very rough draft

To do:
-address query in paragraph concerning Jet Stream
?

6.1 Potential Vorticity Hydraulics

To this point we have dwelt with applications in which the velocity of the current is comparable with the speed of long, internal gravity waves. This situation can arise in channels, along coastlines, in the lee of mountains, or in other special locations, but is less likely to occur in the relatively slow and broad general circulation of the ocean or atmosphere. Even the jet-like currents such as the Gulf Stream and Jet Stream are substantially subcritical with respect to long, gravest-mode, internal gravity waves. *(Need to verify this statement about the Jet Stream)*. On the other hand, Rossby waves and other types of potential vorticity waves are important to the general circulation. As discussed in Section 2.1, these waves depend on lateral gradients of potential vorticity for their restoring mechanism. For the gradients that typically exist in geophysical applications, the waves are generally much slower than long gravity waves and can have long wave speeds in the range of the current velocity.

Hydraulic behavior associated with potential vorticity waves, sometime called *Rossby wave hydraulics,* has been identified in a variety of idealized models, including those of free jets, fronts and coastal currents. The subject is reviewed by Johnson and Clarke (2001). One of the difficulties with this subject, at least at the time of this writing, is that there is very little concrete evidence of this behavior in observed flows. However, the field is relatively young and the phenomena may be present but not yet unrecognized. This chapter will present several examples of the type of behavior predicted. One of the main departures from the hydraulics of gravity-driven systems is that the motion of the fluid is primarily sideways (or along isopycnals) and thus classical jumps jumps, spilling flows, and other features that require significant vertical motion are not present.

Some insight can be gained from the nonrotating flow considered in Section 2.9, where a free-surface gravity wave and a discrete spectrum of potential vorticity waves were present. There, an infinite family of hydraulically controlled flows were found, one having a hydraulic transition with respect to the gravity mode and the others with respect to a particular mixed gravity/potential vorticity mode. The hydraulic transition for the gravity mode was manifested primarily by a change in depth as the fluid crossed the sill, whereas the transition for the higher potential vorticity modes involved lateral displacements of streamlines. In order to examine the potential vorticity dynamics more carefully, it will be helpful to consider simpler systems in which the gravity wave is absent and just one or two of the potential vorticity modes is present. The gravity wave can be eliminated by making the quasigeostrophic approximation, discussed below, or by considering a homogeneous fluid bounded above by a rigid lid. The number of potential vorticity modes can be limited by considering flows with piecewise linear potential vorticity distributions.

Since gravity waves will be relatively unimportant, we need to rethink the standard hydraulic scaling in which $(gD)^{1/2}$ and $L/(gD)^{1/2}$ are chosen as scales for the longitudinal velocity and time. For larger scale flows, the Earth's rotation and the variation of rotation with latitude are of central importance and we need to select scales based on the Coriolis parameter $f = 2\Omega\sin\theta$, where Ω is Earth's the angular velocity and θ is latitude. For the applications in mind, which include ocean and atmosphere fronts, jets and coastal currents, the variation in θ is small compared to its full range and it is sufficient to approximate f according to

$$
\begin{aligned}
&f(\theta) = f(\theta_o) + 2\Omega(\theta - \theta_o)\cos\theta_o \\
&= f_o + \beta^* y^*
\end{aligned}
$$

where $y^* = R(\theta - \theta_o)$, $\beta^* = 2\Omega\cos(\theta_o)/R$, and R is the Earth's radius. If L represents the meridional extent of the current, then $\beta^* L/f_o \ll 1$ for this *beta plane* approximation (6.1.1) to be valid. An obvious time scale is f_o^{-1}, while either $f_o L$ or $\beta^* L^2$ could be chosen as a velocity scale.

