2.4 Steady flow from a deep reservoir.

Motivated largely by suggestions from H. Stommel in the early 1970s, Whitehead, Leetma and Knox (1974), hereafter WLK, developed the first analytical model of hydraulic behavior in a steady, rotating-channel flow with topography. Their model utilizes rectangular cross-sectional geometry and is based on the 'zero potential vorticity' limit [$q \rightarrow 0$ with w=O(1)]. Since $q=D/D_\infty$ and $w=w^*f/(gD)^{1/2}$, the channel width is comparable to the Rossby deformation radius based on the local depth scale D, while D itself is much less than the potential depth. The situation envisioned by WLK is that the flow is fed from a very deep and quiescent upstream reservoir with depth D_∞ and fluid is drawn up and over a relatively shallow sill. With q=0 the absolute vorticity of the fluid is zero ($\partial v / \partial x + 1 = 0$) and the depth profiles are given by (2.2.9) for attached flow and (2.3.14) for separated flow. These profiles are valid only as long as the local depth remains much smaller than the reservoir depth. The calculation cannot therefore be extended indefinitely far upstream from the shallow section of channel.

For the case of attached flow, y-variations of the current are governed by the steady versions of (2.2.15) and (2.2.16), which simply state that the volume flow rate Q and average $\bar{B}$ of the sidewall Bernoulli functions are conserved. The flow rate is defined in terms of average and difference wall depths ($\bar{d}$ and $\hat{d}$) by

$$Q = 2\hat{d}\bar{d} \,. \tag{2.4.1}$$

In the limit $q \rightarrow 0$, the average Bernoulli function is

$$\bar{B} = \frac{2\hat{d}^2}{w^2} + \frac{w^2}{8} + \bar{d} + h \,. \tag{2.4.2}$$

Eliminating $\hat{d}$ between the last two relations yields

$$\frac{Q^2}{2\bar{d}^2 w^2} + \frac{w^2}{8} + \bar{d} + h - \bar{B} = 0 \quad , \tag{2.4.3}$$

which is of the form of the standard hydraulic relation $\mathcal{G}(\bar{d}; w, h; Q, \bar{B}) = 0$ sought by Gill(1977). Here $\bar{d}$ represents the single variable characterizing the flow cross-section; if $\bar{d}$ is known, $\hat{d}$ can be computed from (2.4.1) and the remaining cross-sectional properties from (2.2.29) and (2.2.30). Critical states are found by taking $\partial \mathcal{G} / \partial \bar{d} = 0$, resulting in

$$Q = \bar{d}_c^{\,3/2} w \,, \tag{2.4.4}$$

where the subscript c denotes a critical value. From (2.4.4) it follows that $2\hat{d}_c / w = \bar{d}_c^{\,1/2}$, or

$$\bar{v}_c = \bar{d}_c^{\,1/2}, \tag{2.4.5}$$

in view of the relation $2\hat{d} / w = \bar{v}$ derived in Section 2.2. As expected, Gill's criterion for critical flow matches the direct propagation speed calculation (2.2.31).

Possible locations where critical flow can occur are found by taking $[\partial \mathcal{G} / \partial y]_{\bar{d}=const.} = 0$, which leads to

$$\left(\frac{w}{4} - \frac{Q^2}{\bar{d}_c^{\,2} w^3}\right)\frac{\partial w}{\partial y} + \frac{\partial h}{\partial y} = 0. \tag{2.4.6}$$

In the WLK model w is constant and critical flow therefore requires that $\partial h / \partial y = 0$, as at a sill. In a channel of constant elevation h and with variable w, critical flow requires that either $\partial w / \partial y = 0$, as at a width contraction, or that the expression in parenthesis vanish. In the latter case (2.4.1) and (2.4.4) imply separation of the flow from the left wall ($\bar{d}_c = \hat{d}_c$). However this possibility can be eliminated, as explored in Exercise 1 of this section.

