A Two-Layer Model of the Abyssal Circulation

by

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Abstract

A two-layer Fofonoff model is used to model source-sink flow on the beta-plane as an idealized representation of deep abyssal flow generated by input from both northern and southern oceanic sources on the western boundary of the oceanic basin. The focus of the study is the manner in which boundary layer flow on the western boundary can be sustained in the absence of forced westward interior flow, a requirement of Greenspan's theorem for largely inertial dynamics.

Depending on the location of the source the circulation can generate the needed westward interior flow to support an inertial boundary current by either first penetrating into the interior before joining the boundary current, or it can generate a recirculation in the layer which has the required westward flow to support the boundary current.

1. Introduction

The oceanic meridional overturning circulation is a key component of the Earth's climate system and key dynamic elements of the circulation can be masked by graphical representations of the circulation as a simple overturning flow in latitude and depth ignoring the 3-dimensional nature of the flow. In particular, the western boundary current is often thought of as a passive pipe flow connecting sources in the polar regions with sinks either at the equator in the opposite hemisphere as in the classical model of Stommel and Aarons (1960). For dynamics that are not fundamentally viscous the existence of such western boundary currents requires a sustaining interior flow directed westward (Greenspan, 1962). That presents a problem when there is no external forcing, as in the thermocline level, wind forced circulation above the abyss. Indeed, recent observations (Bower *et. al.*) suggest the flow does not proceed simply along the western boundary but intrudes into the interior before turning to feed the western boundary current on its journey southward.

A simple model of that process has been suggested (Pedlosky, 2018 hitherto P18). That model uses the simple Fofonoff model Fofonoff (1954) which Bretherton and Haidvogel (1976) have suggested is the asymptotic dynamical equation for the fluid whose eddy field has minimized its enstrophy as a consequence of the 2-dimensional cascade of enstrophy to small enough scales where dissipation will expunge it. Their suggestions; that the result of a strong eddy field on the circulation can be modeled by the Fofonoff model, although they show is not completely accurate, presents a temptingly simple tool to suggest the resulting mean circulation. In P18 that suggestion was exploited to demonstrate the excursion of source driven flows in a single layer model of the abyss. The flow from source on the northwest corner of the basin to sink at its southwest corner naturally intruded into the interior before turning westward and building the western boundary current connecting source to sink.

One deficiency of the model in P18 is that it is limited to a single homogeneous layer of fluid. In order to include vertical variations at least a two-layer model would be required. In particular, we know that the overturning circulation is forced by sinking in the North Atlantic of North Atlantic Deep Water (NADW) while at the same time Antarctic Bottom Water (AABW) flows northward and greater depths. In the following development in section 2, the equations for the two-layer Fofonoff model are presented and discussed while solutions are found in section 3 for sources and sinks at different levels issuing from the same location and solutions for sources in the north in one layer and a source in the south in the second layer. A quasi-geostrophic (qg) model is utilized (Pedlosky 1987).

2. The model

The basic presumption of the Fofonoff model is that solutions of the steady, frictionless quasi-geostrophic potential vorticity equation can be easily found *if* the potential vorticity is a linear function of the geostrophic streamfunction. If Q_n is the potential vorticity in the nth layer and ψ_n is the geostrophic stream function in that layer, then the relation

$$Q_n = a_n^{\,2}\psi_n, \tag{2.1}$$

automatically satisfies the quasi-geostrophic potential vorticity equation for steady, frictionless flow. The index n will refer to each of the two layers with n =1 referring to the upper layer and n =2 the lower layer. In the standard qg model this implies that

$$q_n = \nabla^2\psi_n + (-1)^n F_n(\psi_1 - \psi_2) + \beta y = a_n^{\,2}\psi_n,\;\; n = 1,2 \tag{2.2}$$

The first term on the after the first equality sign in (2.2) is the relative vorticity, represented by the Laplacian of the streamfunction and involves second derivative in x and y. The coordinate x measures distance in the east-west direction while y measures distance to the north. The basin is rectangular with a north- south distance L and an east-west length Lx_e where x_e is non dimensional. The term βy in (2.2) represents the meridionally variable part of the planetary vorticity.

The parameters $F_n = \dfrac{f_o^{\,2}}{g' H_n}$, where f_o is the constant part of the planetary vorticity, g' is the reduced gravity and H_n is the constant mean thickness of the nth layer. The parameter $a_n^{\,2}$ is a positive but different constant for each layer. The positivity of those two constants ensures that the solutions found will be stable to finite amplitude perturbations (Arnol'd, 1965). Note that the $a_n^{\,2}$ while restricted to being positive, are otherwise arbitrary. In P18 these constants were related to the source strength and we will do something similar here. First, though, it will be useful to non-dimensionalize the equations. Using L to scale both x and y, and S_o to scale the streamfunction in each layer, (2.2) becomes

