The Destabilization of Rossby Normal Modes by Meridional Baroclinic Shear

by

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Abstract

The Rossby normal modes of a two-layer fluid in a meridional channel of width L_* are altered by the presence of a meridional flow with a small vertical shear. The stability of the modes in the presence of the weak shear is considered. It is found that the joint presence of the Rossby modes and the vertical shear leads to baroclinic instability even for arbitrarily small values of the shear.

The results are used to explain previous numerical calculations of the persistent instability of meridional flows when the ratio $\beta L_D^2 / V > 1$, where V is the magnitude of the shear, β is the planetary vorticity gradient and L_D is the deformation radius. If the flow were zonal it would be stable for such weak shears.

The growth rates are weak when $\beta L_D^2 / V >> 1$ and each unstable mode exists in a narrow range of meridional wavenumber. The asymptotic results qualitatively agree with the earlier numerical results at moderate values of the same parameter.

1. Introduction

In a recent paper Walker and Pedlosky (2002), hereafter WP, examined the instability, within the two-layer model, of a meridional flow on the beta plane in a meridional channel of width L_*. If the channel were unbounded and the meridional flow were of infinite lateral extent earlier results (Pedlosky, 1987) show that the flow would always be unstable to a wave-like perturbation independent of the zonal direction (x) . Such a perturbation would thus be insensitive to the stabilizing presence of β. The finite width of the channel forces an x variation in the perturbation stream function field and thus a meridional velocity which senses the planetary vorticity. Nevertheless, WP found that the meridional shear flow was unstable for all values of β examined, even for those values of $\beta L_D^2 / V > 1$ (symbols have conventional meanings and are defined below) for which the flow would be stable if it were zonal instead of meridional. In addition, the instability extended to wave lengths shorter than the classical short-

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wave cut-off of the two-layer model. WP speculated that the extended range of instability in shear and wave number was due to the destabilization by the presence of weak shear of the Rossby normal modes present in the channel. (Note that for a *plane wave* in an unbounded region a stability threshold does exist if the wave vector is not purely meridional). However, the difficulty of the numerical analysis when the shear is weak and the growth rates are small precluded a satisfying verification of this hypothesis.

In this paper I present an asymptotic perturbation analysis valid for very weak shear to demonstrate the persistence of at least weak instabilities for small values of the shear.

In addition to its explanatory quality with regard to the earlier results in WP it is suggestive that the potential energy present in the weak shear can be tapped by the Rossby mode. Although the basic flow considered here is considerably simpler than the circulation in a complete sub-tropical gyre it is possible that the instability outlined here can provide a mechanism to maintain Rossby modes in such gyres against the inevitable effects of dissipation whose presence has often been cited as a reason for the unlikelihood for the existence of Rossby normal modes.

In section 2 I formulate the basic problem and exhibit the perturbation analysis. Section 3 is a presentation of results and section 4 is a brief discussion of the results and their significance.

2. Formulation

Consider a two layer, quasi-geostrophic model on the beta plane (Pedlosky, 1987). For simplicity we will take each of the layers to have the same basic thickness, H, in the absence of motion. Imagine a channel of width L_* , oriented north–south, as shown in Figure 1.

There is a uniform northward flow, V, in only the upper layer of the two layers. In order for such a flow to be a consistent solution of the potential vorticity equation there must be a vorticity source on the beta plane to maintain the flow. One can imagine a uniform wind stress curl being responsible for maintaining V. Small perturbations that are wave-like in y disturb the basic flow and the quasi-geostrophic potential vorticity equation is linearized in order to describe the initial evolution of the perturbation field. In *dimensionless units* the perturbation equations are:

$$\begin{aligned} (v-s)\Big[\phi_{1xx} - l^2\phi_1 + (\phi_2 - \phi_1)\Big] + \frac{1}{il}\phi_{1x} + v\phi_1 = 0 \\ (\quad - s)\Big[\phi_{2xx} - l^2\phi_2 + (\phi_1 - \phi_2)\Big] + \frac{1}{il}\phi_{2x} - v\phi_2 = 0 \end{aligned} \qquad (2.1\text{ a,b})$$