It is possible to obtain a 'weir' formula relating Q to the reservoir conditions. In the nonrotating example of Section 1.4 the formula was obtained by equating the Bernoulli functions at the sill and reservoir. Following the same procedure, we use (2.4.4) to evaluate (2.4.3) at the sill, leading to

$$\frac{3}{2}\left(\frac{Q}{w}\right)^{2/3} + \frac{w^2}{8} = \bar{B} - h_m, \tag{2.4.7}$$

where h_m is the sill elevation. Next, we need to evaluate $\bar{B}$ in the reservoir, being careful to avoid using the definition (2.4.2), which is not valid there. Instead we simply note that the Bernoulli function in the hypothetical quiescent reservoir must be

$$B = D_\infty + h_\infty \tag{2.4.8}$$

where h_∞ is the reservoir bottom elevation. Since B is uniform throughout the reservoir, $\bar{B} = B$ and therefore

$$\frac{3}{2}\left(\frac{Q}{w}\right)^{2/3} + \frac{w^2}{8} = \Delta z \tag{2.4.9}$$

where $\Delta z = D_\infty + h_\infty - h_m$ is the elevation of the reservoir interface above the channel bottom at the critical section. Rearranging (2.4.9) and writing the result in dimensional form gives

$$Q^* = \left(\frac{2}{3}\right)^{3/2} w^* g^{1/2}\left[\Delta z^* - \frac{w^{*2} f^2}{8g}\right]^{3/2}. \qquad (2.4.10)$$

As $f \to 0$ the limit (1.4.12) for nonrotating flow from a deep reservoir is realized.

If the flow in the channel becomes separated, we switch to the natural variables v_e and w_e (see Figure 2.1). The y-structure of the flow is then described by the steady forms of (2.3.16) and (2.3.17):

$$\tfrac{1}{2} w_e^2 (v_e - \tfrac{1}{2} w_e)^2 = Q \qquad (2.4.11)$$

and

$$\frac{v_e^{\,2}}{2} + h = \bar{B}\,. \qquad (2.4.12)$$

Note that the channel width w does not enter these relations. Changes in the position of the right-hand wall cause lateral displacements of the entire flow with no change in the shape of the interface.

Equation (2.4.11) expresses energy conservation along the free edge of the separated current. Since the depth is zero there, changes in the kinetic energy of the flow must be directly balanced by changes in bottom elevation. It is tempting to treat the left-hand side of this equation as a Gill-type hydraulic function $\mathcal{G}(v_e;h)$ since it contains the single flow variable v_e. However, taking $\partial\mathcal{G}/\partial v_e = 0$ results in v_e=0, whereas the true critical condition based on direct calculation has been shown to be v_e=w_e. On the other hand, if one substitutes for v_e in (2.4.11) using (2.4.12), then the functional relation

$$\mathcal{G}(w_e;h;\bar{B},Q) = w_e(2^{1/2}(\bar{B}-h)^{1/2} - \tfrac{1}{2}w_e) - (2Q)^{1/2} = 0 \qquad (2.4.13)$$

is obtained, and taking $\partial\mathcal{G}/\partial w_e = 0$ yields the desired result:

$$v_{ec} = w_{ec}\,. \qquad (2.4.14)$$

Furthermore, taking $[\partial\mathcal{G}/\partial y]_{w_e=const.} = 0$ leads to the condition that $dh/dy = 0$ at a critical section.