$$\nabla^2\psi_n + (-1)^n F_n(\psi_1 - \psi_2) + (\beta L^3 / S_o)y = a_n{}^2\psi_n, \quad n = 1,2 \qquad \text{(2.3 a,b)}$$

where now the constants $a_n{}^2 = a*_n{}^2\, L^2$ where the asterisk refers to the dimensional form of the variable. In (2.3) we expect the beta term to be important in the dynamics as evidenced in P18 and that the term on the right hand side specifying the dependence of the potential vorticity on the streamfunction to be similarly important. We therefor choose a balance the two terms in the upper layer such that,

$$S_o = \beta L^3 / a_1{}^2 = \beta L / a*_1{}^2 \qquad \text{(2.4)}$$

The parameters measuring the stratification, i.e. the ratio of the basin scale to the deformation radius are now non-dimensional , $F_n = f_o{}^2 L^2 / g' H_n$, so that the final form of (2.3) becomes

$$\begin{aligned} &\nabla^2\psi_1 + F_1(\psi_1 - \psi_2) + a_1{}^2 y = a_1{}^2\psi_1, \\ &\nabla^2\psi_2 + F_2(\psi_2 - \psi_1) + a_1{}^2 y = a_1{}^2\psi_2, \end{aligned} \qquad \text{(2.5 a,b)}$$

The boundary conditions for (2.5 a,b) will be the vanishing of the streamfunction on the northern, eastern and southern boundary. Sources of fluid emanating from the northwest and/or southwest corners of the basin are modeled by specifying constant, non zero values of $\psi_n = -S_n$ on he western boundary at $x = 0$. If the S_n are positive they represent a source flowing southward from the northwest boundary. If a source is negative it models a source emanating from a southern corner flowing northward.

For large values of the constants $a_n{}^2$ the solutions are of boundary layer type. It is first useful to determine what the interior solutions, Ψ_n, of (2.5) are outside the boundary layers. To find those solutions we merely ignore the relative vorticity, i.e. the Laplacian term. We easily obtain,

$$\Psi_1 = \frac{a_1{}^2 y\left[a_2{}^2 + F_1 + F_2\right]}{\left[a_1{}^2 a_2{}^2 + a_1{}^2 F_1 + a_2{}^2 F_2\right]}, \qquad \Psi_2 = \frac{a_1{}^2 y\left[a_1{}^2 + F_1 + F_2\right]}{\left[a_1{}^2 a_2{}^2 + a_1{}^2 F_1 + a_2{}^2 F_2\right]}, \qquad (2.7\text{ a,b})$$

representing a zonal flow, independent of x and y in each layer, moving westward.

The ratio of those velocities in the two layers is just a function of the $a_n{}^2$ and F_n. Typically, the *smaller* the ratio $a_1{}^2 / a_2{}^2$, the *larger* the ratio of the interior flow in the upper layer is to the flow in the lower layer. In P18, in the large $a_n{}^2$ limit, the flow emerges from the sources in the basin corner and flows to the eastern boundary in the boundary layers and only then to the western boundary and so it was evident that the source strength, in that limit, should match the total westward transport in the interior. That allowed a simple relationship between the source strength and the Fofonoff constant. Similarly in this problem the non-dimensional transports in each layer should be related to the total westward transport in each layer and that nondimensional value would equal S_n yielding the equalities,

$$S_1 = \frac{a_1^2\left[a_2^2 + F_1 + F_2\right]}{\left[a_1^2 a_2^2 + a_1^2 F_1 + a_2^2 F_2\right]}, \qquad S_2 = \frac{a_1^2\left[a_1^2 + F_1 + F_2\right]}{\left[a_1^2 a_2^2 + a_1^2 F_1 + a_2^2 F_2\right]}, \qquad (2.8\text{ a,b})$$

We can then, in principle, use (2.8 a, b) to solve for the $a_n{}^2$ in terms of the S_n. That is a difficult and messy task. Instead, I will take the inverse of that by specifying the $a_n{}^2$ (and the F_n) which then directly yield the S_n.

The solution to (2.5 a,b) subject to the boundary conditions that the streamfunction vanish on the northern, southern and eastern boundaries will be found in the form

$$\psi_n = \sum_{j=1}^{N} \psi_{jn}(x) \sin j\pi y \ , \text{n=1,2} \qquad (2.9)$$

where N is chosen larger enough (typically N =100). The problem for the ψ_{jn} becomes,

$$\begin{aligned} \frac{d^2\psi_{1j}}{dx^2} - K_{1j}^2\psi_{1j} &= -F_1\psi_{2j} + 2a_1^2(-1)^j / j\pi, \\ \frac{d^2\psi_{2j}}{dx^2} - K_{2j}^2\psi_{2j} &= -F_2\psi_{1j} + 2a_1^2(-1)^j / j\pi \end{aligned} \qquad (2.10 \text{ a,b})$$

