Chapter 2: The Hydraulics of Homogeneous Flow in a Rotating Channel.

The original models of rotating, hydraulically-driven currents were motivated by observations of deep overflows. The spillage of dense fluid over the sills of the Denmark Strait, the Faroe Bank Channel and other deep passages was suggestive of hydraulic control and early investigators hoped that weir formulas might be of use in estimating the volume transport. To this end the whole volume of dense, overflowing fluid was treated as a single homogeneous layer with reduced gravity. For the Denmark Strait overflow (Figures I6 and I7) this layer typically includes all fluid denser than σ_θ=27.9. Even with this simplification, it was apparent that formulas such as (1.4.12), or their generalizations for nonrectangular cross sections, were not applicable. For one thing, the interface bounding the dense layer no longer has uniform elevation across the channel. Rotation brings more fundamental complications into play, including vortex dynamics and new types of waves. Some of these processes can arise in nonrotating flows but the presence of rotation makes them unavoidable. For these reasons, some of the early investigators questioned whether the concept of hydraulic control was at all applicable to rotating flows.

We shall trace the development of the early theories for rotating-channel flow and show that hydraulic control and many of the other features reviewed in the first chapter remain present in one form or another. However, a number of novel features will arise, some which are not entirely understood. Many are associated with the nonuniformity of the flow in the transverse (across-channel) direction. Examples include the confinement of the volume transport to sidewall boundary layers, the formation of velocity reversals and recirculations, and the separation of the stream from one of the sidewalls. The waves involved in rotating hydraulic control are also different-some are trapped to the side walls of the channels, others are manifested by the meandering of the free edge of a separated flow, still others involve fluctuations of the horizontal velocity but not the free interface.

Our treatment of steady models will be preceded by a discussion of waves. As before, an understanding of these waves in the linear limit is a prerequisite for introduction of the concepts of subcritical and supercritical rotating flow. The nonlinear theory of these waves leads to an understanding of steepening and spreading, the process by which rotating hydraulic jumps, bores and rarefaction waves are formed. Under the usual assumption of gradual variations of the flow along its predominant direction, three types of waves arise. The first is the Kelvin wave, an edge wave closely related to the long gravity waves of the last chapter. The second is the frontal wave, which replaces the Kelvin wave when the edge of the flow is free to meander independently of sidewall boundaries. The third wave is the potential vorticity wave, a disturbance that exists when gradients of potential vorticity exist within the fluid. Nearly all analytical models of deep overflows assume that the potential vorticity is uniform within the flow, thereby eliminating this wave. We will touch on only one model that does not. Coastal currents and surface jets are more dependent on potential vorticity dynamics and will be covered in later chapters.

2.1 The Semigeostrophic Approximation in a Rotating Channel.

We consider homogeneous flows confined to a channel rotating with constant angular speed *f/2* in the horizontal plane. As we will occasionally switch back and forth between dimensional and nondimensional variables, star superscripts are used to denote the dimensional versions. Thus (x^*,y^*) denote cross-channel and along-channel directions, (u^*,v^*) the corresponding velocity components, and (d^*,h^*) the fluid depth and bottom elevation. Provided the scale of x^*- and y^*-variations of d^* are large compared to the typical depth, the shallow water equations continue to apply. The dimensional version of these equations is

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}-fv^*=-g\frac{\partial d^*}{\partial x^*}-g\frac{\partial h^*}{\partial x^*}+F^{(x)*} \qquad (2.1.1)$$

$$\frac{\partial v^*}{\partial t^*}+u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}+fu^*=-g\frac{\partial d^*}{\partial y^*}-g\frac{\partial h^*}{\partial y^*}+F^{(y)*} \qquad (2.1.2)$$

$$\frac{\partial d^*}{\partial t^*}+\frac{\partial(u^*d^*)}{\partial x^*}+\frac{\partial(v^*d^*)}{\partial y^*}=0\,. \qquad (2.1.3)$$

Forcing and dissipation is contained in $\mathbf{F}=(F^{(x)*},F^{(y)*})$. For positive *f*, the channel rotation is counterclockwise looking down from above, as in the northern hemisphere. These equations apply to a homogeneous layer with a free surface or to the active lower layer of a '$1\frac{1}{2}$-layer' or 'equivalent barotropic' model. In the latter, *g* is reduced in proportion to the fractional density difference between the two layers. In such cases the upper boundary of the active layer will be referred to as 'the interface'.