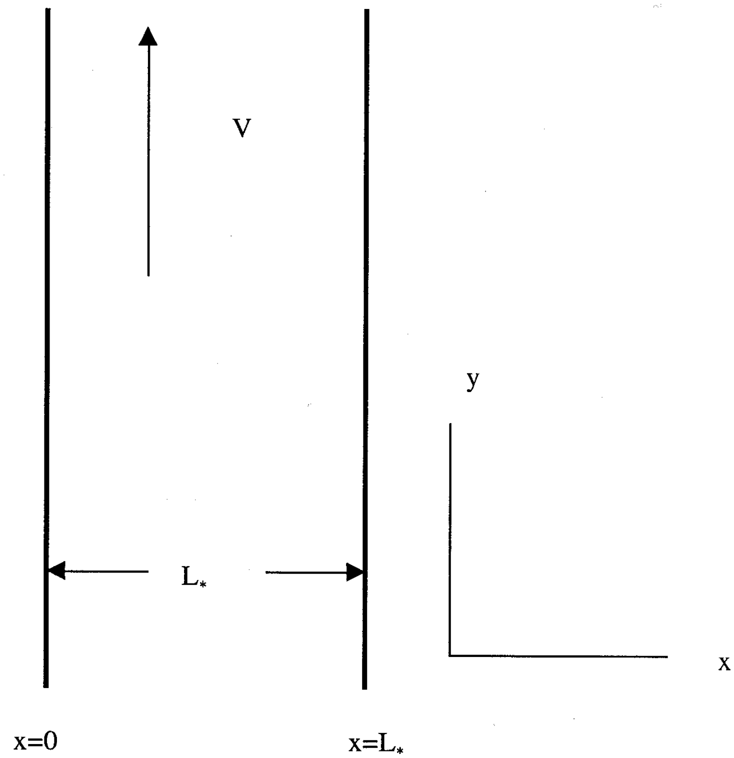

Figure 1. The meridional channel containing the flow. The upper layer basic state velocity is V and the channel width is L.

In (2.1) the cross channel coordinate x has been scaled with the deformation radius, $(g^{'}H)^{1/2}/f_o$ where the reduced gravity and Coriolis parameter are in standard notation. Subscripts x denotes derivatives with respect to x. The along channel wave number l is similarly scaled with the inverse deformation radius. Both the meridional basic state velocity and the complex phase speed of the wave perturbation, s, are scaled with the characteristic Rossby long wave speed $\beta L_D{}^2$. For details of the derivation the reader is referred to WP. If the basic state shear is weak so that $v = V/\beta L_D{}^2 << 1$ an expansion in an asymptotic series in v is suggested.

First, however, there is a suggestion from the numerical results of WP that the unstable modes, if they exist, will have scales that can be short compared to a deformation radius and wavelengths in the y-direction that are also short. This suggests that the following transformations for x, l and c are useful:

$$\xi = x / v^{1/2},$$

$$a = l v^{1/2} \qquad (2.2\text{ a,b,c})$$

$$s = v(s_o + v s_1 + ...)$$

where we have expanded the phase speed in a series in the small parameter *v,* the perturbation equations are now:

$$(1 - s_o - v s_1 + ...)\left[\phi_{1\xi\xi} - a^2 \phi_1 + v(\phi_2 - \phi_1)\right] - \frac{i}{a}\phi_{1\xi} + v\phi_1 = 0$$

$$(2.3\text{a,b})$$

$$(\quad - s_o - v s_1 + ...)\left[\phi_{2\xi\xi} - a^2 \phi_2 + v(\phi_1 - \phi_2)\right] - \frac{i}{a}\phi_{2\xi} - v\phi_2 = 0$$

Note that in these stretched coordinates the boundaries of the channel are at $\xi = 0$ and $\xi = L_* \big/ (L_D v^{1/2}) \equiv L.$