where $K^2{}_{jn} = j^2\pi^2 + F_n + a^2{}_n$. It is convenient to write the solution in terms of an x independent portion that balances the final term on the right hand side of (2.10 a,b) plus a homogeneous solution, viz.,

$$\psi_{j1} = -\frac{2a_1^{\,2}(-1)^j}{j\pi}\left(\frac{K^2_{\ 2j}+F_1}{K^2_{\ 1j}K^2_{\ 2j}-F_1F_2}\right)+A_j\frac{\sinh\alpha_j x}{\sinh\alpha_j x_e}+B_j\frac{\sinh\alpha_j\left[x-x_e\right]}{\sinh\alpha_j x_e}$$
$$+C_j\frac{\sinh\beta_j x}{\sinh\beta_j x_e}+D_j\frac{\sinh\beta_j\left[x-x_e\right]}{\sinh\beta_j x_e},$$

$$\psi_{j2} = -\frac{2a_1^{\,2}(-1)^j}{j\pi}\left(\frac{K^2_{\ 1j}+F_2}{K^2_{\ 1j}K^2_{\ 2j}-F_1F_2}\right)-A_j\frac{(\alpha_j^{\,2}-K_{1j}^{\ 2})}{F_1}\frac{\sinh\alpha_j x}{\sinh\alpha_j x_e}-B_j\frac{(\alpha_j^{\,2}-K_{1j}^{\ 2})}{F_1}\frac{\sinh\alpha_j\left[x-x_e\right]}{\sinh\alpha_j x_e} \qquad (2.11\text{ a,b})$$
$$-C_j\frac{(\beta_j^{\,2}-K_{1j}^{\ 2})}{F_1}\frac{\sinh\beta_j x}{\sinh\beta_j x_e}-D_j\frac{(\beta_j^{\,2}-K_{1j}^{\ 2})}{F_1}\frac{\sinh\beta_j\left[x-x_e\right]}{\sinh\beta_j x_e},$$

In the above equation

$$\alpha_j^{\,2} = \frac{(K_{1j}^{\ 2}+K_{2j}^{\ 2})}{2}+\frac{1}{2}\left[(K_{1j}^{\ 2}-K_{2j}^{\ 2})^2+4F_1F_2\right]^{1/2},$$
$$\beta_j^{\,2} = \frac{(K_{1j}^{\ 2}+K_{2j}^{\ 2})}{2}-\frac{1}{2}\left[(K_{1j}^{\ 2}-K_{2j}^{\ 2})^2+4F_1F_2\right]^{1/2} \qquad (2.12\text{ a,b})$$

In all cases the constants α_j^2 and β_j^2 are positive. Satisfying the homogeneous boundary conditions on the streamfunction in both layers on $x = x_e$, and the condition that the streamfunction matches S_1 and S_2 on x =0 determines the constants A_j, B_j, C_j, D_j in (2.11) a,b and those relations are given in Appendix A.

To calculate the circulation pattern, a_1^2 and a_2^2 must be given from which the Source strengths are calculated using (2.8 a,b). The stratification parameters F_1, F_2 are also given. A range of values has been used and typical examples will be discussed in the next section. For positive values of the sources S_1, S_2 the flow emanates from the northwest corner of the basin and exits through the southwest corner. We can also choose a source strength to be negative which implies that for that layer the source is at the southwest corner and the sink at the northwest corner. Recall that the scale for the source strength by which the nondimensional strengths S_1 and S_2 must be multiplied is

$$S_0 = \frac{\beta_* L^3}{a_1^2} .$$

3. Results

Figure 1 shows a typical case where the source in both layers is in the northwest corner with the absorbing sink in the southwest corner. The flow predicted by the theory is typical for such arrangements. Some of the streamfunction values are given to show the flow direction. The flow in each layer shown in panels 1a and 1 b enters the basin in the northwest corner and rather than proceeding directly down the western boundary, flows into the interior of the basin and then sweeps clockwise to form a anti-cyclonic gyre that feeds a western boundary current as it then flows southward to the sink. The source-sink flow generates its own westward interior flow as in P18 to satisfy Greenspan's theorem allowing a western boundary current south of the source that gradually increases in strength. The flow strength and direction differ somewhat from layer to layer and figure 1c shows the difference, $\psi_2 - \psi_1$ which is proportional to the deformation of the interface between the two layers. It is suggestively similar to the circulation observed by Bower *et al.*(2009) insofar as the southward flow is not limited to the western boundary current. There is a weak western boundary current near the source that increases in strength with distance from the source southward. Mathematically this occurs because the interior streamfunction near $y = 1$ is, by construction, close to matching the boundary condition on $x = 0$ in each layer leaving little discrepancy to be filled by the western boundary current and this is true for both layers by construction.

