2.2 Uniform Potential Vorticity: Boundary Layers and Kelvin waves.

The models of hydraulic behavior in rotating channels that first appeared in the literature in the 1970's consider flow with uniform potential vorticity:

$$q^* = \frac{f + \partial v^* / \partial x^*}{d^*} = \frac{f}{D_\infty} = const. \tag{2.2.1}$$

If q^* is materially conserved, such flows arise as a result of evolution from an initial state of uniform q^*. Or, a steady flow with uniform q^* would arise if the streamlines could be traced back into a reservoir or other source region with uniform q^*. Either case must be free of potential vorticity altering dissipation or forcing, a situation that is probably not representative of deep ocean overflows. Also, there is little in the way of observations that indicate how the potential vorticity varies from one streamline to the next in deep ocean overflows. The primary observational difficulty lies in the estimation of the relative vorticity. Despite these objections, uniform potential vorticity models are quite tractable and give valuable insight into a dynamically restricted but intriguing arena. The main deficiency of such models is that they lack the ability to support potential vorticity waves. However, such waves tend to propagate much more slowly than Kelvin waves, which are admitted. A possible, though unverified, reality is that the potential vorticity waves are all carried downstream by the flow and that upstream influence is carried entirely by the Kelvin waves.

Before entering into a detailed discussion of benchmark theories for steady flow (particularly Sections 2.4 and 2.5), we make some observations about uniform potential vorticity flow in general. First note that the potential depth D_∞ is now constant throughout the fluid and that the potential vorticity, in nondimensional terms, is $q = (D/f)q^* = D/D_\infty$. The depth scale D is simply a measure of the typical or average depth in a region of interest: often a section of the channel near a sill or narrows. The dimensionless potential vorticity is therefore a measure of the departure of this average depth from the potential depth. In many hydraulics problems the average depth varies over several orders of magnitude between the sill or narrows and the upstream reservoir. If the average depth in the sill region is chosen as D then the nondimensional depth d may be regarded as O(1) there. In this case a *small* value of q would indicate that the typical sill depth is << the potential depth, implying the sill region to be one of significant relative vorticity. In addition the small value of q can be exploited to further simplify the governing equations in what is called the 'zero potential vorticity' approximation. Caution is advised, however, as these properties only hold where d remains O(1). The values of d in the reservoir might be >>1, implying a completely different flow regime.

In most of what follows, the rotating channel is assumed to have rectangular cross-section with vertical side walls at $x=\pm w(y)/2$, as shown in Figure 2.2.1. For this geometry equation (2.1.14) becomes

$$\frac{\partial^2 d(x,y,t)}{\partial x^2} - qd(x,y,t) = -1. \qquad (2.2.2)$$

The boundary conditions at the edges of the stream depend upon whether the fluid depth remains non-zero over the entire cross-section, as in Figure 2.2.1a, or whether the depth vanishes at one or more values of x, as in Figure 2.2.1b. As it turns out, there are a limited number of possible configurations. These may be found by first noting that if $d\rightarrow 0$ at some point (x_o,y) than one of the following conditions must hold:

1) (x_o,y) lies at a side wall, as in Figure (2.2.2a),
2) d is also zero in a neighborhood to the left ($x<x_o$) or the right ($x>x_o$) of (x_o,y), as in Figure (2.2.2b), or
3) $\partial d/\partial x$ is discontinuous at (x_o,y), as in Figure (2.2.2c).

Most importantly, d cannot vanish smoothly in the interior of the current as shown in Figure (2.2.2d). The curvature $\partial^2 d/\partial x^2$ is clearly positive at such a point, whereas (2.1.14) indicates that it must be negative. In other words the fluid depth cannot smoothly vanish in the interior of the current; it must first vanish at a sidewall. Gill (1977) first showed this result for steady channel flow with uniform q but the same clearly applies to time-dependent flows with possibly nonuniform q since (2.2) continues to apply. For northern hemisphere rotation, there are only two configurations we will generally have to consider and they are the ones shown in Figures 2.2.1a and 2.2.1b.

