6.3 Zonal jets on a beta plane.

The ocean and atmosphere are host to a variety of eastward jet-like flows including the Gulf and Jet Streams, the Kurishio, the Pacific equatorial counter current, and jets associated with frontal systems in the Antarctic Circumpolar current. It is natural to ask whether these might be subject to hydraulic effects associated with the Rossby waves, or more general potential vorticity waves, that arise and that otherwise propagate westward. Hydraulic transitions might occur where the currents pass over ridges or other types of bottom topography or, in the case of the Circumpolar current, where it passes through the Drake Passage. If hydraulic control occurs, it would imply that the effects of the topography could be felt far upstream and downstream.

The suggestion of hydraulic behavior first appears in work by Rossby (1949) who investigated an eastward jet with a half width w^* and velocity u^* independent of y^*. Rossby showed that the jet has two possible widths for the same volume and momentum fluxes. The 'alternate' states were thought to be the two end states of a hydraulic jump of unknown form. Rossby thought that a jump might provide a mechanism for atmospheric blocking. He also showed that the alternate states merge when $3u_o/\beta^* w^{*2} = 1$, which could then be interpreted as a critical condition. Armi (1989) argued that smooth, subcritical-to-supercritical transitions might also be possible by showing that two *conjugate* jet states, i.e. states having the same volume flux and energy, can exist. The jet has a velocity profile that decays from a maximum velocity U_o along the centerline to zero at the edges. The conjugate states found to be identical for $U_o/\alpha' \beta^* w^{*2} = 1$, where α' is depends on the pressure distribution across the jet. Armi was apparently able to establish an critically-controlled jet by circulating fluid through a laboratory annulus and observing that $U_o/\beta^* w^{*2} = 1$ near the point of withdrawal and <1 upstream.

Both of the studies mentioned above base their model on the assumption that the velocity profiles of the conjugate or alternate state are similar. As shown by Woods (1993) a complete solution with self-similarity can be found provided that special upstream conditions exist and that the jet is equatorial ($f=\beta^* y^*$). His theory puts the dynamical elements of Armi's and Rossby's work in a more consistent setting, although both of these models were centered at midlatitude. The hypothetical form of Woods's solution is suggested in Figure 6.3.1, which shows a barotropic, eastward jet, centered on and symmetrical about the equator ($y^*=0$). The zonal velocity decays from a maximum at the center to a value of zero at the edges $y^*=\pm w^*$, where the flow is joined to a quiescent ambient region. The jet impinges on a ridge over which the depth d^* decreases gradually from upstream value D. Self similarity means that the fractional compression or expansion of streamlines at any x^* is the same for each streamline across the velocity profile. In other words $\psi^* = \psi^*[y^*/w^*(x^*)]$.

To investigate whether a dynamically consistent flow of the assumed form exists, we consider the barotropic potential vorticity equation on an equatorial beta plane ($f_o=0$). A convenient set of dimensionless variables is

$$y = y^*/w_o,\ \psi = \psi^*/Q,\ u = u^* D w_o / Q,\ w = w^*/w_o, \text{ and } d = d^*/D,$$

where w_o is the upstream half-width and $2Q$ is the volume transport. The steady form of (6.1.8) is then

$$\frac{\beta y + d^{-1}\dfrac{\partial^2\psi}{\partial y^2}}{d} = P(\psi). \qquad (6.3.1)$$

where $\beta = \beta^* w_o^3 D / Q$ and $ud=-\partial\psi/\partial y$. The contribution to the relative vorticity from $\partial v/\partial x$ has been neglected in view of the assumed gradually varying nature of the flow.

The assumed symmetry of the flow about the x-axis means attention may be confined to the upper half plane. The boundary conditions are then

$$\psi(0) = 0,\ \left(\frac{d\psi}{dy}\right)_{y=w} = 0, \qquad (6.3.2a,b)$$

and

$$\psi(1) = -1. \qquad (6.3.3)$$

We seek a solution of the form $\psi = \psi(\zeta)$, where $\zeta = y / w(x)$. The potential vorticity can be written in terms of ζ as

$$\frac{\beta w\zeta + \dfrac{1}{dw^2}\dfrac{d^2\psi}{d\zeta^2}}{d}.$$

Since this quantity is conserved along streamlines (lines of constant ζ), its upstream (w=d=1) value can be equated with it value at any x along the same streamline:

$$\beta\zeta + \frac{d^2\psi}{d\zeta^2} = \frac{\beta w(x)\zeta + \dfrac{1}{d(x)w(x)^2}\dfrac{d^2\psi}{d\zeta^2}}{d(x)},$$

or

$$\frac{d^2\psi}{\partial\zeta^2} = \frac{\beta(d-w)dw^2}{(1-d^2w^2)}\zeta.$$

The solution satisfying the two homogeneous boundary conditions (6.3.2a,b) is

$$\psi = \frac{\beta(d-w)dw^2}{6(1-d^2w^2)}\zeta(\zeta^2-3) \qquad (6.3.4)$$

This solution is not yet of the form sought since the coefficients *w* and *d* depend explicitly on *x*. However, if the final boundary condition (6.3.3) is enforced, then

$$\frac{\beta(d-w)dw^2}{6(1-d^2w^2)} = \frac{1}{2} \qquad (6.3.5)$$

and therefore the desired form:

$$\psi = \frac{\zeta}{2}(\zeta^2-3) \qquad (6.3.6)$$

is achieved.