We also note that to lowest order in the shear, *v*, the two layers are decoupled. This is consistent with the results of WP which showed that unstable modes exist for $\beta L_D{}^2 / V > 1$, with strikingly different cross channel scales in the x-direction. The perturbation streamfunction is also expanded in a series in *v*, as is *a,* thus:

$$\phi_n = \phi_n{}^{(o)} + v\phi_n{}^{(1)} + ..., \ n = 1,2$$

$$(2.4)$$

$$a = a_o(1 + v\alpha + ...)$$

At lowest order this yields the problem

$$(1 - s_o)\left[\frac{d^2}{d\xi^2}\phi_1{}^{(o)} - a_o{}^2\phi_1{}^{(o)}\right] - \frac{i}{a_o}\phi_{1\xi} = 0,$$

$$(2.5\text{a,b})$$

$$(\quad - s_o)\left[\frac{d^2}{d\xi^2}\phi_2{}^{(o)} - a_o{}^2\phi_2{}^{(o)}\right] - \frac{i}{a_o}\phi_{2\xi} = 0$$

whose solutions, subject to the boundary conditions of vanishing stream function at the channel walls are:

$$\phi_1^{(o)} = A_1 e^{i\xi/[2a_o(1-s_o)]} \sin m\xi,$$
$$\phi_2^{(o)} = A_2 e^{-i\xi/[2a_o s_o]} \sin n\xi \qquad (2.6a,b)$$

where

$$m \equiv M\pi / L, \quad M = 1,2,3...$$
$$n = N\pi / L, \quad N = 1,2,3... \qquad (2.7\ a,b)$$

Since L is the channel width scaled inversely with $\nu^{1/2}$ it follows that for consistency M and N, each an integer, must be large, O$\left(\nu^{-1/2}\right)$ so that m and n are O(1) in accordance with the assumptions of the asymptotics. We note that the solutions (2.6) are each the Rossby normal modes of the upper and lower layers respectively in which a carrier wave is modulated by a solution of the Helmholtz equation (Pedlosky, 1987). The normal mode structure in the upper layer is altered only by the Doppler shift of the meridional flow in the upper layer. The solutions (2.7) reflect the fact that the horizontal structure of the mode may be quite different from one layer to the other if M and N are different. The condition that both (2.6 a,b) are solutions corresponding to the same frequency or phase speed is simply that the two solutions for s_o,

$$s_o = \frac{1}{2a_o(a_o^2 + n^2)^{1/2}} \qquad (2.8)$$

from (2.6b) and

$$s_o = 1 - \frac{1}{2a_o(a_o^2 + m^2)^{1/2}} \qquad (2.9)$$

from (2.6a), must yield identical results.

The condition that s_o be the same in (2.8) and (2.9), i.e. that we are dealing with a single composite mode, yields the condition,

$$2a_o = \frac{1}{\left(a_o^2 + m^2\right)^{1/2}} + \frac{1}{\left(a_o^2 + n^2\right)^{1/2}}. \qquad (2.10)$$

For any pair (*m,n*) there is a single solution for the (scaled) y- wave number a_o. Figure 2 shows a map in the M, N, plane of a_o for the case v =*.02*, L* =10. Although M and N are integers the figure treats the variables as continuous for graphical clarity. It is important to note that to this order the phase speed is strictly real so that instability will be apparent only at the next order in the expansion in *v*.

At next order the perturbation equations are:

$$(1-s_o)\left[\phi_{1\xi\xi}{}^{(1)} - a_o{}^2\phi_1{}^{(1)}\right] - \frac{i}{a_o}\phi_{1_\xi}{}^{(1)} = -(1-s_o)\left[\phi_2{}^{(o)} - \phi_1{}^{(o)}\right]$$

$$+s_1\left[\phi_{1\xi\xi}{}^{(o)} - a_o{}^2\phi_1{}^{(o)}\right] - \phi_1{}^{(o)} \qquad (2.11a)$$

$$+2(1-s_o)\left[\alpha a_o{}^2\phi_1{}^{(o)}\right] - \frac{i}{a_o}\alpha\phi_{1\xi}{}^{(o)}$$

$$-s_o\left[\phi_{2_{\xi\xi}}{}^{(1)} - a_o{}^2\phi_2{}^{(1)}\right] - \frac{i}{a_o}\phi_{2_\xi}{}^{(1)} = s_o(\phi_1{}^{(o)} - \phi_2{}^{(o)}) + \phi_2{}^{(o)}$$