For q=constant (2.2.2) can easily be solved and a convenient representation of the solution is given by

$$d(x,y,t) = q^{-1} + \hat{d}(y,t)\frac{\sinh(q^{1/2}x)}{\sinh(\frac{1}{2}q^{1/2}w)} + (\bar{d}(y,t) - q^{-1})\frac{\cosh(q^{1/2}x)}{\cosh(\frac{1}{2}q^{1/2}w)}, \qquad (2.2.3)$$

following a form first used by Gill(1977). The geostrophic velocity associated with this depth is

$$v = q^{1/2}\hat{d}(y,t)\frac{\cosh(q^{1/2}x)}{\sinh(\frac{1}{2}q^{1/2}w)} + q^{1/2}(\bar{d}(y,t) - q^{-1})\frac{\sinh(q^{1/2}x)}{\cosh(\frac{1}{2}q^{1/2}w)}, \qquad (2.2.4)$$

in view of (2.1.12).

The quantities $\hat{d}$ and $\bar{d}$ represent half the difference and sum of the depth along the sidewalls:

$$\bar{d} = \tfrac{1}{2}[d(\tfrac{1}{2}w,y,t) + d(-\tfrac{1}{2}w,y,t)] \qquad (2.2.5)$$

and

$$\hat{d} = \tfrac{1}{2}[d(\tfrac{1}{2}w, y, t) - d(-\tfrac{1}{2}w, y, t)]. \qquad (2.2.6)$$

Using (2.2.4) these can also be related to the average and difference of the wall velocities:

$$\bar{v} = \tfrac{1}{2}[v(\tfrac{1}{2}w, y, t) + v(-\tfrac{1}{2}w, y, t)] = q^{1/2}T^{-1}\hat{d} \qquad (2.2.7)$$

and

$$\hat{v} = \tfrac{1}{2}[v(\tfrac{1}{2}w, y, t) - v(-\tfrac{1}{2}w, y, t)] = q^{1/2}T(\bar{d} - q^{-1}), \qquad (2.2.8)$$

where

$$T = \tanh(\tfrac{1}{2}q^{1/2}w) \qquad (2.2.9)$$

The dimensional form of the depth profile (2.2.3) is

$$d^* = dD = D_\infty + \hat{d}^* \frac{\sinh(x^*/L_d)}{\sinh(w^*/2L_d)} + (\bar{d}^* - D_\infty)\frac{\cosh(x^*/L_d)}{\cosh(w^*/2L_d)}, \qquad (2.2.10)$$

where

$$L_d = \frac{(gD_\infty)^{1/2}}{f} \qquad (2.2.11)$$

is the Rossby radius of deformation based on the potential depth D_∞. In the limit $w^*/L_d \to \infty$ (2.2.10) becomes

$$d^* = D_\infty + [d(w^*/2, y^*, t^*) - D_\infty]e^{(x^* - \frac{1}{2}w^*)/L_d} + [d^*(-w^*/2, y^*, t^*) - D_\infty]e^{-(x^* + \frac{1}{2}w^*)/L_d}, \qquad (2.2.12)$$

and thus the solution takes on a boundary layer structure with the side-wall depths tending to the interior value D_∞ over the length L_d. The situation is depicted in Figure 2.2.3.

In the other extreme, a sufficiently narrow channel should allow us to return to the one-dimensional equations describing non-rotating flow. This limit is more subtle than what one might guess. Consider a channel whose width w^* is $\le L_d$ so that w^* becomes the cross-channel length scale. Then a good indication of the strength of rotation is the change in the interface elevation across the channel divided by the average depth, equal to $\hat{d}^*/\bar{d}^*$ in the present model. An estimate for this quantity can be obtained by approximating $\partial^2 d/\partial x^2$ by $\hat{d}/w^2$ in (2.2.2). Dimensionalization of the result leads to

$$\frac{\hat{d}^*}{\bar{d}^*} = O\left(\frac{w^{*2}}{L_d^2}\right) + O\left(\frac{w^{*2} f^2}{gD}\right)$$

implying that the channel width must be << the Rossby radii of deformation based on both the potential depth D_∞ *and* the local depth D in order for rotation to be ignored. In most applications D will be $O(D_\infty)$ or less and therefore a good rule of thumb is to ignore rotation if w^* is << $(gD)^{1/2}/f$. In nondimensional terms, the latter is equivalent to w<<1.