The situation becomes radically different if the boundary condition in the lower layer is altered. If the source in the lower layer is in the *southwest* corner and the boundary current flows northward then

$$S_1 = \frac{a_1^2\left[a_2^2 + F_1 + F_2\right]}{\left[a_1^2 a_2^2 + a_1^2 F_1 + a_2^2 F_2\right]},$$
$$S_2 = -\frac{a_1^2\left[a_1^2 + F_1 + F_2\right]}{\left[a_1^2 a_2^2 + a_1^2 F_1 + a_2^2 F_2\right]}, \qquad (3.1a,b)$$

So that the value of the streamfunction on the boundary in lower layer is negative to indicate flow *entering* the lower layer in the southwest corner to flow northward. The solution in this case is shown in Figure 2.The flow in the lower layer is entirely different in structure. The mass flux in the source now flows northward entirely in the western boundary current with no excursion for the boundary layer flow into the interior. This is typical for all parameter values examined. To satisfy Greenspan's theorem an interior *recirculation* in the lower layer has been produced that at all latitudes has, near the western boundary, a westward component as required. Recall that the governing equations we are employing are the result of an enstrophy cascade by eddies and it is natural to consider the recirculation as the physical consequence of the associated eddy field. Mathematically, our model has produced this difference due to the mismatch between the interior flow solution (2.7b) and the boundary value (3.1a,b) of the streamfunction.

3. Discussion

The Fofonoff model, as shown by Bretherton and Haidvogel (1976), is suggested as the governing equations for the mean flow of a turbulent fluid on the beta plane whose enstrophy has been minimized by the turbulent cascade of entropy to high wave numbers where it is dissipated. The simplicity of the model allows analytical solutions of complex flows that would otherwise require complex numerical calculations. In the case we have examined the equations have been used to model source sink flow on the beta plane as a simple representation of the deep flow in an eddy rich ocean for parameter values for which the dynamics of the western boundary current is largely inertial rather than viscous. The constraint of Greenspan's theorem manifests itself in a twofold fashion. In one case, as when both layers have sources in the northwest region of the deep basin, the resulting flow intrudes into the interior of the basin rather than flowing directly from those sources to their sinks in the southwest corner of the basin. This is reminiscent of the observations of floats placed in the North Atlantic Deep Water entering the Grand Banks region that shows an analogous behavior. When, instead, the lower layer had its source in the southwest corner and flows northward, the boundary current remains attached to the western boundary but the model does so by producing a basin-wide clockwise recirculation that provides the required interior westward flow near the western boundary.

The sources have been placed in all cases on the western boundary to emphasize the fact that the inertial western boundary current, must for fundamental dynamical reason, be more than a passive north-south pipe flow. Either the boundary current itself must flow into the interior to provide the sustaining interior westward flow or, as in the case of the southern source, a interior recirculation must provide that requirement. Consistent with the interpretation of the Fofonoff model as the end state of an eddy driven flow we identify the recirculation in Figure 2 as eddy driven.

There is an element of arbitrariness in model that is a weakness. In principle, the beta term in (2.5 a,b) is determined up to constant that βy could be replaced by $\beta(y-1)$ in which case the roles of the sources in the north and south could be switched and the resulting behavior altered so that it is the southern source that leads in interior intrusions and the northern source in the lower layer that requires the recirculation. Indeed, the beta term could be replaced by any term of the form $\beta(y-y_r)$ generating a family of solutions.

I have chosen the current arrangement as being the more suggestively realistic one. In particular, it has the virtue of emphasizing that the abyssal circulation, i.e. the lower branch of the overturning circulation is a 3-dimensional flow with all the richness normally to be expected. When the source in the lower layer is from the south it is obviously not a model of flow entering the basin from the southern hemisphere since the qg model does not include cross equatorial dynamics. It is instead an illustration of an alternative western boundary current driven by a source that does *not* penetrate into the interior but is instead sustained by a interior eddy recirculation that provides the needed western flow in the interior at the western boundary.

Appendix A

Coefficients in solution (2.11 a,b)

$$D_j = \frac{2a_1^2(-1)^j}{j\pi}\left[\frac{K_{1j}^2+F_2}{K_{1j}^2K_{2j}^2-F_1F_2}F_1+\frac{(\alpha_j^2-K_{1j}^2)(K_{2j}^2+F_{1})}{K_{1j}^2K_{2j}^2-F_1F_2}\right]\frac{1}{\left(\beta_j^2-\alpha_j^2\right)}+ \tag{A.1}$$

$$+(-1)^j F_1 S_2\frac{(1-(-1)^j)}{j\pi\left(\beta_j^2-\alpha_j^2\right)}+S_1\frac{(\alpha_j^2-K_{1j}^2)\,(1-(-1)^j)}{j\pi\left(\beta_j^2-\alpha_j^2\right)}$$

$$B_j = \frac{-2a_1^2(-1)^j}{j\pi}\left[\frac{K_{2j}^2+F_1}{K_{1j}^2K_{2j}^2-F_1F_2}\right]-2S_1\frac{(1-(-1)^j)}{j\pi}-D_j, \tag{A.2}$$

$$A_j = \left[\frac{(\beta_j^2-K_{1j}^2)}{F_1}\frac{(K_{2j}^2+F_2)}{K_{1j}^2K_{2j}^2-F_1F_2}+\frac{(K_{1j}^2+F_2)}{K_{1j}^2K_{2j}^2-F_1F_2}\right]\frac{1}{(\beta_j^2-\alpha_j^2)} \tag{A.3}$$

$$C_j = -A_j+\frac{2a_1^2(-1)^j}{j\pi}\frac{\left[K_{2j}^2+F_1\right]}{(K_{1j}^2K_{2j}^2-F_1F_2)} \tag{A.4}$$

