2.10 Transport Bounds

We have seen how difficult it is to calculate the volume flux Q of a hydraulically controlled, rotating flow when idealizations such as uniform potential vorticity and rectangular cross section are relaxed. Although calculations are still possible through numerical means, one might first ask whether any general statements about Q can be made without regard to the details of q and h. An approach developed by Killworth and McDonald (1993) and Killworth (1994) is to seek bounds on Q in terms of simple measures of the upstream flow and the channel geometry. Given some information about the available energy, one simply attempts to find the maximum Q that can be forced through a section of a channel with a given geometry. Although the bounds are formulate without reference to hydraulic control, the result bears a remarkable similarity to hydraulic laws developed in early sections.

The topographic cross section is arbitrary and it is only assumed that the bottom is wetted continuously across, so that the flow occurs in one coherent stream. In contrast to the situation in typical hydraulic models, $B(\psi)$ need not be conserved from one section to the next. However, it most meaningful to imagine that all the streamlines that cross through the section originate in an upstream basin where the maximum B is equal to E. This maximum applies only to those basin streamlines that make their way to the sill section. If non-conservative processes are then limited to a quadratic bottom drag, $B(\psi)$ can only decrease along a particular ψ and the maximum B at any downstream section must be equal to or less than E. These ideas require some modification if the section streamlines that originate far downstream (Section 2.9) or are part of a local closed gyre (Section 2.7). Although the section may lie anywhere, the tightest bound is obtained at the sill, meaning the section with the greatest minimum bottom elevation, h_{min}. The smallest possible value that B (nondimensionally $v^2/2+d+h$) can have occurs when depth d and velocity v are zero at $h=h_{min}$. It follows that

$$h_{\min} \le B(\psi) \le E \qquad (2.10.1)$$

In addition to geostrophy, the chief assumption made is that the potential vorticity of the flow is non-negative.

Now consider a hypothetical flow at the sill section (Figure 2.10.1a). The layer thickness is assumed to go to zero at the edges $x=-a$ and $x=b$ of the stream, but the sidewalls could just as well be vertical. The surface or interface may have segments of negative slope indicating negative velocities. The bound on Q is formulated by first making a sequence of changes to the flow, each of which maintains or increases the original flux.

The first step is to excise any segments of reverse flow along the sidewalls, so that new edges of the current lie at $x=b$ and $x=-a'$ (Figure 2.10.1b). We then place a vertical wall at $x=b$ and, to the left of $x=-a'$, we alter the bottom topography such that it becomes flat and has the elevation $h_{\min}$ (Figure 2.10.1c). Over this flat portion we add a positive region of flow that smoothly brings the layer depth to zero at a point x=-a. The width of the side region is arbitrary. None of the alterations thus far could increase the volume flux. The flux of the altered flow is given by

$$\int_{-a}^{b} dvdx = \int_{-a}^{b} (z_s - h)vdx = \tfrac{1}{2}\left(z_s^2(b) - z_s^2(-a)\right) - \int_{-a}^{b} h\frac{\partial z_s}{\partial x}dx \geq Q \qquad (2.10.2)$$

where $z_s = d + h$.

We next eliminate any interior minima in z_s slicing off the top of the mound of water to the left of any such minima (Figure 2.10.1c,d). The segment extending from x=x_1 to x=x_2 in the figure is therefore replaced by a quiescent region, and the same is done to the left of any remaining minima. To prove that this operation cannot increase the flux note that for the Figure 2.10.1.c flow we have

$$z_s(x_2) = B\left(\psi(x_2)\right) \qquad (2.10.3)$$

and

$$\tfrac{1}{2}v(x_1)^2 + z_s(x_1) = B\left(\psi(x_1)\right). \qquad (2.10.4)$$

Since the surface elevation is the same at the end points

$$B\left(\psi(x_1)\right) - B\left(\psi(x_2)\right) = \tfrac{1}{2}v(x_1)^2 > 0 \qquad (2.10.5)$$

Finally, the previous assumption of positive potential vorticity q along with the relationship $dB/d\psi$=q means that B must increase with ψ and thus

$$\psi(x_2) - \psi(x_1) \leq 0. \qquad (2.10.6)$$

The flux to be removed is must be non-positive.