Another version of the momentum equations that will prove useful is obtained by multiplying (2.1.1) and (2.1.2) by d^* and using (2.1.3). The resulting 'depth-integrated' momentum equations are

$$\frac{\partial(d^*u^*)}{\partial t^*}+\frac{\partial}{\partial x^*}\left(d^*u^{*2}+g\frac{d^{*2}}{2}\right)+\frac{\partial(u^*v^*d^*)}{\partial y^*}-fv^*d^*=-gd^*\frac{\partial h^*}{\partial x^*}+d^*F^{(x)*},$$

and

$$\frac{\partial(d^*v^*)}{\partial t^*}+\frac{\partial}{\partial y^*}\left(d^*v^{*2}+g\frac{d^{*2}}{2}\right)+\frac{\partial(u^*v^*d^*)}{\partial x^*}+fu^*d^*=-gd^*\frac{\partial h^*}{\partial y^*}+d^*F^{(y)*}.$$

For large-scale oceanic and atmospheric flows away from the equator and away from fronts and boundary layers, the forcing and dissipation terms and the terms expressing acceleration relative to the rotating earth are generally small in comparison to the Coriolis acceleration. The horizontal velocity for these types of flows is approximately geostrophic or, in the context of our shallow water model,

$$fv^* \cong g\frac{\partial(d^*+h^*)}{\partial x^*} \qquad \text{and} \qquad fu^* \cong -g\frac{\partial(d^*+h^*)}{\partial y^*}.$$

These relations suggest that geostrophic flow moves parallel to lines of constant pressure, with high pressure to the right in the northern hemisphere. This situation was quite different for the flows treated in Chapter 1, in which the velocity is aligned with the pressure gradient and flow is accelerated from high to low pressure. For the deep overflows and strong atmospheric down-slope winds the acceleration of the flow down the pressure gradient is also a characteristic feature, suggesting a departure from the geostrophic balance.

To explore this issue further it is helpful to nondimensionalize variables. Define D and L as a typical depth scale and along-channel length scales and take $(gD)^{1/2}$ as a scale for v^*. This choice is made in the anticipation that the gravity wave speed will continue to be a factor in the dynamics of hydraulically controlled states and that such states will require velocities as large as this speed. A natural scale for t^* is therefore given by $L/(gD)^{1/2}$. As a width scale, we pick $(gD)^{1/2}/f$, which is the Rossby radius of deformation based on the depth scale D. For readers not familiar with the theory of rotating fluids, the Rossby radius of deformation is the distance that a long gravity wave [with speed $(gD)^{1/2}$] will travel in half of a rotation period. Motions with much smaller length scales are generally not influenced by rotation. The Rossby radius appears as a natural width scale for boundary currents and boundary-trapped waves. With these choices, the cross-channel velocity scale $(gD)/fL$ is suggested by balancing the second and third terms in (2.1.3). The dimensionless variables are therefore

$$x=\frac{x^*f}{(gD)^{1/2}},\quad y=\frac{y^*}{L},\quad t=\frac{t^*(gD)^{1/2}}{L}$$

$$v=\frac{v^*}{(gD)^{1/2}},\quad u=\frac{fLu^*}{gD},\quad d=\frac{d^*}{D},\quad h=\frac{h^*}{D},\quad \mathbf{F}=\frac{L\mathbf{F}^*}{gD}. \tag{2.1.4}$$

Substitution into (2.1.1-3) leads to

$$\delta^2\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)-v=-\frac{\partial d}{\partial x}-\frac{\partial h}{\partial x}+\delta F^{(x)} \tag{2.1.5}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+u=-\frac{\partial d}{\partial y}-\frac{\partial h}{\partial y}+F^{(y)} \tag{2.1.6}$$

$$\frac{\partial d}{\partial t}+\frac{\partial (ud)}{\partial x}+\frac{\partial (vd)}{\partial y}=0\,, \tag{2.1.7}$$

where $\delta=(gD)^{1/2}/fL$ is the ratio of the width scale of the flow to *L*: a horizontal aspect ratio.

