black arrows and green worms, respectively). The domain is a square basin 2000 km by 2000 km driven by a basin-scale wind having negative curl, as if a subtropical gyre. Only the northwestern quadrant of the model domain is shown here. The main circulation features in the upper layer of the model are a thin western and northern boundary current that flows clockwise and a well-developed westward recirculation just to the south of the northern boundary current. This westward flow is baroclinically unstable and oscillates with a period of about 60 days. How would you characterize the Eulerian and Lagrangian representations of this circulation? In particular, do you notice any systematic differences?

11.2 SOFAR float and current meter data from the Sargasso Sea

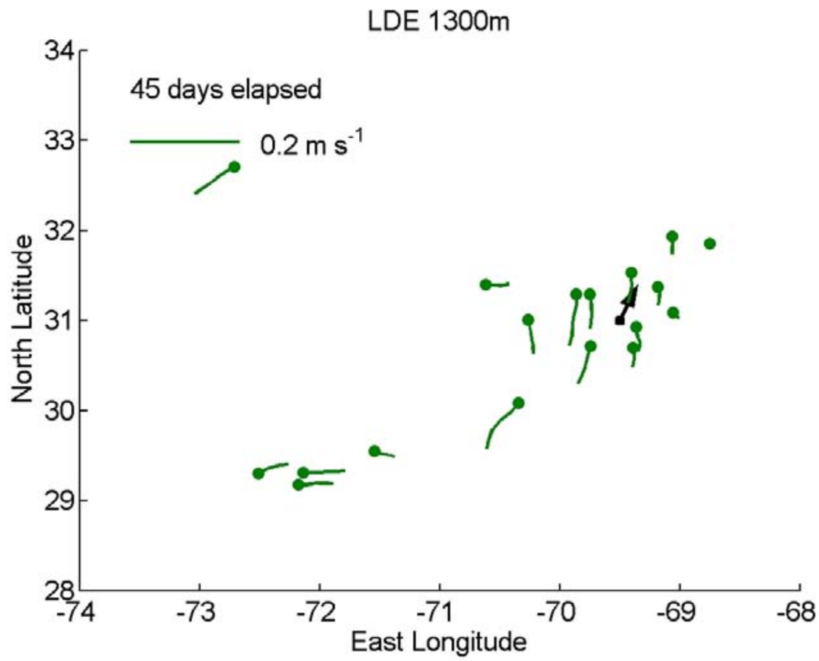

Figure 15: SOFAR float trajectories (green worms) and a single current meter observation (black vector) from the Local Dynamics Experiment conducted in the Sargasso Sea. The float trajectories are five-day segments, and the current vector is scaled similarly. When the flow is large scale and when the floats surround the current meter mooring, the Lagrangian (green) and the Eulerian (black) velocities appear to be indistinguishable. At times when the velocity is changing direction the comparison is quite difficult to make, given the smoothing implicit in the float trajectory segments. The northeast to southwest oscillation seen here appears to be a barotropic Rossby wave; for an analysis of the potential vorticity balance following the motion see Price, J. F. and H. T. Rossby, 'Observations of a barotropic planetary wave in the western North Atlantic', *J. Marine Res.*, **40**, 543-558, 1982. These data and much more are available online from http://ortelius.whoi.edu/ Some animations of the extensive float data archive from the North Atlantic may be found at http://www.phys.ocean.dal.ca/ lukeman/projects/argo/

11.3 Orbital velocity and parcel trajectories in a gravity wave

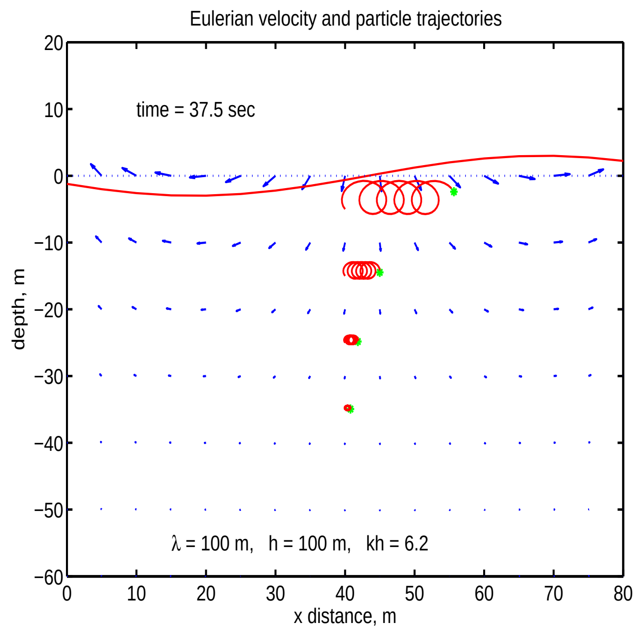

Figure 16: A surface gravity wave propagating from left to right as computed on linear theory. The Eulerian (orbital) velocities are the array of vectors, and parcel trajectories are the red, growing lines that were computed numerically. This wave has an amplitude of 3 m, and a wavelength of 100 m in a water depth of 100 m and so is rather steep wave. The wave frequency $\omega \approx 1$ s^{-1} so that the PVD trajectories have an amplitude almost equal (numerically, in these MKS units) to that of the Eulerian velocity.