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Figure Captions

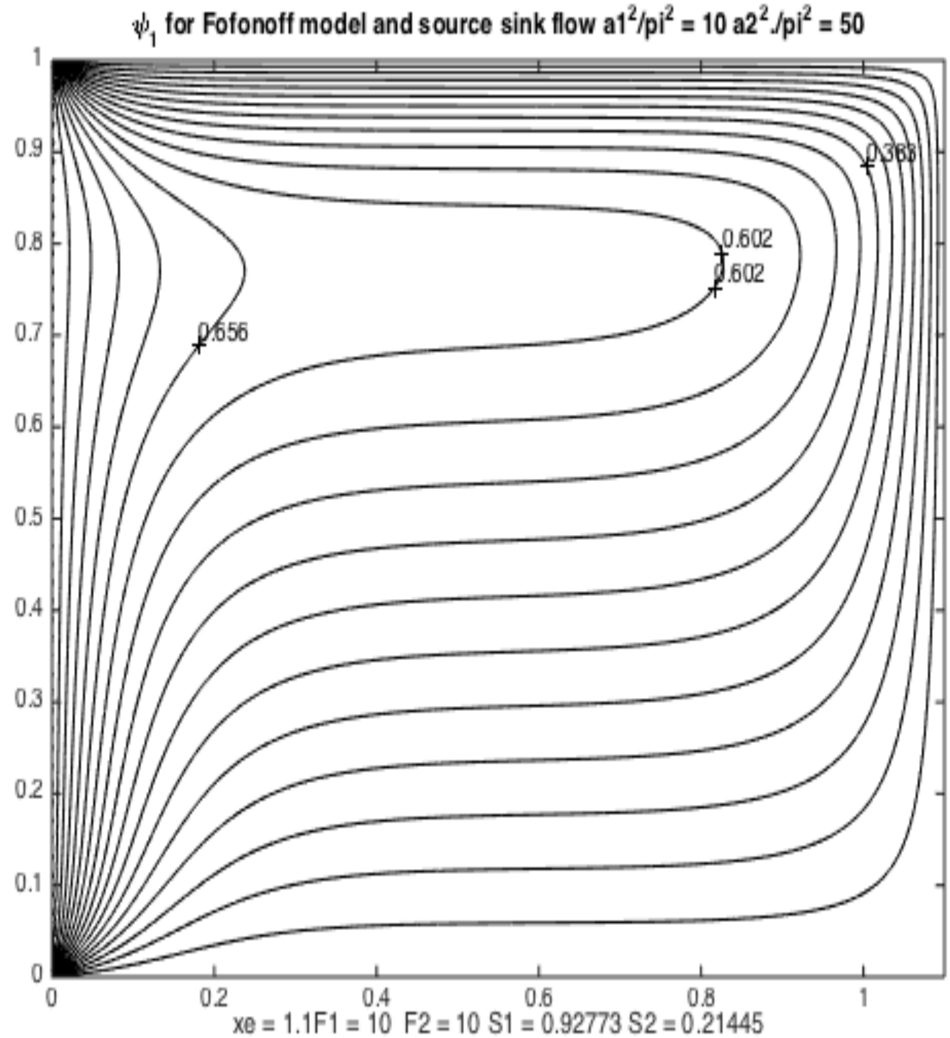
Figure 1. The circulation with a source in the northwest corner and a sink in the southwest corner, $a_1^2 = 10\pi^2, a_2^2 = 50\pi^2, S_1 = 0.92773, S_2 = 0.21445$ $F_1 = 10, F_2 = 10$.

a) ψ_1 **b)** ψ_2 **and c)** $\psi_2 - \psi_1$ **.**

Figure 2. For the same parameter values as in Figure 1 except that S_2 is negative indicating northward flow of the source on the western boundary. Note that now the boundary current flows at full strength from the southern source to the north and is supported at each latitude by an interior flow that is a pure recirculation.