The failure of the criterion $\partial\mathcal{G}/\partial v_e = 0$ to yield the correct critical condition in its application to (2.4.12) is tied into the peculiar dynamics of the frontal wave and the choice of v_e as the dependent variable. Consider the depth profile under critical conditions, as shown by the solid curve in Figure 2.4.1. The slope of the free surface is zero at the wall and v_e=w_e. Suppose that the profile is slid an infinitesimal distance to the right or left without changing its shape, as suggested by the dashed line. Since $\partial d / \partial x = 0$ at the wall the altered flow has the same wall depth, and therefore the same volume flux, as before. In addition the sideways displacement does not alter the value of B at the free edge, since v_e is unchanged. In summary neither $\bar{B}$ nor Q is altered by the sideways displacement and the disturbance, which only involves changes in w_e, qualifies as a stationary long wave. [1] Thus the condition $\partial \mathcal{G} / \partial v_e = 0$, which checks for disturbances in v_e that leave $\mathcal{G}$ unchanged, misses the critical condition. In essence, satisfaction of Gill's criterion for a $\mathcal{G}$ written in terms of a particular dependent variable is a sufficient, but not necessary, condition for criticality. To avoid such cases one must be sure to use *all* the constraints available in the formulation. The multivariate version (1.5.14) of the critical condition therefore provides the safest route.

The weir formula for this case may be obtained using a similar procedure as above, resulting in

$$Q^* = \frac{g(\Delta z^*)^2}{2f} \qquad (2.4.15)$$

for the separated case.

Equation (2.4.14) suggests that the separation first occurs at the critical section when $w_c = v_{ec}$ or, in view of (2.4.5), when $w_c = 2\bar{d}_c^{1/2}$. Furthermore, since v=0 at the right wall in this case, energy conservation implies that the level of the interface at the right wall is the same as that of the reservoir, so that $2\bar{d}_c = \Delta z$. Elimination of $\bar{d}_c$ between these last leads to $\Delta z = w_c^2 / 2$, and therefore

$$w_c^* \begin{cases} < \sqrt{2}(g\Delta z^*)^{1/2}/f & \text{(non-separated)} \\ > \sqrt{2}(g\Delta z^*)^{1/2}/f & \text{(separated)} \end{cases} \qquad (2.4.16)$$

That is, the flow at the critical section is separated if the channel width is greater than √2 times the Rossby Radius of deformation based on Δz^*. Thus, a decrease in reservoir surface elevation relative to the sill encourages critical separation of the flow.

WLK carried out an experiment designed to test the transport relations (2.4.10 and 2.4.15) and the separation criterion (2.4.16). The apparatus consists of a cylindrical tank divided into two basins by a vertical wall (Figure 2.4.2). Well above the bottom, a short channel with rectangular cross-section is fitted through an opening in the wall. An alcohol-water mixture is filled up to the level of the bottom of the channel in both basins, and above this lies a layer of kerosene with slightly lower density. A pump transfers the lower fluid from the left-hand basin to the right, where it wells up through a packed bed

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[1] It should also be noted that the same argument is applicable to a separated flow with an arbitrary potential vorticity distribution. Such a flow is hydraulically critical if the velocity at the right wall vanishes.

of rocks. This fluid rises, passes through the channel, and spills into the right-hand basin. Photos of the overflow as seen looking upstream into the channel appear in Figure 2.4.3. For the case shown, the flow in the channel is non-separated. The height Δz^* of the upstream interface above the channel bottom was measured by an optical device. The value of Q^* is not measured directly.

The experiment is initiated by establishing a hydraulically controlled flow with $f=0$ and measuring the corresponding $\Delta z^*=\Delta z_o^*$. In principle, Δz_o^* should equal $\frac{3}{2}Q^{*2/3}g^{-1/3}w^{*-2/3}$. The turntable is then spun up to a particular f and, once a new steady state had been established, a new Δz^* is measured. The transport Q^* is determined only by the pumping rate and remains constant throughout the spin-up, so that the reservoir interface elevation is forced to adjust to drive the same amount of fluid across the sill.