The end result of this surgery is a water surface rising monotonically to the right, so the stream has positive or zero velocity everywhere across the channel with flux greater than the original flux. A bound on the flux of the altered flow can be formulated beginning with the (2.10.2) definition of transport:

$$\tfrac{1}{2}\left(z_s{}^2(b) - z_s{}^2(-a)\right) - \int_{-a}^{b} h\frac{\partial z_s}{\partial x}dx, \qquad (2.10.7)$$

which for the altered state cannot be less than the original Q. Since $\partial z_s / \partial x$ is non-negative, the integral in the above expression cannot be less than

$$\int_{-a}^{b} h_{\min}\partial z_s/\partial x dx = h_{\min}(z_s(b)-z_s(-a)) = h_{\min}(z_s(b)-h_{\min}), \qquad (2.10.8)$$

It immediately follows

$$\begin{aligned} Q &\le \tfrac{1}{2}\left(z_s^{\,2}(b)-z_s^{\,2}(-a)\right)-\int_{-a}^{b} h\frac{\partial z_s}{\partial x}dx \\ &\le \tfrac{1}{2}\left(z_s^{\,2}(b)-h_{\min}^2\right)-h_{\min}\left(z_s(b)-h_m(a)\right)=\tfrac{1}{2}\left(z_s(b)-h_{\min}\right)^2 \end{aligned} \qquad (2.10.9)$$

Now $z_s(a)$ cannot exceed the maximum value E of the Bernoulli function, and therefore $Q \le \frac{1}{2}\left(E-h_{\min}\right)^2$. Also, if we associate with E an equivalent surface elevation $h_{\min}+\Delta z_E$, then the transport bound becomes $Q \le \frac{1}{2}\Delta z_E^2$ or, in dimensional terms:

$$Q^* \le \frac{g\left(\Delta z_E{}^*\right)^2}{2f}. \qquad (2.10.10).$$

There are a number of examples, all with rectangular cross sections and all with separated sill flow, for which the right-hand side of (2.10.10) gives the exact flux. The first is the case of flow from an infinitely deep and quiescent basin across a sill (Section 2.4). Here $\Delta z_E{}^*$ is just the reservoir head, Δz of (2.4.15), and is a constant over the upstream basin. The same applies for the case of uniform, but finite potential vorticity flow (see 2.6.7b). The largest value of the Bernoulli function at the sill lies along the right-wall streamline and the sidewall depth there is just $\Delta z_R{}^*$. Since the flow stagnates at the right wall, $\Delta z_R{}^*$ equals the required $\Delta z_E{}^*$. We also argued in Section 2.6 that any separated sill flow that stagnates along the right wall is critical and that the corresponding flux is given by (2.6.7). The bound would be exact for this class of flows as well. In all these cases the flow is either positive or zero at the edges, so that no fluid need be excised from the end points (Figure 2.10.1a,b). Also, since the bottom is horizontal, the shaving off of mounds of fluid (Figure 2.10.1c) does not alter the volume flux. The sequence of steps taken to formulate the bound therefore cannot decrease the transport. The cases cited serve notice that the bound (2.10.10) is achievable.

The fact that the bound (2.10.10) is achieved in two examples with rectangular cross-sections suggests that departures from this geometry should tend to reduce the flux. However, if the geometry is sufficiently irregular that the flow becomes divided into two or more streams, then the combined flux can exceed the bound, though (2.10.10) continues to hold for each individual stream (Whitehead, 2003). Simply put, the formation of multiple streams is similar to the existence of multiple openings through which fluid may drain from the basin.

Killworth and McDonald (2003) have shown that the bound can be extended to a fluid with N active layers, each with its own uniform density, and all lying below a deep and inactive upper fluid. The volume flux Q_n in layer n is according to

$$F_n \le \frac{g_n}{2g}\left(E_n - h_{\min}\right)^2, \qquad (2.10.11)$$

where g_n is the reduced gravity and E_n is the maximum Bernoulli function for that layer, the latter defined with the same restriction as the single-layer case.