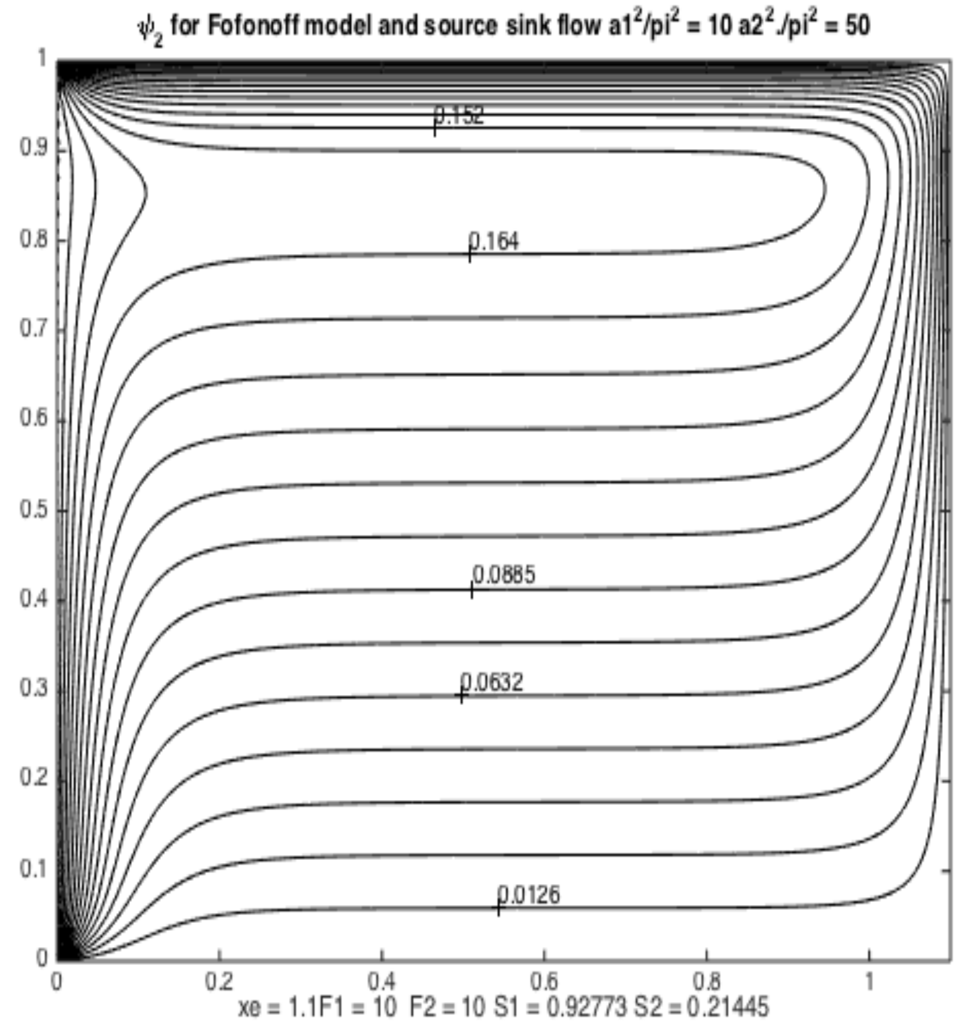
Figures

a)