The limit $\delta\rightarrow 0$ leads to a geostrophic balance in the cross-channel (*x*-) direction but not the along channel direction. The along-channel velocity *v* is geostrophically balanced but the cross-channel velocity *u* is not. The flow in this limit is therefore referred to as *semigeostrophic*. The semigeostrophic approximation requires that variations of the flow along the channel are gradual in comparison with variations across the channel. In particular, the interface must slope steeply across the channel but only mildly along the channel. The along-channel velocity component *v* is therefore directed nearly perpendicular to the pressure gradient. As (2.1.6) suggests, the (weaker) along-channel pressure gradient *does* lead to acceleration in the same direction, but this occurs over a distance *L* large compared to the cross-stream scale δL.

As in most other descriptions of rotating fluids, vorticity and potential vorticity are conceptually and computationally central. For shallow homogeneous flow, the discussion is simplified by the fact that the horizontal velocity is *z*-independent, so that the fluid moves in vertically coherent vertical columns. The vorticity or potential vorticity of the fluid can therefore be discussed in terms of the vorticity of a material column. If the curl of the shallow water momentum equations (i.e. $\partial(2.1.2)/\partial x^*-\partial(2.1.1)/\partial y^*$) is taken and (2.1.3) is used to eliminate the divergence of the horizontal velocity from the resulting expression, the following conservation law for potential vorticity can be obtained:

$$\frac{d^*q^*}{dt^*}=\frac{\mathbf{k}\cdot\nabla^*\times\mathbf{F}^*}{d^*}. \tag{2.1.8}$$

Here $\frac{d^*}{dt^*}=\frac{\partial}{\partial t^*}+u^*\frac{\partial}{\partial x^*}+v^*\frac{\partial}{\partial y^*}$, $\mathbf{k}$ is the vertical unit vector, and

$$q^*=\frac{f+\zeta^*}{d^*}\quad . \tag{2.1.9}$$

The *relative vorticity* $\zeta^*=\frac{\partial v^*}{\partial x^*}-\frac{\partial u^*}{\partial y^*}$ is the vorticity of a fluid column as seen in the rotating frame of reference. The *absolute vorticity* is the total vorticity ζ^*+f of the column. The *potential vorticity* q^* is simply the absolute vorticity divided by the column thickness d^*. If the forcing and dissipation has no curl ($\nabla^*\times\mathbf{F}^*=0$) the potential vorticity of the material column is constant. Conservation of potential vorticity is a consequence of angular momentum conservation; if the column thickness d^* increases, conservation of mass requires the cross-sectional area of the column to decrease, and the column must spin more rapidly to compensate for a decreased moment of inertia.

It is sometimes convenient to represent the potential vorticity q^* as

$$q^* = \frac{f+\zeta^*}{d^*} = \frac{f}{D_\infty},$$

where D_∞ is known as the *potential depth*. Each infinitesimal fluid column has its own time-independent potential depth. To interpret this quantity, consider a column with relative vorticity ζ^* (also $=q^*d^*-f$ by the definition of q^*). Next stretch or squeeze the column thickness d^* to the value f/q^*, so that ζ^* vanishes. This new thickness *is* the potential thickness D_∞. This interpretation is limited by the fact that D_∞ may be negative, making it physically impossible to remove ζ^* by stretching. Most of the applications we will deal with have positive potential depth.

The nondimensional versions of (2.1.8) and (2.1.9) are

$$\frac{dq}{dt} = \frac{\dfrac{\partial F^{(y)}}{\partial x} - \delta \dfrac{\partial F^{(x)}}{\partial y}}{d} \qquad (2.1.10)$$

and

$$q = \frac{1+\dfrac{\partial v}{\partial x} - \delta^2 \dfrac{\partial u}{\partial y}}{d}. \qquad (2.1.11)$$

In the semigeostrophic limit $\delta \to 0$, we have

$$v = \frac{\partial d}{\partial x} + \frac{\partial h}{\partial x} \qquad (2.1.12)$$

and

$$q = \frac{1+\dfrac{\partial v}{\partial x}}{d}. \qquad (2.1.13).$$

These two relations can be combined, yielding an equation for the x-variation in depth

$$\frac{\partial^2 d}{\partial x^2} - qd = -1 - \frac{\partial^2 h}{\partial x^2}. \tag{2.1.14}$$

If q=constant the above equation can easily be solved, reducing the calculation to a two-dimensional problem (in y and t). This situation arises if q is initially constant throughout the fluid and no forcing or dissipation is present.