The appropriate scaling can now be deduced by reconsidering the shallow water equations (2.1.1-2.1.3) with no forcing or dissipation:

$$
\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \mathbf{u}^* + (f_o + \beta^* y^*)\mathbf{k} \times \mathbf{u}^* = -g\nabla\eta^* \qquad (6.1.1)
$$

and

$$
\frac{\partial \eta^*}{\partial t^*} + \nabla \cdot \left[\mathbf{u}(D + \eta^* - h^*)\right] = 0. \qquad (6.1.2)
$$

If these equations apply to a deep, homogeneous layer with slightly greater density than the overlying fluid, then g should be taken as the reduced gravity and η^* as the upwards displacement of the upper interface. Application to a buoyant surface layer can also be made by interpreting η^* as the downwards displacement of the lower interface. Nonrotating hydraulics (Chapter 1) involves balances between the advection terms, the local time-derivative terms, and the pressure gradients terms. Semigeostrophic hydraulics includes these terms, at least in the predominant direction of the flow, and adds the Coriolis acceleration. The scaling $U=(gD)^{1/2}$ is preserved. For the slower, broader flows subject to beta plane hydraulics, U is typically $\ll (gD)^{1/2}$ and another scaling must be sought. There are two classes of flows that one is likely to encounter depending on the size of the Rossby number $R_o = U/f_o L$. The first is characterized by $R_o = O(1)$ and includes strong jets such as the Gulf and Jet Streams and some equatorial currents. Hydraulic models of these flows are typically treated using a barotropic, rigid-lid model and this is described at the end of this section. The second class includes broad-scale flows and weaker jets in which both horizontal velocity components are in near geostrophic balance. Such flows can be treated using the quasigeostophic approximation, in which the velocity scale U is chosen as gN/f_oL, where N is a scale η^*. It can also be seen from (6.1.1) that the ratio of the advection terms to the Coriolis acceleration is the order of the Rossby number $R_o=U/fL$, and this ratio must clearly be small in the presence of nearly geostrophic motion. The final term to consider in (6.1.1) is the local time derivative; its ratio to the Coriolis term is $O(T^{-1}f^{-1})$. In order that the geostrophic balance be preserved to lowest order, the time scale T must be chosen much longer than an inertial period ($T>>f^{-1}$). A convenient choice is $T= R_o^{-1} f^{-1}$.

Using the dimensionless variables $\eta=\eta^*g/ f_oUL$, $(u,v)=(u^*,v^*)/U$, $(x,y)=(x^*,y^*)/L$, and $t=t^*fR_o$, (6.1.1) becomes

$$R_o\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)+(1+R_o\beta y)\mathbf{k}\times\mathbf{u}=-\nabla\eta \qquad (6.1.3)$$

where $\beta=\beta^*L^2/U$. The parameters R_o and β are both <<1 and will be regarded as equal in order. Further progress can then be made by expanding the dependent variables in powers of R_o:

$$u=u^{(0)}+R_ou^{(1)}+\cdots\ ,$$
$$v=v^{(0)}+R_ov^{(1)}+\cdots,$$

and

$$\eta=\eta^{(0)}+R_o\eta^{(1)}+\cdots.$$

The leading order velocity is geostrophic:

$$v^{(o)}=\frac{\partial\eta^{(o)}}{\partial x} \qquad (6.1.4)$$

and

$$u^{(o)}=-\frac{\partial\eta^{(o)}}{\partial y}, \qquad (6.1.5)$$

showing that $\eta^{(0)}$ acts as a streamfunction.

The dimensionless version of the continuity equation (6.1.2) is

$$R_o\frac{\partial\eta}{\partial t}+\nabla\cdot\left[\mathbf{u}\left(S^{-1}\left(1-\frac{h^*}{D}\right)+R_o\eta\right)\right]=0. \qquad (6.1.6)$$

where $S=\dfrac{f_o^{\,2}L^2}{gD}$, the square of the ratio of the horizontal length scale to the Rossby radius of deformation $L_d=(gD)^{1/2}/f_o$. The lowest order approximation is

$$\nabla\cdot\left[\mathbf{u}^{(0)}\left(1-\frac{h^*}{D}\right)\right]=-\mathbf{u}^{(0)}\cdot\nabla\left(\frac{h^*}{D}\right)=0\,,$$

which uses the fact that $\nabla\cdot\mathbf{u}^{(0)}=0$. If h^*/D is O(1), geostrophic flow must move along contours of constant h^*. This topographic steering would imply that a current would have to move around an isolated topographic feature such as a ridge. Hydraulic effects tend to occur when the flow passes over topography and this is permissible in the current framework only when h^*/D is small. We therefore assume that $h^*/D=O(R_o)$ and so define $h=h^*/(R_o D)$.