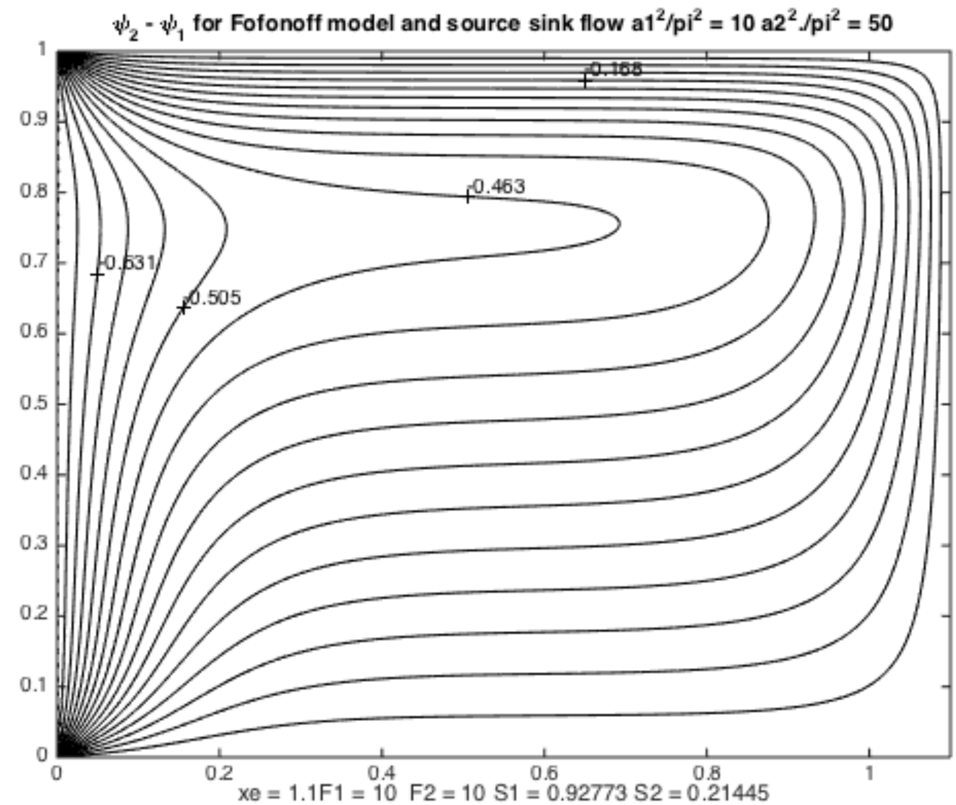


ψ1 for Fofonoff model and source sink flow a1²/pi² = 10 a2²./pi² = 50
0.363
0.602
0.602
0.656
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
xe = 1.1F1 = 10 F2 = 10 S1 = 0.92773 S2 = 0.21445

b)


ψ2 for Fofonoff model and source sink flow a1²/pi² = 10 a2². /pi² = 50
0.152
0.164
0.0885
0.0632
0.0126
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
xe = 1.1F1 = 10 F2 = 10 S1 = 0.92773 S2 = 0.21445

c)



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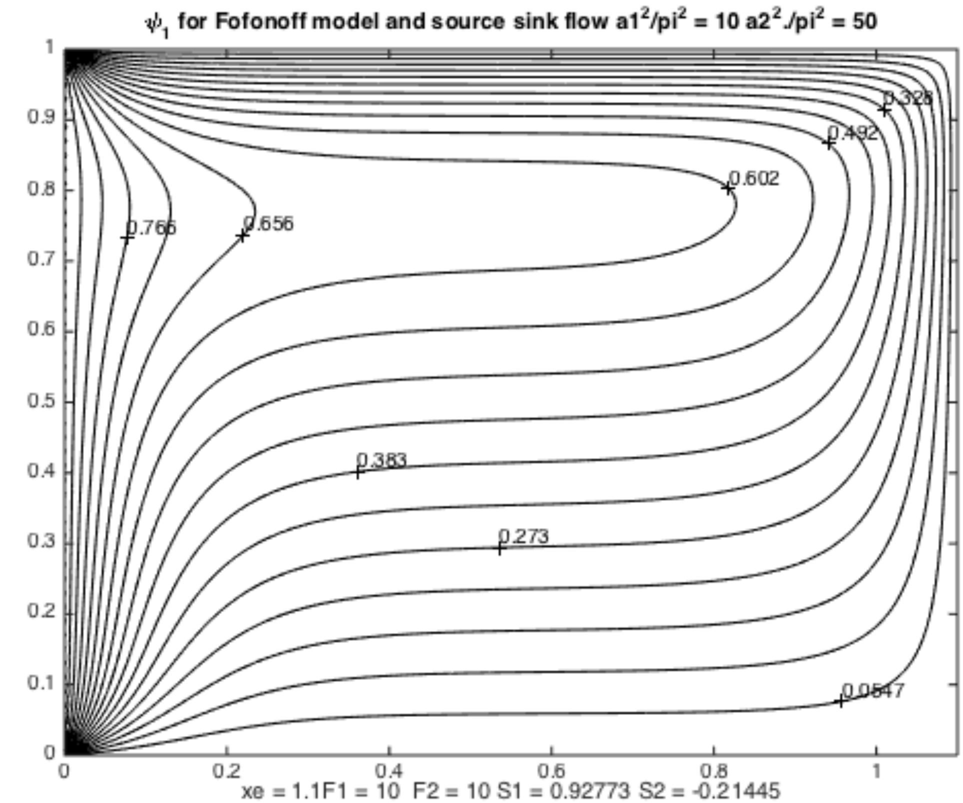
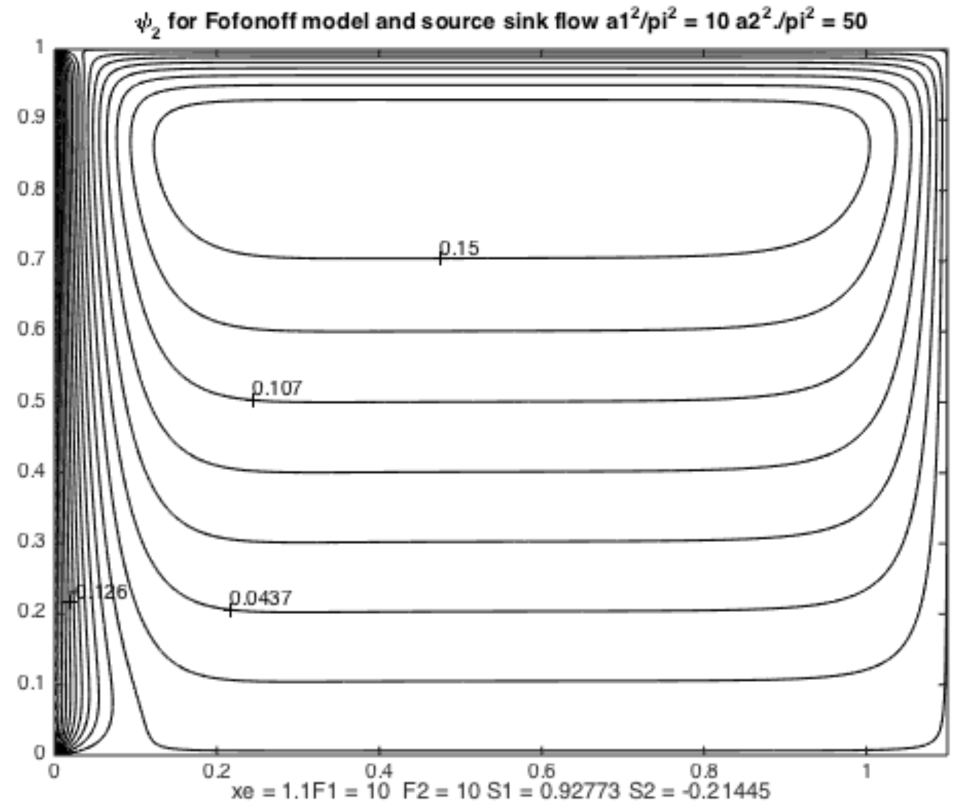
a) ψ_1 **b)** ψ_2 **and c)** $\psi_2 - \psi_1$ **.**