The importance of two deformation radii may seem confusing to the reader who has observed that L_d appears to be the only intrinsic lateral length scale in the depth and velocity profiles (*e.g*. 2.2.12). The situation can be clarified by reference to Figure 2.2.3 which shows a cross-section for a case where w^* is somewhat larger than L_d. As suggested by (2.2.12) the depth near the center of the channel is $\cong D_\infty$. The overall depth scale D is also $\cong D_\infty$. Rotation is clearly important here as both L_d and $(gD)^{1/2}/f$ are $<w^*$. However, we could consider a second flow consisting of a short section of the depth profile near the left wall, as shown in the inset to Figure 2.2.3. An imaginary right wall can be inserted a short distance from the left wall so that a new channel is formed. Although the width of the hypothetical channel is << L_d rotation continues to be important because the *new* depth scale D is $<<D_\infty$ and the deformation radius based on the new D is $\cong$ this width.

Recalling that the argument $q^{1/2}w$ in the velocity and depth profiles (2.2.3,4) is equivalent to w^*/L_d, and that $w=w^*f/(gD)^{1/2}$ we can now distinguish between two 'narrow' channel limits. In the first, $w^*<<L_d$ but rotation is still important ($w=O(1)$), so that

$$q << 1 \quad \text{and} \quad w = O(1).$$

This case is sometimes called the *zero potential vorticity limit* and will be explored shortly. The second, more severe limit is that of negligible rotation*:*

$$w << 1 \text{ and } q \leq O(1).$$

The distinguishing features of these limits are hidden by the fact that $q^{1/2}w$ is goes to zero in each case.

The y- and t- dependence of the solutions can be obtained by first evaluating the y-momentum equation (2.1.6) at both the channel sidewalls. After some manipulation, which is left as Exercise (2.2.3), the wall version of the momentum equations can be written as

$$\frac{\partial v(\pm w/2, y, t)}{\partial t} + \frac{\partial B(\pm w(y)/2, y, t)}{\partial y} = 0 \qquad (2.2.13)$$

where

$$B = \tfrac{1}{2}v^2 + d + h \qquad (2.2.14)$$

is the Bernoulli function. (Note that the term $u^2/2$ is negligible and therefore missing from B in the semigeostrophic approximation.)

The difference and sum of the two sidewall equations in (2.2.13) are next taken and the relations $\bar{v} = q^{\frac{1}{2}}T^{-1}\hat{d}$ and $\hat{v} = q^{1/2}T(\bar{d} - q^{-1})$ used to obtain

$$2q^{-1/2}\frac{\partial(T\bar{d})}{\partial t} + \frac{\partial Q}{\partial y} = 0 \tag{2.2.15}$$

and

$$q^{1/2}\frac{\partial(T^{-1}\hat{d})}{\partial t} + \frac{\partial \overline{B}}{\partial y} = 0\,, \tag{2.2.16}$$

where

$$\begin{aligned}\overline{B} &= [B(w/2, y, t) + B(-w/2, y, t)]/2 \\ &= \tfrac{1}{2}q[T^{-2}\hat{d}^2 + T^2(\bar{d} - q^{-1})^2] + \bar{d} + h\end{aligned} \tag{2.2.17}$$

is the average of the Bernoulli function on the two side walls and

$$Q = 2\bar{d}\hat{d} \tag{2.2.18}$$

is the volume transport over the channel cross-section. The latter follows from multiplication of the geostrophic relation by v and integration of the result across the channel:

$$Q = \int_{-\frac{1}{2}w}^{\frac{1}{2}w}(vd)dx = \int_{-\frac{1}{2}w}^{\frac{1}{2}w} d\frac{\partial d}{\partial x}dx = \tfrac{1}{2}[d^2(\tfrac{1}{2}w, y, t) - d^2(-\tfrac{1}{2}w, y, t)] = 2\hat{d}\bar{d}\,. \tag{2.2.19}$$

Thus (2.2.15) is a statement of mass conservation and (2.2.16) is an expression of momentum conservation. (The latter is made more evident if (2.2.7) is used to show that the first term in (2.2.16) is just $\partial\bar{v}/\partial t$.)

The wave propagation characteristics of the solution become clear if the evolution equations (2.2.15) and (2.2.16) are written in Riemann invariant form. When the channel width and bottom elevation are constant, conserved Riemann invariants can be found by following a general procedure laid out in Appendix B (or see Exercise 1 of Section 1.3). As shown by Pratt (1983) the results can be written

$$\frac{d_{\pm}R_{\pm}}{dt} = 0 \tag{2.2.20}$$

where

$$\frac{d_\pm}{dt} = \frac{\partial}{\partial t} + c_\pm \frac{\partial}{\partial y}, \qquad (2.2.21)$$