$$+s_1\left[\phi_{2_{\xi\xi}}{}^{(o)} - a_o{}^2\phi_2{}^{(o)}\right] - 2s_o a_o{}^2\alpha\,\phi_2{}^{(o)} \qquad (2.11b)$$

$$-\frac{i}{a_o}\alpha\phi_{2\xi}{}^{(o)}.$$

It is only at this order that the coupling between the two layers enters the perturbation equations and the potential vorticity gradient associated with the vertical shear is explicitly included in the analysis. If baroclinic instability is to occur these must be essential ingredients. The instability properties will be contained in behavior of s_1 whose imaginary part will yield the growth rate for the perturbation (after multiplication by the y wave-number).

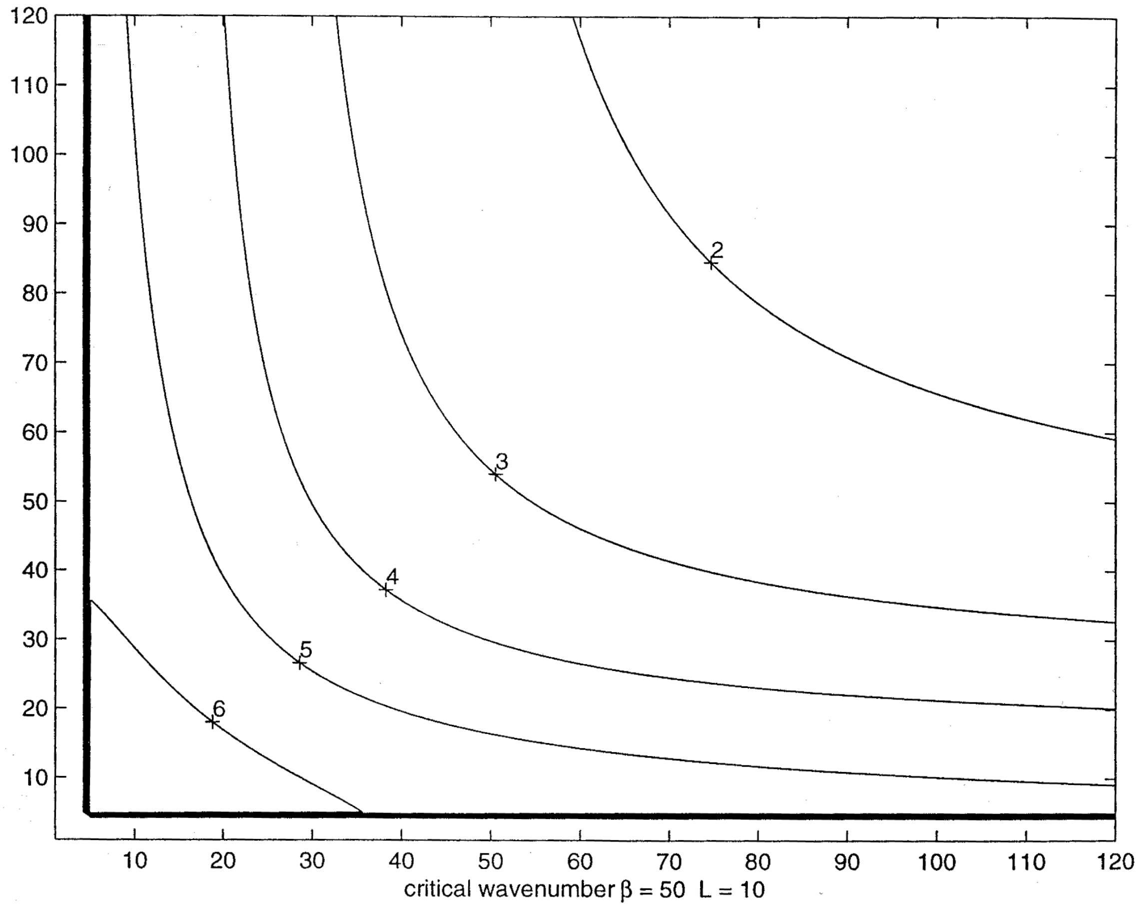

Figure 2. Contours of the critical wave number a_o in the M, N plane for $\nu = .02$, L =10.