Another form of the shallow water momentum equations (2.1.1) and (2.1.2) that will prove very helpful is

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + (f + \zeta^*)\mathbf{k} \times \mathbf{u}^* = -\nabla B^* + \mathbf{F}^*. \tag{2.1.15}$$

Here

$$B^* = \frac{u^{*2} + v^{*2}}{2} + g(d^* + h^*) \tag{2.1.16}$$

is the two-dimensional Bernoulli function. Its dimensionless form is $B = B^*/gD = \frac{1}{2}(\delta^2 u^2 + v^2) + d + h$. In the semigeostrophic approximation $B = v^2/2 + g(d+h)$, which is the same expression we used in the one-dimensional flows treated in Chapter 1.

If the flow is steady $(\partial / \partial t^* = 0)$ then the continuity equation (2.1.3) implies the existence of a transport stream function $\psi^*(x,y)$ such that

$$v^* d^* = \frac{\partial \psi^*}{\partial x^*} \text{ and } -u^* d^* = \frac{\partial \psi^*}{\partial y^*} \tag{2.1.17}$$

The total volume transport Q^* is the value of ψ^* on the right-hand edge of the flow (facing positive y^*) minus ψ^* on the left wall. If, in addition, there is no forcing or dissipation ($\mathbf{F}^*$=0) then (2.1.15) can be written

$$\frac{(f + \zeta^*)}{d^*} \mathbf{k} \times \mathbf{u}^* d^* = -\nabla B^*, \tag{2.1.18}$$

or $q^* \nabla \psi^* = \nabla B^*$. Thus the Bernoulli function is conserved along streamlines:

$$B^* = B^*(\psi^*)$$

and

$$q^* = \frac{dB^*}{d\psi^*}. \qquad (2.1.19)$$

This remarkable link between energy and potential vorticity is one of the central constraints used in hydraulic theories for two-dimensional flow. As shown by Crocco (1937), the relationship (2.1.19) holds in more general settings.

In the examples of one-dimensional, steady sill flows presented in Sec. 1.4, the flow in the channel or reservoir upstream of the sill was completely specified by two *constants*, the transport Q^* and the Bernoulli function B^*. In the rotating, two-dimensional generalization of these examples there are three conserved quantities: the *functions* $B^*(\psi^*)$, $q^*(\psi^*)$ and the total volume transport Q^* [equal to the difference between the values of ψ^* on the channel side walls]. As shown by (2.1.19) these three quantities are not independent. If $B^*(\psi^*)$ is specified and the range of ψ^* which actually exists within the channel is given, then $q^*(\psi^*)$ is completely determined within the channel.

Some knowledge of waves is crucial to one's understanding of the hydraulics of rotating-channel flows. In general, it is possible to place the relevant waves in three classes. The first two are the Kelvin and Poincaré waves, both of which depend on the combined effects of rotation and gravity. The third class, potential vorticity waves, can exist in flows with neither gravity nor background rotation. Their dynamics involve vortex induction mechanics that can arise when the potential vorticity of the fluid flow varies spatially. We now discuss some of the linear properties of each class when the waves arise as small perturbations from a resting state in a channel geometry. Nonlinear steepening and other finite amplitude effects will be treated in later sections.