For attached flow, the ratio $\Delta z^*/\Delta z_o^*$ can be determined using (2.4.9). The dimensional version of the result is

$$\frac{\Delta z^*}{\Delta z_o^*}=\frac{f^2 w^{*2}}{8g\Delta z_o^*}+1. \qquad (2.4.17)$$

For the separated sill flow, the value of Δz^* is given by (2.4.15) and thus

$$\frac{\Delta z^*}{\Delta z_o^*}=\left(\frac{2}{3}\right)^{3/4}\left(\frac{4w^{*2}f^2}{g\Delta z_o^*}\right)^{1/4}. \qquad (2.4.18)$$

The transition between the two cases occurs when $w^*=w_e^*=(2g\Delta z^*/f^2)^{1/2}$, which corresponds to $\frac{3w^{*2}f^2}{8g\Delta z_o^2}=1$ or $\frac{\Delta z^*}{\Delta z_o^*}=\frac{4}{3}$. Figure 2.4.4 shows a plot of $\frac{\Delta z^*}{\Delta z_o^*}$ as a function of $\left[3w^{*2}f^2/8g\Delta z_o^*\right]^{1/2}$ as determined by (2.4.17) or (2.4.18). For attached flow (to the left) the agreement is quite good. For separated flow, the agreement is still fairly good.[2]

Although the transport formulas (2.4.10) and (2.4.15) suggest that increasing f leads to smaller Q^*, this conclusion is only valid if the upstream interface level remains fixed. In reality, the effect of rotation on transport depends on how the flow is driven; in the WLK experiment Q^* is maintained at a fixed rate while f is varied.

The WLK experiment was designed to approximate the zero potential vorticity limit by causing the channel flow to be drawn from a deep, quiescent reservoir. Clearly,

[2] The ‘sill’ in the laboratory tank is actually a finite length of uniform channel . Even under conditions described as being ‘separated’, the depth at the left wall is actually nonzero at the upstream end of this channel and zero at the downstream end. So there is some question as to where the critical section is and whether the flow is entirely separated at this section.

the long wave approximation is violated at the entrance of the channel, where an abrupt change in geometry occurs. Also, a gyre in the deep upstream fluid was observed to form, making the assumption of quiescence doubtful. Despite the violations of underlying assumptions, agreement between predicted and observed transports is generally good. In fact the WLK model is one of the few theories that has been subjected to careful laboratory verification. As we will show in Section 2.6, the relation (2.4.15) provides the transport under more general conditions, provided Δz^* is suitably interpreted. It will also be shown (Section 2.10) that the same relation provides a bound on transport in even more general circumstances.

Exercises

1) Consider a channel with variable *w* and constant *h*. Equation (2.4.6) suggests that a critical section in such a channel can occur where $\partial w/\partial y = 0$ or where $w^2 = 2Q/\bar{d}_c$.

(a) Show that the latter implies $\bar{d}_c = \hat{d}_c$ (the flow is separating from the left wall).

(b) Suppose that for increasing *y*, w(*y*) decreases monotonically to a minimum value at y=0, then increases monotonically for positive *y*. Further suppose that critical separation of the flow occurs in $y=y_s<0$. Since the width of a separated flow with $\partial h/\partial y = 0$ is constant, the flow must immediately reattach downstream of $y=y_s$. Now consider two sections slightly upstream and downstream of $y=y_s$ having the same value of $d(-w/2,y)$. By mass conservation the values of $d(w/2,y)$ must also be the same. Show, however, that under these conditions the values of the $\bar{B}$ at the two sections must be different, so that the solution is invalid.

(c) Alternatively, suppose that critical separation occurs in the broadening section of the channel (y>0). Stability considerations demand that the flow slightly upstream must be subcritical. Then there are two possibilities. First the flow upstream remains subcritical through the contraction and, by symmetry of the subcritical solution with respect to *w*, must undergo critical separation at that upstream section having the same *w* as the downstream section. However, we have just ruled out such separation in (b). The other possibility is that the flow is critical at y=0 (and experiences some type of hydraulic jump in $0<y<y_s$). In this case show that Q cannot be conserved between y=0 and $y=y_s$. (*Hint:* Use the properties of separated and non-separated critical flows to evaluate the sidewall velocities and depths.)

2) Suppose that the channel draining the reservoir in the WLK model has constant *w*. Further suppose that the flow separates from the left wall upstream of the sill. Given the values of w, h_m, and Δz, at what value of *h* does separation occur?