To find s_1 it is only necessary to remove resonant terms from the right-hand sides of (2.11a,b). This is easily done by multiplying (2.11a) by the function

$$f_1 = e^{-i\xi/(2a_o[1-s_o])} \sin m\xi$$

which has the form of the complex conjugate of the normal mode of the upper layer, and multiplying (2.11b) by

$$f_2 = e^{i\xi/(2a_o s_o)} \sin n\xi$$

and integrating over the width of the channel.

After some algebra this leads to two equations relating the amplitudes of the normal modes of the two layers, viz.:

$$A_2 = A_1 \frac{\left[-\frac{s_o}{2} - \frac{s_1}{4a_o^2(1-s_o)^2} + \frac{1}{2}(1-s_o)\alpha\left(4a_o^2 + 2m^2\right)\right] L}{\gamma_{12}(1-s_o)} \qquad (2.12a)$$

and

$$A_1 = A_2 \frac{\left[\frac{s_1}{4a_o^2 s_o^2} - \frac{(1-s_o)}{2} + \frac{s_o \alpha}{2}(4a_o^2 + 2n^2)\right] L}{\gamma_{12}^{*} s_o} \qquad (2.12b)$$

where the coupling constant γ_{12} is given by:

$$\gamma_{12} = -i \frac{mn(k_m + k_n)\left[e^{-i(k_m+k_n)L}(-1)^{M+N} - 1\right]}{\left[(m-n)^2 - (k_m+k_n)^2\right]\left[(m+n)^2 - (k_m+k_n)^2\right]} \qquad (2.13)$$

where:

$$k_m = \left(m^2 + a_o^2\right)^{1/2}$$

$$k_n = \left(n^2 + a_o^2\right)^{1/2}.$$

Note that the asterisk in (2.12b) denotes the complex conjugate of the coupling coefficient γ_{12}.

Eliminating A_1 and A_2 between (2.12 a,b) leads to a quadratic equation for s_1.

$$s_1^2 + s_1 B + C = 0,$$

$$B = 2a_o^2 s_o^2 (1-s_o)^2 \left\{ \begin{array}{l} \frac{1}{s_o} - \frac{1}{(1-s_o)} + \\ \frac{2s_o}{(1-s_o)^2}\alpha(2a_o^2 + n^2) - \frac{2(1-s_o)}{s_o^2}\alpha(2a_o^2 + m^2) \end{array} \right\} \qquad (2.14\ a,b,c)$$

$$C = |\gamma_{12}|^2\, 16 s_o^3 (1-s_o)^3 / L^2$$
$$-4a_o^4 s_o^2 (1-s_o)^2 \left[(1-s_o) - 2s_o\alpha(2a_o^2 + n^2)\right] \bullet \left[s_o - 2(1-s_o)\alpha(2a_o^2 + m^2)\right]$$

whose solution determines the stability of the mode whose structure to first-order is given by (2.6 a,b).

3. Results

In terms of our original non-dimensional variables the growth rate of an unstable mode is given by lc_i where c_i is the imaginary part of the wave's phase speed. We have already noted that to lowest order in our expansion in powers of v the phase speed is real and an imaginary part, if it exists, will depend on the imaginary part of s_1. It then follows that the growth rate σ will be

$$\sigma = lc_i = av^{3/2} imag(s_1) \tag{3.1}$$

where the scaled wave number a is itself given by the expansion (2.4) so that the growth rate is a function of α. In the following figures the growth rate is given in terms of its original variables, i.e. l and in terms of v or its inverse $\tilde{\beta}=1/v$.