The velocity profile corresponding to (6.3.6) is

$$u = -d^{-1}\frac{\partial\psi}{\partial y} = \frac{3}{2wd}\left(1-\frac{y^2}{w^2}\right), \qquad (6.3.7a)$$

or

$$u^* = \frac{3Q}{2w^*d^*}\left(1-\frac{y^{*2}}{w^{*2}}\right) \qquad (6.3.7b)$$

If order to track the width of the current as the jet passes over the topography, it is necessary to relate *w* to *d*. This relation is given by (6.3.5), which can be written as in the form of a Gill functional:

$$\mathcal{G}(w;d;\beta) = \beta dw^3 - (\beta+3)d^2w^2 + 3 = 0, \qquad (6.3.8)$$

Critical states correspond to $\partial\mathcal{G}/\partial w = 0$, or

$$\frac{3\beta w_c}{2(\beta+3)d_c} = 1, \qquad (6.3.9)$$

where the subscript ‘c’ denotes the value at the critical section.

Plots of *d* as a function of *w* for different β (Figure 6.3.1) give the standard sort of hydraulic curve with minima at w=w_c. All the curves are connect to the assumed upstream point *d*=*w*=1. The depth range in the plots has been restricted to $0 \le d \le 1$, corresponding to a ridge (rather than topographic trough). The left branch of each curve presumably represents supercritical states since it corresponds to relatively small widths and relatively large velocities. For β>6, the assumed upstream flow lies on the subcritical branch of the appropriate curve and a hydraulically controlled solution can be constructed in the usual way. For β<6, the upstream flow lies the supercritical branch, apparently leading to an (unstable) supercritical-to-subcritical transition over the topography. However an equally valid solution for the same transport can be constructed by choosing the subcritical value of *w* for *d*=1 as the upstream state. [After all, the arguments leading to the solution are equally valid if *d*=*w*=1 is chosen as the downstream state.] At the value β=6 the upstream flow is exactly critical, as can be verified using (6.3.9). In this case, any slight increase in topographic elevation would lead to upstream influence. In dimensional terms β=6 corresponds to $4u*(0)/\beta*w_o^2$=1, which is also the result found by Armi (1989) for a jet with a quadratic velocity profile.

The mechanics of upstream influence in the equatorial jet is unknown. Given a hydraulically controlled flow, with a subcritical upstream state, what would happen if the minimum depth d_c over the ridge were decreased? If the resulting upstream disturbance preserves the self similar character of the flow, then adjustments in the upstream state would be contained entirely in the value of $\beta = \beta * w_o^3 D / Q$. Since β* and *D* are fixed, the implied changes would involve the upstream width w_o or volume flux 2*Q*, or both. Figure 6.3.1 implies that for β<6, a small decrease in the depth over the ridge would lead to a decrease in β, whereas the opposite is true for β>*6*. It is not clear, however, what this would mean for the width or flux. The possibility that *Q* could be altered distinguishes this model from quasigeostophic model the previous section, for which the volume flux is fixed.

Exercises

1. Calculate the form drag exerted against the equatorial jet by the ridge as a result of a subcritical to supercritical transition.

Figure Captions

Figure 6.3.1 Definition sketch for the self similar, equatorial jet of Woods (1993).

Figure 6.3.2 Solution curves for the Woods (1993) jet for various values of $\beta = \beta * w_o^3 D / Q$.