Consider the shallow water equations linearized about a state of rest with d=1. Setting $d = 1 + \eta$ (η<<1), and assuming u<<1 and v<<1, equations (2.1.5-2.1.7) with $\boldsymbol{F}$,h=0 are

$$\delta^2 \frac{\partial u}{\partial t} - v = -\frac{\partial \eta}{\partial x}, \qquad (2.1.20)$$

$$\frac{\partial v}{\partial t} + u = -\frac{\partial \eta}{\partial y} \qquad (2.1.21)$$

and

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad (2.1.22)$$

The corresponding potential vorticity equation, which can be obtained directly from the above or simply by linearization of the nondissipative version of (2.1.8), is

$$\frac{\partial}{\partial t}\left[\frac{\partial v}{\partial x}-\delta^2\frac{\partial u}{\partial y}-\eta\right]=0\,.$$

Integrating this relation in time lead to

$$\frac{\partial v}{\partial x}-\delta^2\frac{\partial u}{\partial y}-\eta=\frac{\partial v_o}{\partial x}-\delta^2\frac{\partial u_o}{\partial y}-\eta_o\,, \qquad (2.1.23)$$

where $(\ )_o$ indicates an initial value. The last equation indicates that the linearized potential vorticity, equal to the relative vorticity $\partial v/\partial x-\delta^2\partial u/\partial y$ plus the stretching contribution $-\eta$, is preserved at each (x,y).

The left hand side of (2.1.23) can be expressed in any of the three variables u,v, or η by using (2.1.20-2.1.22) to eliminate the remaining two. For example the equation for η is

$$\frac{\partial^2\eta}{\partial x^2}+\delta^2\frac{\partial^2\eta}{\partial y^2}-\delta^2\frac{\partial^2\eta}{\partial t^2}-\eta=\frac{\partial v_o}{\partial x}-\delta^2\frac{\partial u_o}{\partial y}-\eta_o\,. \qquad (2.1.24)$$

For an arbitrary initial disturbance the resulting flow will consist of two components. The first is a steady flow whose potential vorticity is given by the potential vorticity of the initial disturbance. This flow is obtained by finding a steady solution to (2.1.24). The second component consists of waves that are generated as a result of the unbalanced part of the initial flow. Individually, these waves are solutions to the homogeneous version of (2.1.24) subject to the boundary condition

$$\frac{\partial^2\eta}{\partial x\partial t}=-\frac{\partial\eta}{\partial y}\quad (x=\pm w/2), \qquad (2.1.25)$$

obtained by evaluating (2.1.20) and (2.1.21) at the sidewalls and eliminating v from the result.

Assuming traveling waves of the form $\eta=\mathrm{Re}\left[aN(x)e^{i(ly-\omega t)}\right]$, where ω is the frequency and l is the longitudinal wave number, one finds two distinct solutions (Gill, 1982 or Pedlosky1987), both of which were discovered by Kelvin (1879). The first is named after Poincaré (1910) and has an oscillatory structure in x:

$$N_n(x)=\cos(k_n x)+b_n\sin(k_n x) \qquad (2.1.26)$$

where $k_n=n\pi/w$, and $b_n=-\omega_n k_n/l$ (n=odd) or $b_n=l/\omega_n k_n$ (n=even). The frequency satisfies the dispersion relation

$$\delta^2\omega^2 = \frac{n^2\pi^2}{w^2} + \delta^2 l^2 + 1\,(n=1,2,3,...),$$

the dimensional form of which is

$$\omega^{*2} = gD(\frac{n^2\pi^2}{w^{*2}} + l^{*2}) + f^2\,, \qquad (2.1.27)$$

where D is the background depth.

Poincaré waves can be better understood by first considering a long gravity wave propagating in an arbitrary direction on an infinite, nonrotating plane. The form of the wave is given by $\eta^* = \mathrm{Re}\left[a^* e^{i(k^*x^*+l^*y-\omega^*t^*)}\right]$, where k^* and l^* represent the wave numbers. The dispersion relation for this wave is given in dimensional terms by (2.1.27) with f=0 and with k^* replaced by the discrete wave number $(n^2\pi^2/w^*)^2$. Next consider a second wave with wave numbers $(-k^*, l^*)$ and therefore having the same frequency as the first wave. If the second wave also has the same amplitude a as the first, a superposition of the two waves leads to a u^* field proportional to $\mathrm{Re}\left[a(e^{i(k^*x^*+l^*y^*-\omega^*t^*)} - e^{i(-k^*x^*+l^*y^*-\omega^*t^*)})\right] = \mathrm{Re}\left[2aie^{i(l^*y^*-\omega^*t^*)}\sin k^*x^*\right]$. Since u^* is zero whenever k^*x^* is an integer multiple of π, the waves satisfy the side-wall boundary conditions in a channel with side walls at $x^* = \pm w^*/2$ provided that k^* is chosen to be $2n\pi/w^*$. These waves are sometimes called oblique gravity waves and their cross-channel structure is said to be *standing*. Poincaré waves are rotationally modified versions of these waves.