a)


ψ1 for Fofonoff model and source sink flow a1²/pi² = 10 a2².//pi² = 50
0.766
0.656
0.602
0.492
0.328
0.383
0.273
0.0547
xe = 1.1F1 = 10 F2 = 10 S1 = 0.92773 S2 = -0.21445

b)


ψ2 for Fofonoff model and source sink flow a1^2/pi^2 = 10 a2^2./pi^2 = 50
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.15
0.107
0.126
0.0437
0
0.2
0.4
0.6
0.8
1
xe = 1.1F1 = 10 F2 = 10 S1 = 0.92773 S2 = -0.21445

c)

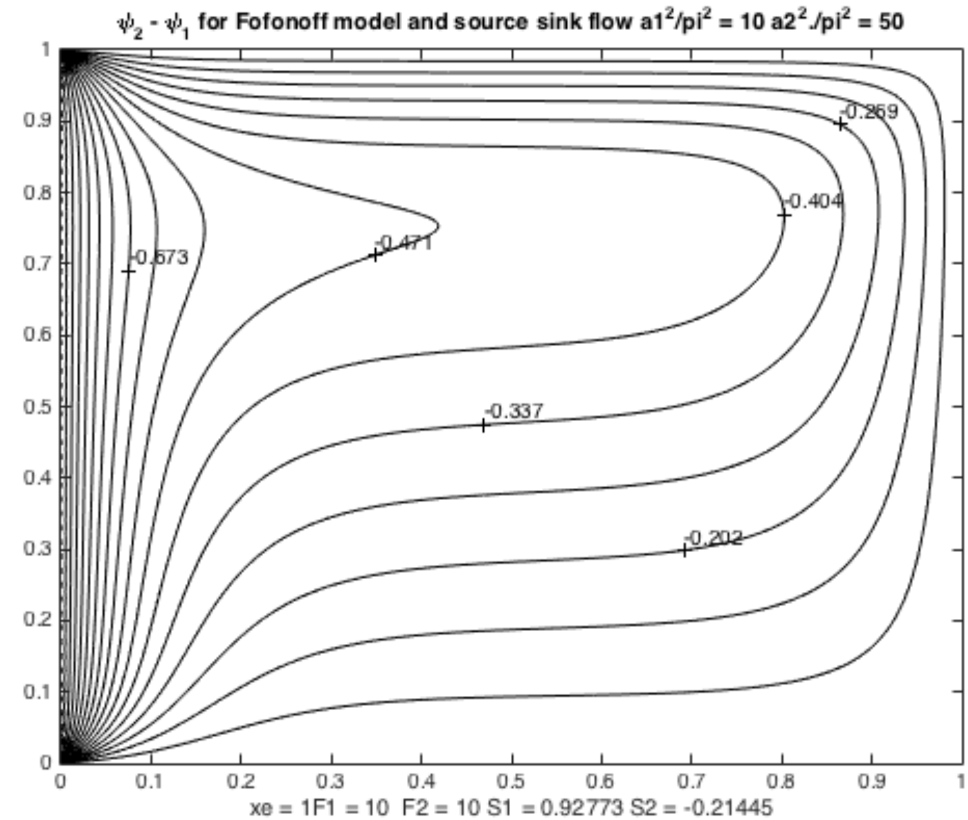


Figure 2 For the same parameter values as in Figure 1 except that S_2 is negative indicating northward flow of the source on the western boundary. Note that now the boundary current flows at full strength from the southern source to the north and is supported at each latitude by an interior flow that is a pure recirculation.