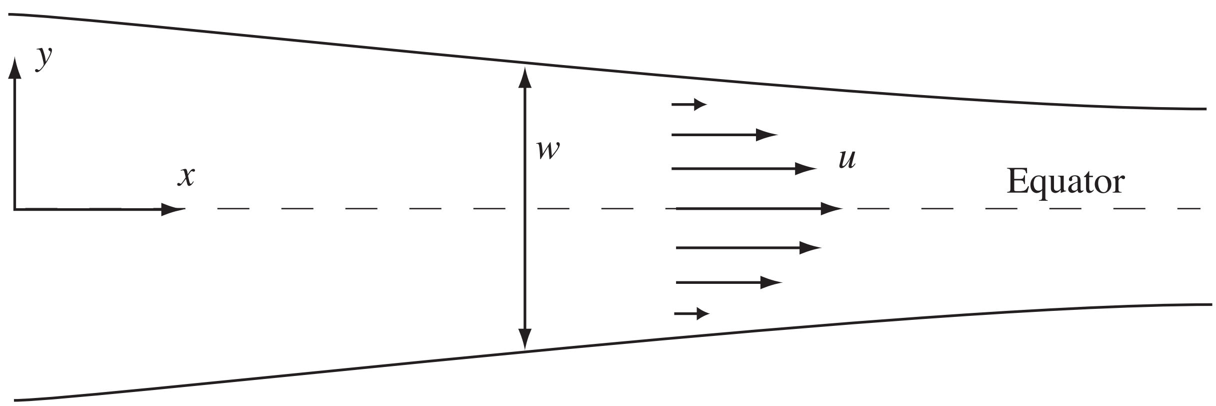

Figure 6.3.1

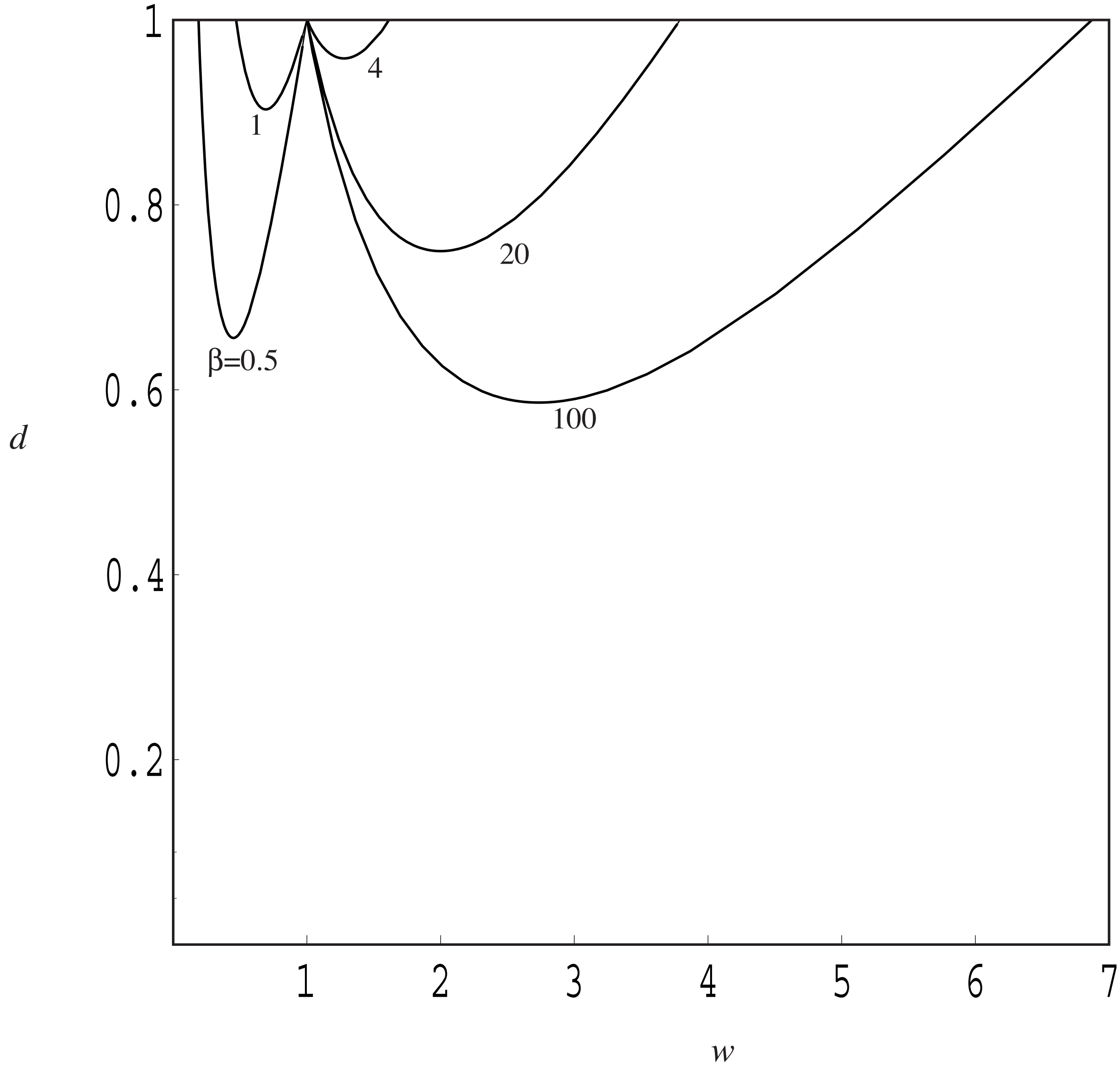

Fig 6.3.2