At the O(R_o) level of expansion, (6.1.3) and (6.1.6) are

$$\frac{\partial\mathbf{u}^{(0)}}{\partial t}+\mathbf{u}^{(0)}\cdot\nabla\mathbf{u}^{(0)}+\beta y\mathbf{k}\times\mathbf{u}^{(0)}=-\mathbf{k}\times\mathbf{u}^{(1)}-\nabla\eta^{(1)}$$

and

$$\frac{\partial\eta^{(0)}}{\partial t}+\nabla\cdot\left[\mathbf{u}^{(0)}\left(\eta^{(0)}-S^{-1}h\right)\right]=-\nabla\cdot\left[S^{-1}\mathbf{u}^{(1)}\right]$$

Taking the curl of the first equation and using the second equation to eliminate $\mathbf{u}^{(1)}$ from the result leads to the quasigeostrophic potential vorticity equation:

$$\left(\frac{\partial}{\partial t}+u^{(0)}\frac{\partial}{\partial x}+v^{(0)}\frac{\partial}{\partial y}\right)\left(\nabla^2\eta^{(o)}-S\eta^{(o)}+h+\beta y\right)=0\,. \qquad (6.1.6)$$

The same result could have been obtained directly from the shallow water potential vorticity equation (2.1.8) through the same scaling and expansion. The variable part of potential vorticity $(\zeta^*+f)/d^*$ (which is constant to a first approximation) is approximated by the second expression (6.1.6). The relative vorticity is $\nabla^2\eta^{(o)}$, the stretching term resulting from departures from constant layer thickness is $-S\eta^{(o)}+h$, and the departure from constant background rotation is βy.

Now consider a plane wave of the form

$$\eta^{(0)}=\mathrm{Re}\left[Ne^{i(kx+ly-\sigma t)}\right]$$

propagating over a horizontal bottom and in the presence of a uniform zonal flow of velocity U_o. It is left as an exercise to show that the wave frequency is given by

$$\sigma = U_o k - \frac{\beta k}{k^2 + l^2 + S}.$$

The wave crests and troughs move to the east at the rate

$$\frac{\sigma}{k} = U_o - \frac{\beta}{k^2 + l^2 + S}$$

A wave that is long $(k^2 << l^2 + S)$ in the x-direction propagates in that direction at the speed $U_o - \beta/(l^2 + S)$. In order for that wave to be arrested by the background flow, it is necessary for that flow to be eastward (U_o>0). In addition the magnitude of U_o must be at least as large as β/S , or

$$U_o^* / \beta^* L_d^2 = O(1). \qquad (6.1.7)$$

The dimensionless parameter can be thought of as a beta-plane Froude number, and (6.1.7) is a prerequisite for the occurrence of hydraulic effects in the quasigeostrophic model. The specific conditions for the criticality of a particular flow with respect to a potential vorticity wave will generally be much more involved. In some applications β* may be replaced by a potential vorticity gradient due to topography or background shear.

An alternative approach that illustrates Rossby-wave hydraulics without the complication of gravitational effects is the rigid-lid, barotropic model. No restriction is placed on the size of R_o or h^*/D, but stratified systems are excluded. The governing equation is obtained from (2.1.8) by regarding the depth $d^*(x^*,y^*)$ as fixed. In the absence of forcing and dissipation the result is

$$\frac{d^*}{dt^*}\left(\frac{f_o + \beta^* y^* + \frac{\partial v^*}{\partial x^*} - \frac{\partial u^*}{\partial y^*}}{d^*}\right) = 0. \qquad (6.1.8)$$

Since the Rossby radius of deformation is effectively infinite, the horizontal length scale L is typically set by the topography or potential vorticity distribution. Velocity and time scales are then chosen as $\beta^* L^2$ and $\beta^* L$.

Most of the models of Rossby-wave hydraulics involve zonal flows and it is standard to use x^* as the predominant direction of flow. We will therefore switch from the earlier convention of using y^* as the flow axis.