$$\begin{aligned} c_\pm &= q^{1/2} T^{-1} \hat{d} \pm \bar{d}^{1/2} [1 - T^2 (1 - q\bar{d})]^{1/2} \\ &= \bar{v} \pm \bar{d}^{1/2} [1 - T^2 (1 - q\bar{d})]^{1/2}, \end{aligned} \qquad (2.2.22)$$

$$R_\pm = q^{1/2} T^{-1} \hat{d} \pm \int^{\bar{d}} r(\alpha) d\alpha, \qquad (2.2.23)$$

and

$$r(\bar{d}) = \bar{d}^{-1/2} [1 - T^2 (1 - q\bar{d})]^{1/2}. \qquad (2.2.24)$$

As in the one-dimensional, nonrotating model discussed earlier, there are two wave modes characterized by the Riemann invariants $R_\pm$ that are conserved following the corresponding characteristic speeds $c_\pm$. However the structure of the waves is quite different and it is worthwhile taking a few moments to examine this structure in some detail. First consider the expression (2.2.22) for $c_\pm$, which tempts one to interpret $\bar{v}$ as an advective speed due to the current and $\pm \bar{d}^{1/2} [1 - T^2 (1 - q\bar{d})]^{1/2}$ as a propagation speed relative to the current. This interpretation is not entirely correct, as can be seen by taking the limit a$q^{1/2} w \to \infty$ (or $T \to 1$), resulting in $c_\pm = \bar{v} \pm \bar{d} q^{1/2}$. Substituting $\hat{v} + q^{-1/2}$ for $\bar{d} q^{1/2}$, which follows from (2.2.8) leads to $c_\pm = \bar{v} \pm (\hat{v} + q^{-1/2})$, or $c_+ = v(w/2, y, t) + q^{-1/2}$ and $c_- = v(-w/2, y, t) - q^{-1/2}$. The expression $\pm \bar{d}^{1/2} [1 - T^2 (1 - q\bar{d})]^{1/2}$ therefore contains a hidden advection component. If (2.2.4-2.2.6) are used to express the wall velocities in terms of the wall depths, $c_\pm$ for wide channels can further be simplified to (in dimensional form):

$$c_+^* = v^*(w^*/2, y^*, t^*) + (gD_\infty)^{1/2} = \left(\frac{g}{D_\infty}\right)^{1/2} d^*(w^*/2, y^*, t^*) \qquad (2.2.25)$$

and

$$c_-^* = v^*(-w^*/2, y^*, t^*) - (gD_\infty)^{1/2} = -\left(\frac{g}{D_\infty}\right)^{1/2} d^*(-w^*/2, y^*, t^*) \qquad (2.2.26)$$

The two waves have now separated into independent Kelvin waves trapped along the two side walls (Bennett, 1973). The cross-sectional depth profile is given by (2.2.12). For the wave trapped on the right-hand wall (facing in the positive *y*-direction) the characteristic speed is positive provided the depth at that wall is non-zero (the flow is attached). For finite depth flow the right wall Kelvin wave propagates towards positive *y*, while the left wall counterpart propagates towards negative *y*. All wide channel flows of finite depth are therefore subcritical in the sense that the two long waves propagate in opposite directions. In order to reverse the propagation speed of one of the waves it is necessary for the flow to separate or for the channel width to be less than several L_d, forcing the Kelvin waves to overlap.

The right-wall Kelvin wave will steepen if the wall depth, and therefore c_+, decreases with positive y. It can be shown that $R_+ = c_+$ (and $R_- = c_-$) and therefore the right wall depth is conserved following the characteristic speeds. From (2.1.6) it can also be shown that the cross-channel velocity in an evolving right-wall wave is given by

$$u^* = \frac{g}{2fd^*}(1-e^{(x^*-w^*/2)/L_d})e^{(x^*-w^*/2)/L_d}\frac{\partial[d^*(w^*/2,y^*,t^*)-D_\infty]^2}{\partial y}. \qquad (2.2.27)$$

The quantity $d^*(w^*/2,y^*,t^*)-D_\infty$ is the height of the free surface at the right wall above the resting interior depth, and can be thought of as a wave amplitude. The fact that the square of this amplitude appears in (2.2.27) is an indication that the cross-channel velocities are generated by nonlinear advection. In fact, it is well known that linear Kelvin waves have $u \equiv 0$. As a wave steepens, the y-derivative of the square of the amplitude increases and strong cross-channel velocities are generated. For a wave of elevation propagating into a quiescent region, the square of the amplitude decreases with y and the steepening generates strong u^* away from the wall. The combination of diminishing along-channel scales and increasing u^* may be expected to eventually lead to violations of the semi-geostrophic (and perhaps even the hydrostatic) approximation.