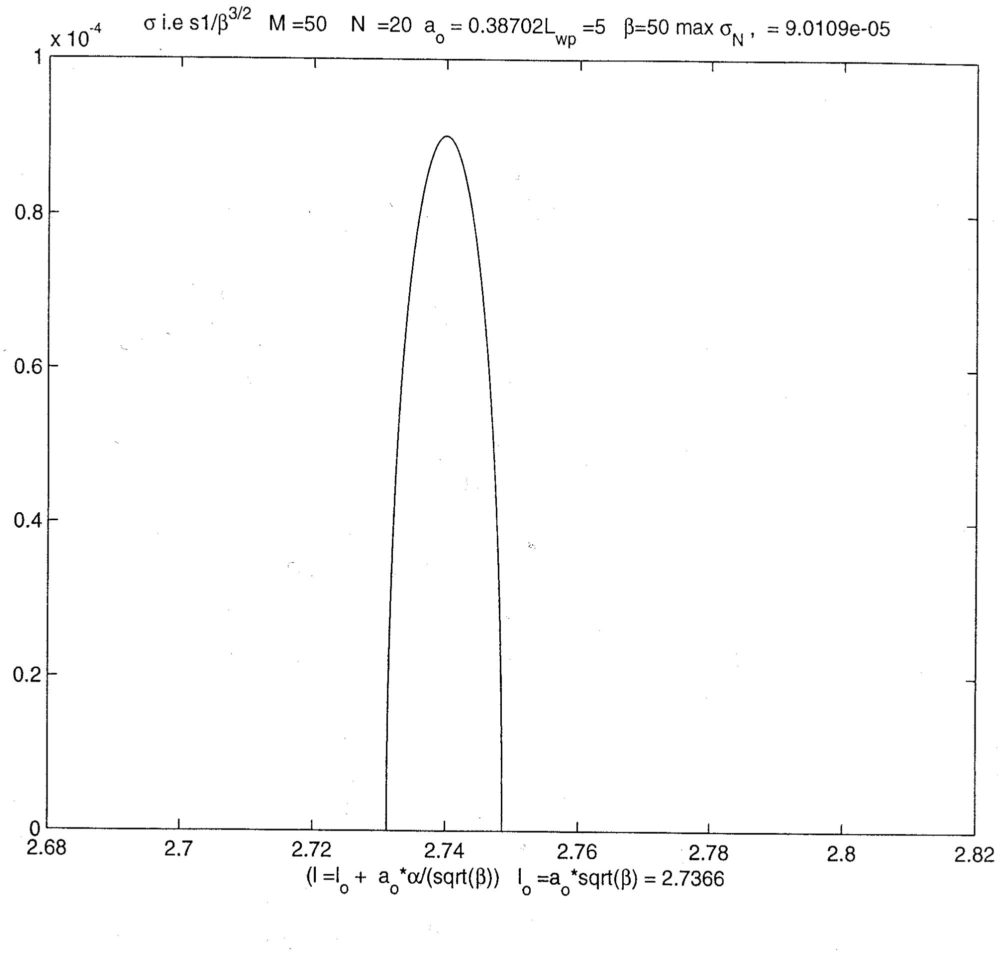

Figure 3. Growth rate versus wave number for v =.02, M = 50, N = 20, L = 5. Note the narrow window in l for which $\sigma_I > 0$.

Figure 3 shows the growth rate as a function of l for v=.02($\tilde{\beta} = 50$) for the mode corresponding to M=50 and N =20 for a channel of width 5 L_D. There is a narrow range of wavenumbers in which the growth rate is positive and the magnitude of the growth rate is quite small, for these values of the order of 10^{-4} (in units βL_D). There is a symmetry in M,N so that the mode with M and N reversed has the same growth rate. Any combination (M,N) will have some range over which its growth rate will be non zero so that a plot of growth rate versus l for small v would show a bewildering superposition of intervals of unstable wave numbers. This is precisely what WP found numerically but the smallness of the growth rates precluded a clear picture for v less than 0.5. For example, the growth rate curve for M = 25, N = 25 is shown in Figure 4. The maximum growth rate is slightly larger than in the previous example. Indeed, as shown below, the maximum corresponds to the diagonal solution M=N but the differences are not great enough to truly privilege that mode. Note too that the interval of wave number is different than the previous example. As WP found, these modes are unstable for values of l which exceed the classical short wave cut-off of $2^{1/4}$.