Figure Captions

Figure 2.4.1 Cross section of a critical, separated current (solid curve) and a new steady flow with the same Bernoulli function and volume flux, created by a sideways displacement of the current (dashed line).

Figure 2.4.2 Cross section of cylindrical tank used in WLK experiments. (From Whitehead et al. 1974).

Figure 2.4.3 View of overflow through the rectangular channel in on of the WLK experiments. The observer faces upstream. ((From Whitehead et al. 1974).

Figure 2.4.4 Comparison of laboratory data and predictions from the Whitehead et al. (1974) experiment and theory. (From Whitehead et al. 1974).

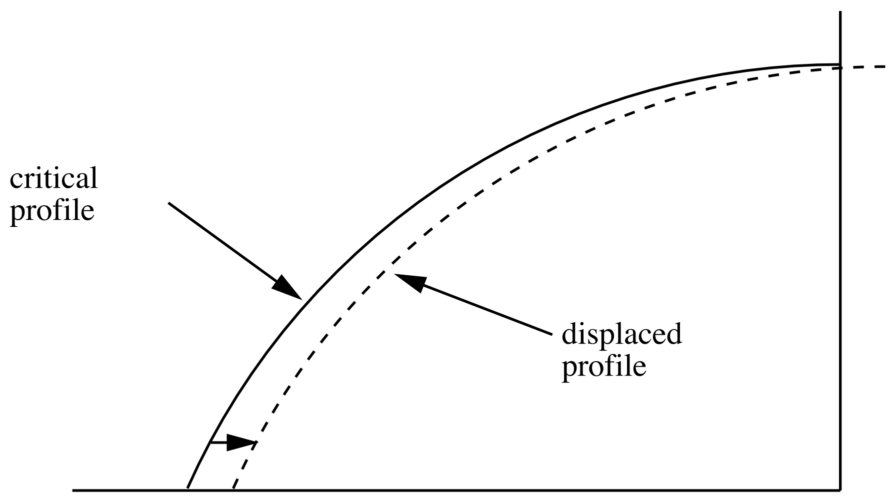

Figure 2.4.1

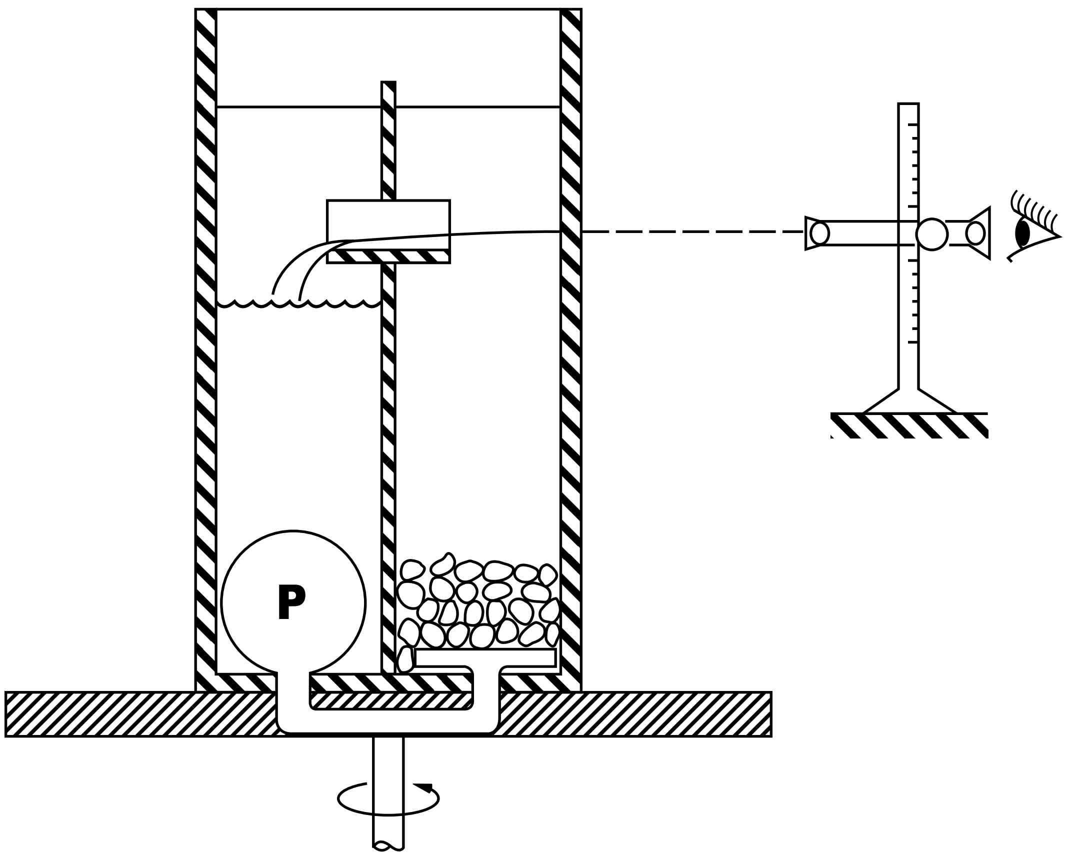

Figure 2.4.2

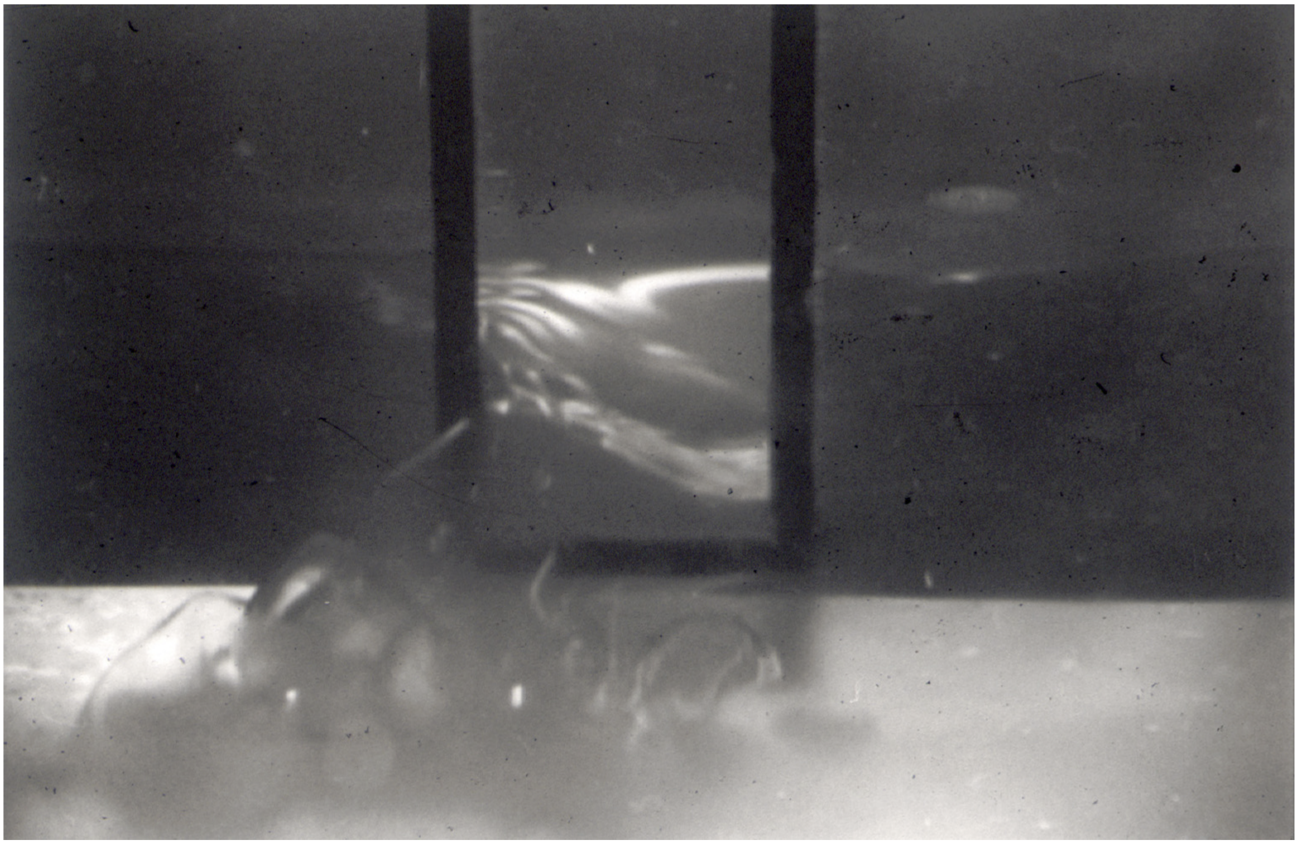

Figure 2.4.3 version 2

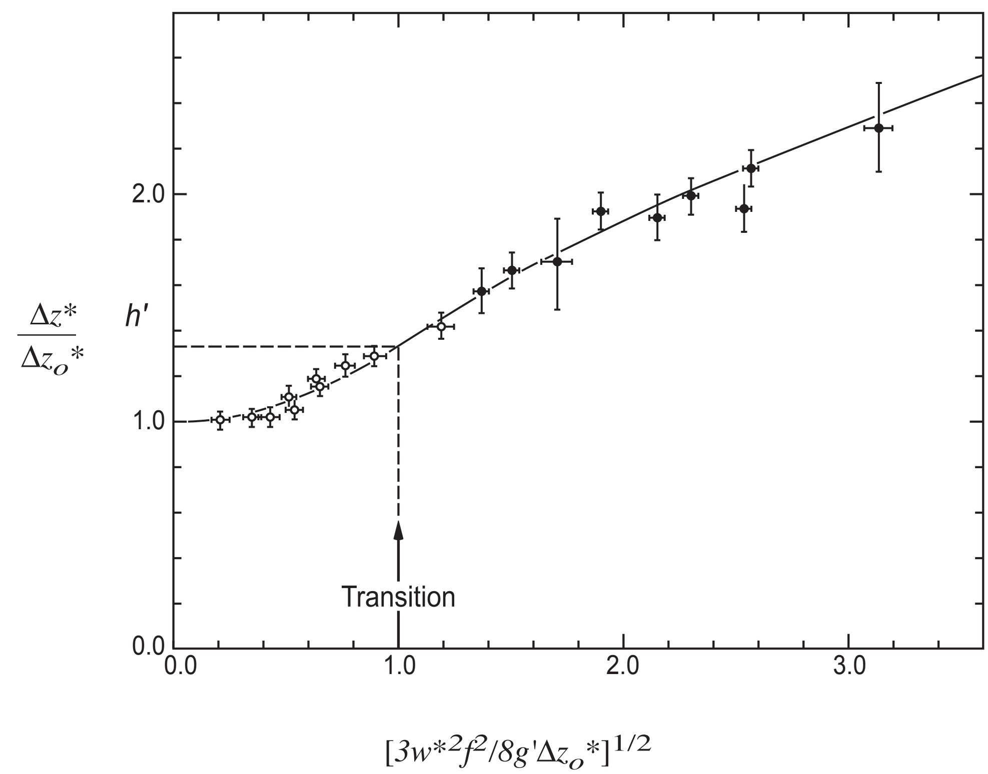
$\frac{\Delta z^*}{\Delta z_o^*}$
h'
2.0
1.0
0.0
Transition
0.0
1.0
2.0
3.0
$[3w^{*2}f^2/8g'\Delta z_o^*]^{1/2}$

Fig. 2.4.4