For channel widths of the order of L_d or less, each Kelvin wave is felt all across the channel. A limiting case of this behavior is 'zero potential vorticity' flow, which is characterized by $q \rightarrow 0$ and w=O(1). Rotation remains important in this limit since the 'local' radius of deformation is comparable to the channel width (i.e. w=O(1)). However the channel width is much smaller than the decay scale of the Kelvin waves: $q^{1/2}w=w^*/L_d<<1$. Since $q = D/D_\infty <<1$, the limit occurs if the channel flow is fed by fluid originating from a relatively deep, quiescent reservoir. The reservoir depth is therefore the potential depth D_∞ and the process of squashing fluid columns to the scale depth D implies that

$$\frac{f+\partial v^*/\partial x^*}{f} = O\left(\frac{D}{D_\infty}\right) <<1, \qquad (2.2.28)$$

which can be obtained by rearranging (2.2.1). Thus the relative vorticity $\partial v^*/\partial x^*$ of the fluid in the channel is nearly equal and opposite to the planetary vorticity f, and absolute vorticity is much less than f.

The cross-channel structure for this case can be found by taking the limit $q \rightarrow 0$ while leaving w, $\hat{d}$, and $\bar{d}$ fixed in (2.2.3) and (2.2.4), or simply solving the cross-channel structure equation (2.1.14) with q=0. In either case one obtains

$$d = \bar{d}(y,t) + \frac{2\hat{d}(y,t)x}{w} - \frac{x^2-(w/2)^2}{2} \qquad (2.2.29)$$

and

$$v = \bar{v}(y,t) - x, \qquad (2.2.30)$$

where $\bar{v} = 2\hat{d}/w$.The velocity therefore varies linearly across the channel while the depth variation is quadratic. It is important to realize that (2.2.29) and (2.2.30) are not uniformly valid over the entire length of channel. If one moves upstream into the reservoir, $\bar{d} \to \infty$ and the assumption that d remain fixed as $q \to 0$ no longer holds. (Thus the term qd in the cross-channel structure equation remains finite and the full depth and velocity profiles (2.2.3) and (2.2.4) would hold.)

The corresponding characteristic wave speeds and Riemann invariants are

$$c_{\pm} = \bar{v} \pm \bar{d}^{1/2} \qquad (2.2.31)$$

and

$$R_{\pm} = \bar{v} \pm 2\bar{d}^{1/2}, \qquad (2.2.32)$$

the same as the values for nonrotating, one-dimensional flow if the v and d are replaced by the average of their wall values. Steepening and rarefacation of disturbances can therefore be treated the same as in the nonrotating limit. However, when a shock (intersection of characteristic curves) occurs, the subsequent development is potentially much different.

Exercises:

1) If q is uniform and non-negative, and the channel flow is semigeostrophic, show that there can be at most one point of flow reversal (v=0 where $\partial v/\partial x \neq 0$) across any channel section.

2) Derive equation (2.2.15) by integrating the continuity equation across the channel. (Be sure to allow for the possibility that w varies with y.)

3) Derive eq. (2.2.13) by writing the semigeostrophic momentum equations along the side walls of the channel. (Hint: use the kinematic boundary conditions $u(\pm w/2, y, t) = v(\pm w/2, y, t)\partial(\pm w/2)/\partial y$. Also make the replacement

$$\left(\frac{\partial v(x,y,t)}{\partial y}\right)_{x=\pm w/2} = \frac{\partial v(\pm w(y)/2, y, t)}{\partial y} \mp \frac{\partial v}{\partial x}\frac{\partial (w/2)}{\partial y}, \qquad (2.2.33)$$

which expresses the y-derivative of v at constant x in terms of the y-derivative of the wall value of v.)

4) Derive (2.2.20-2.2.24) by following the same procedure for obtaining characteristics and Riemann invariants laid out in Exercise 1 of Section 1.3

Figure Captions

2.2.1) Cross sections for attached and separated flows, facing downstream.

2.2.2) Hypothetical states of separation.

2.2.3) The cross section for the case in which the deformation radius L_D based on D_∞ is small compared to the channel width. The inset shows a segment of the flow that has been cut out and placed in an imaginary channel. The new flow is clearly influenced by rotation even though its width is << L_D.