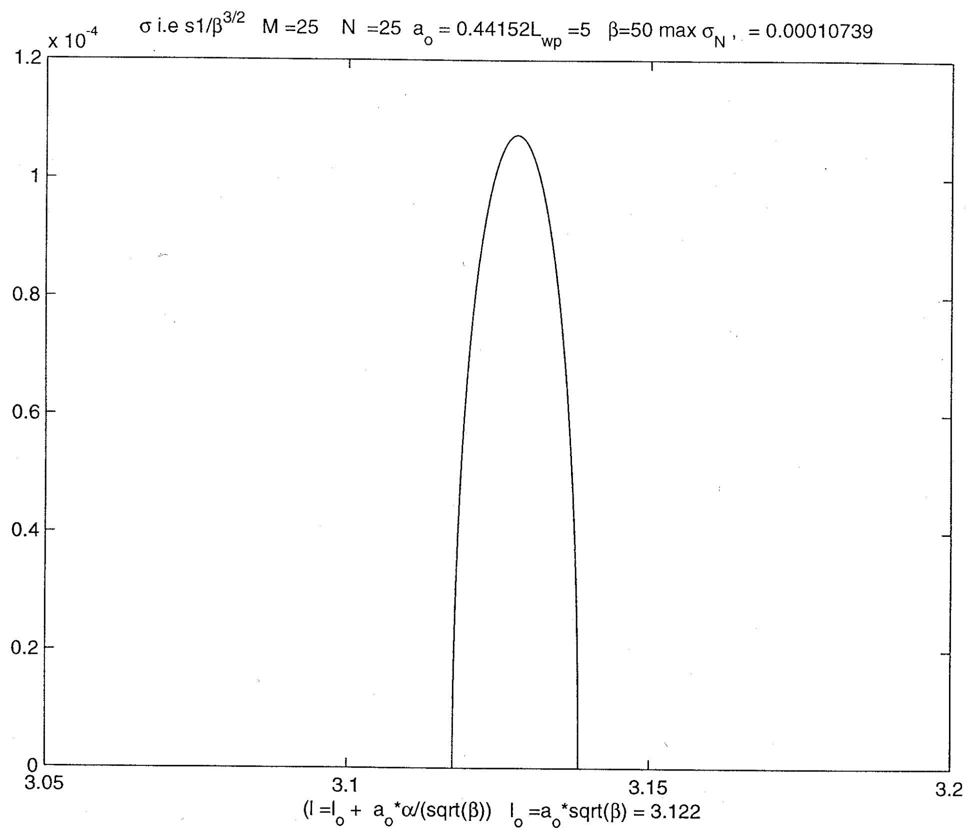

Figure 4. As in Figure 3 except for M =25, N = 25. Note the change in the l interval of instability.

An alternative representation represents the maximum growth rate for each M,N in the M,N plane. One can identify the corresponding y-wavenumber l with the aid of Figure 2. Figure 5 shows just such a contour plot. The maximum occurs very close to the values chosen for Figure 4 and the maximum value is about 1.2×10^{-4} and occurs on the M,N diagonal. Note that on the diagonal $s_o = 0.5$, a result that could be anticipated from symmetry, so that for such diagonal modes the structure of the eigenfunctions is, to lowest order, the same in each layer. There will be a departure at higher order and the off-diagonal unstable modes, whose growth rates are commensurate with the diagonal modes will have very different structures in two layers.