Figure Captions

Figure 2.10.1 Series of surgical procedures used to alter a given flow (a) in order to produce a simpler flow (d) whose transport is known. The transport cannot be decreased in any step and thus the transport (d) acts as a bound. (Based on a figure in Killworth and MacDonald 1993).

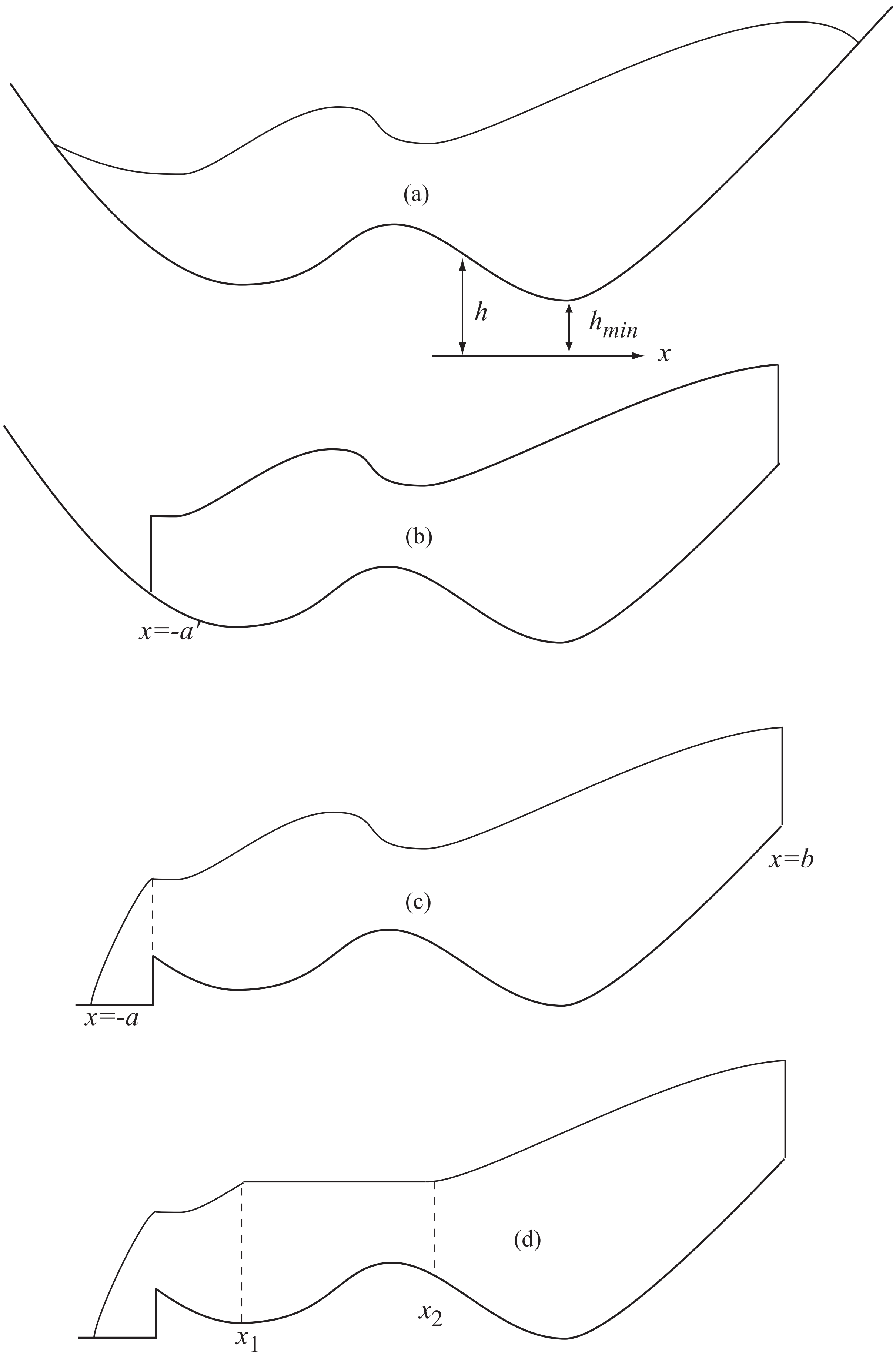

Figure 2.10.1