The second class consists of edge waves named after Kelvin himself. The cross-channel structure and dispersion relation are given by

$$N_\pm(x) = \frac{\sinh(x) \pm \cosh(x)}{\sinh(\frac{1}{2}w)} \qquad (2.1.28)$$

and

$$\omega_\pm = \pm l\,, \qquad (2.1.29)$$

(or $\omega^*_\pm = \pm(gD)^{1/2} l^*$).

Kelvin waves have a boundary layer structure that becomes apparent when the channel width is much wider than the deformation radius. Taking the limit w>>1 (equivalently $w^* >> (gD)^{1/2}/f$) in (2.1.28) leads to

$$N_+^*(x^*) \propto N^*(\tfrac{1}{2}w^*)e^{(x^*-\frac{1}{2}w^*)f/(gD)^{1/2}}$$

and

$$N_{-}*(x*) \propto N*(-\tfrac{1}{2}w*)e^{-(x*+\frac{1}{2}w*)f/(gD)^{1/2}}.$$

The first solution corresponds to a wave propagating in the positive *y*-direction at speed $(gD)^{1/2}$ and trapped to the wall at $x^*=w^*/2$. The trapping distance is the Rossby radius of deformation based on the background depth D. The other wave moves in the opposite direction and is trapped to the wall at $x^*=-w^*/2$. In the limit of weak rotation, $N_{\pm}$ becomes constant and the Kelvin waves reduce to x-independent, long gravity waves propagating along the channel. A further distinguishing property of linear Kelvin waves is that the cross-channel velocity u is identically zero.

Kelvin waves are nondispersive, meaning that the phase speed c^* does not depend on the wave number l^*. The wave frequency $\omega^*=c^*l^*$ is proportional to l^* and therefore the group velocity $\partial\omega^*/\partial l^*$ is equal to c^*. In Chapter 1, we described the resonance that can occur when a background flow is critical $c^*=0$ with respect to a nondispersive wave. A bottom slope or other stationary forcing introduces disturbance energy that cannot propagate away. Eventually the disturbance becomes large enough to break away, leading to fundamental changes in the upstream flow. We expect that Kelvin waves will play an important role in the upstream influence of rotating channels flows.

Poincaré waves are not admitted under semigeostrophic dynamics, a result that can be shown by taking ($\delta\to 0$) in (2.1.27). The limiting condition $(\frac{n^2\pi^2}{w^2}+1=0)$ cannot be satisfied for real n. Since most simple models of the hydraulics of rotating flow in a channel or along a coast use the semigeostrophic approximation, Poincaré waves do not arise. There are, however, a few models where hydraulic effects arise in unbounded flows (e.g. see Section 3.8). These effects involve Poincaré waves with short wave lengths ($l\to\infty$), for which (2.1.27) reduces to $c=\omega/l=\pm 1$ (or $\omega^*=\pm(gD)^{1/2}$). In this limit the waves behave like nonrotating gravity waves and can be considered nondispersive if propagation is somehow limited to a single direction.

Poincaré and Kelvin waves rely on gravity and a free surface to provide a restoring mechanism. Potential vorticity waves, on the other hand, rely on gradients of potential vorticity within the fluid. One can introduce this effect by modifying the above example to include a lateral bottom slope $\partial h^*/\partial x^* = -s = const$. In addition, the Kelvin and Poincaré waves by placing a rigid lid on the top of the fluid. The resting basic state now contains a potential vorticity gradient associated with the variable depth. If D is the layer thickness at midchannel ($x^*=0$) and if the bottom and surface tilt lead to only slight variations of h^* about D, then the potential vorticity of the ambient fluid is

$$q^* = \frac{f+\partial v^*/\partial x^*}{d^*} = \frac{f}{D+sx^*} \cong \frac{f}{D} - \left(\frac{sf}{D^2}\right)x^*.$$