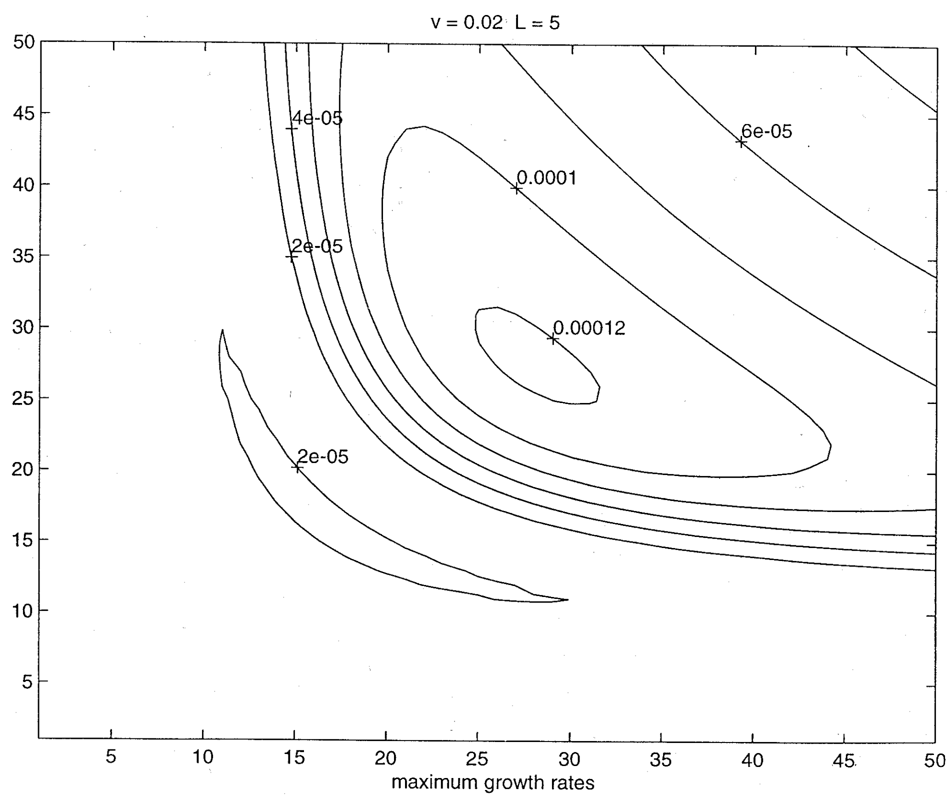

Figure 5. Contours of growth rate in the M–N plane. Note that the maximum occurs along the line M = N.

4. Discussion

The examination of the instability of meridional baroclinic shear flows produces some novel instability characteristics when compared with the classical problem of the instability of zonal flows. Perhaps none is more surprising than the absence of a critical threshold for instability for the shear even in those cases, as studied in this paper, where the geometry of the flow forces the disturbance to sense the effect of β. In spite of considerable effort a necessary condition for instability has not been proven for the system (2.1) and although a negative is not a proof it was thought to suggest that all meridional flows would be unstable. That interesting hypothesis is put on firmer ground by the asymptotic result of this paper which shows that very weak meridional shears can destabilize otherwise neutral Rossby normal modes for the channel. The preexisting mode has its structure slightly altered by the shear allowing the release of the available potential energy in the shear flow. Since the shears we have considered are weak the corresponding energy source is rather feeble and the resulting growth rates are small. One could stretch the asymptotics to values of v which are not very small to obtain larger growth rates but that range is already covered by the detailed numerical analysis of WP which qualitatively agrees with the present results. From the point of view of oceanic applications that may be unnecessary. Although the geometry of the channel is simple compared to the geometry of the subtropical gyre it is still true that instabilities of the flow in the eastern regions of the gyre can be interpreted in terms of the instability of meridional flow (see for example Spall, 2000) and emphasizes the important role of zonal boundaries (here taken as simple meridians) in affecting the variability of the mid-basin flow. This leads to the interesting possibility that instabilities of the type described in this paper can extract sufficient energy from the gyre circulation to maintain Rossby normal modes against dissipation as long as the weak growth rates pertinent to such modes exceed the dissipation rates for such modes. Cessi and Primeau (2001) and LaCasce (2000) have already pointed out the existence of such special Rossby normal modes with exceedingly long dissipative times under the influence of scale selective dissipation. The coupled instability of the Rossby normal modes and the baroclinic shear could provide a mechanism for the generation and maintenance of the normal modes rather that relying on the persistent "ringing" of the system by repeated external forcing.

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References

Cessi, P. and F. Primeau, 2001. Dissipative selection of low-frequency modes in a reduced gravity basin. *Journal of Physical Oceanography*, **31**, 127–137.

LaCasce, J. H., 2000. Baroclinic Rossby waves in a square basin. *Journal of Physical Oceanography*, **30**, 3161–3178.

Pedlosky, J., 1987. *Geophysical Fluid Dynamics*. Springer-Verlag. New York, pp. 710.

Spall, M., 2000. Generation of strong mesoscale eddies by weak ocean gyres. *Journal of Marine Research,* **58**, 97–116

Walker, A. and J. Pedlosky, 2002. On the instability of meridional baroclinic currents. *Journal of Physical Oceanography* (to appear).