Under these conditions the flow will support potential vorticity waves with phase speeds given by

$$c^* = -\left(\frac{sf}{w^*}\right)\frac{1}{(n^2\pi^2/w^{*2}) + l^{*2}}, \quad \text{n=1,2,3,}\cdots$$

In the long wave limit ($w^*l^*\rightarrow 0$) the waves are nondispersive:

$$c^* = -\frac{sfw^*}{n^2\pi^2} = \left(\frac{dq^*}{dx^*}\right)\frac{w^*D^2}{n^2\pi^2}, \quad n\text{=1,2,3,}\cdots, \qquad (2.1.30)$$

where w^* is the channel width and $\frac{dq^*}{dx^*} = -\frac{sf}{D^2}$. This example is discussed fully by Pedlosky (1987). For positive s, $\partial q^*/\partial x^* < 0$ and thus higher potential vorticity is found on the left-hand side (facing positive y^*) of the channel. In this case the propagation tendency of the waves is towards negative y.

The waves produced in the last example are called topographic Rossby waves since the background potential vorticity gradient is due to a sloping bottom. More generally, steady flows with nontrivial depth and vorticity distributions generally have potential vorticity gradients and will support potential vorticity waves, although some of these waves may be unstable. The nondispersive character of the long waves is indicative of their ability to transmit upstream influence, an effect that will be demonstrated in later sections.

Exercises

1) *Dissipation and vorticity flux.*
(a) By taking the curl of the shallow water momentum equations (2.1.15) obtain the vorticity equation

$$\frac{\partial \zeta_a^*}{\partial t^*} + \nabla\cdot(\mathbf{u}^*\zeta_a^*) = \mathbf{k}\cdot(\nabla\times\mathbf{F}^*),$$

where $\zeta_a^* = f + \zeta^*$ is the total (or *absolute)* vorticity of a fluid column.
(b) By writing $\mathbf{k}\cdot(\nabla\times\mathbf{F}^*) = -\nabla\cdot\mathbf{J}_n^*$ where $\mathbf{J}_n^*=\mathbf{k}\times\mathbf{F}^*$, rewrite the vorticity equation in the form

$$\frac{\partial \zeta_a^*}{\partial t^*} + \nabla\cdot(\mathbf{u}^*\zeta_a^* + \mathbf{J}_n^*) = 0\,.$$

Thus the quantity $\mathbf{u}^*\zeta_a^* + \mathbf{J}_n^*$ may be interpreted as the total flux of absolute vorticity, the term $\mathbf{u}^*\zeta_a^*$ accounting for the advective part of the flux and the term $\mathbf{J}_n^*$ accounting for the dissipative flux.

(c) By taking the cross product of **k** with the steady version of (2.1.15) obtain the relation

$$\mathbf{k}\times\nabla B^* = \mathbf{u}^*\zeta_a^* + \mathbf{J}_n^*.$$

By comparing this with the relation $\mathbf{k}\times\nabla\psi^* = \mathbf{u}^*$ interpret B^* as a streamfunction for the total vorticity flux. Further show that the derivative of B^* along streamlines gives a vorticity flux that is entirely due to dissipation, whereas the derivative of B^* in the direction normal to streamlines gives a flux that is partly due to dissipation and partly due to advection.

[The ideas developed in this exercise are due in part to Schär and Smith (1993).]

2) Equation (2.1.24) allows a solution to the linear shallow water equations in terms of η. Show that the equivalent equations for u and v are given by

$$\frac{\partial^2 u}{\partial x^2} + \delta^2\frac{\partial^2 u}{\partial y^2} - \delta^2\frac{\partial^2 u}{\partial t^2} - u = -\frac{\partial}{\partial y}\left[\frac{\partial v_o}{\partial x} - \delta^2\frac{\partial u_o}{\partial y} - \eta_o\right]$$

and

$$\frac{\partial^2 v}{\partial x^2} + \delta^2\frac{\partial^2 v}{\partial y^2} - \delta^2\frac{\partial^2 v}{\partial t^2} - v = \frac{\partial}{\partial x}\left[\frac{\partial v_o}{\partial x} - \delta^2\frac{\partial u_o}{\partial y} - \eta_o\right].$$