1.7 Solution to the Initial-Value Problem

The shock joining relations developed in the previous section make it possible to solve the initial-value problem posed by Long's experiment. The term 'solve' is used advisedly here for we do not actually calculate the evolving flow during its early development. Instead, we wait until the various transients have separated from one another, at which point the flow field consists of steady segments separated by isolated bores and rarefaction waves. The formal solution is thereby guided by the experiment. Piecing together the different steady segments of flow allows a solution to be constructed and, more importantly, allows calculation of the obstacle heights required to initiate partial or total blockage or establishment of a hydraulic jump.

Let us continue to view the problem as the adjustment to the sudden introduction of an obstacle into a uniform stream. As noted in the previous section permanent upstream effects (partial blockage) occur when the obstacle is sufficiently high that the initial flow has insufficient energy to ascend the crest or sill, at least according to a steady-state calculation. The critical obstacle height h_c is given by (1.6.1). Figure 1.7.1a shows the developing upstream flow for $h_m > h_c$. The initial flow (v_0, d_0), also the flow far upstream, is approached by a bore moving at speed c_1, downstream of which lies a new steady flow (v_a, d_a). Equations (1.6.4) and (1.6.5) can be used to link the two steady flows across the bore, leading to

$$(v_0 - c_1)d_0 = (v_a - c_1)d_a \qquad (1.7.1)$$

and

$$(v_0 - c_1)^2 d_0 + g d_0^2/2 = (v_a - c_1)^2 d_a + g d_a^2/2\,. \qquad (1.7.2)$$

In addition, conservation of energy and mass connect the sill flow with the steady flow immediately upstream of the obstacle according to

$$v_a d_a = v_c d_c \qquad (1.7.3)$$

and

$$\frac{v_a^2}{2} + g d_a = \frac{v_c^2}{2} + g(d_c + h_m)\,. \qquad (1.7.4)$$

Adding to these the condition that the sill flow is critical,

$$v_c = (g d_c)^{1/2}\,, \qquad (1.7.5)$$

results in five equations for the unknowns c_1, d_a, v_a, v_c, and d_c.

Figure 1.7.2 shows the locations of the different solution regimes in terms of the dimensionless obstacle height h_m/d_0 and initial Froude number F_0. The curve BAE gives the critical obstacle height h_c/d_0 in terms of F_0 and is determined by (1.6.1). To the left of this curve the obstacle is shorter than the critical height and the steady flow established is completely supercritical or subcritical, depending on the initial Froude number. No upstream influence exists. To the right of this curve the flow upstream influence occurs and the flow adjusts to a hydraulically controlled steady state. As we have shown, the upstream influence takes the form of a bore that partially blocks the flow. Note that *any* bore that propagates upstream must decrease the volume transport, a property that can be deduced from conservation of mass (1.7.1) in the form:

$$v_a d_a = v_0 d_0 + c_1(d_a - d_0)\,. \tag{1.7.6}$$

Since $c_1<0$ and $d_o<d_a$ the final transport is less than the initial transport ($v_a d_a < v_0 d_0$) and we say that the flow is partially blocked. Various properties of the solution including the bore speed and final transport can be obtained by solving (1.7.1-1.7.5) and some of these properties are presented in Baines (1995, Figures 2.10 and 2.12).

Further to the right in the diagram curve BC gives the value of h_m/d_0 needed to completely block the flow. This curve is determined by setting $v_a=0$ and $d_a=h_m$ in (1.7.1) and (1.7.2) and eliminating c_1 between the two equations, resulting in

$$F_0 = \left(\frac{h_m}{d_0} - 1\right)\left(\frac{1 + h_m/d_0}{2h_m/d_0}\right)^{1/2}. \tag{1.7.7}$$

The wedge shaped region EAF in Figure 1.7.2 represents special initial conditions for which two final steady states are possible, depending on how the experiment is performed. To understand this, first consider the curve AF which indicates upstream values of F_d and h_m/d_0 where a stationary bore is possible in the flow approaching the obstacle. For these upstream conditions the steady flow near the obstacle can either be entirely supercritical, or have the stationary bore upstream of the obstacle leading to hydraulically controlled flow over the obstacle. The curve is obtained by setting $c_1=0$ in (1.7.1)-(1.7.5), resulting in

$$\frac{h_m}{d_0} = \frac{(8F_0^2+1)^{3/2}+1}{16F_0^2} - \frac{1}{4} - \frac{3}{2}F_0^{2/3}$$

If one performs the original version of Long's experiment in EAF, no upstream bore is found and the final steady state is the entirely supercritical flow, as in the upper left inset of Figure 1.7.2. The other alternative can be realized by starting with an obstacle of height $h_m>h_c$ (to the right of curve AE) and waiting until a hydraulically controlled flow is established. If the obstacle height is then reduced to a value in the region EAF, the hydraulically controlled solution will persist. A numerical demonstration of the process is shown in Figure 1.7.3. In frame (a) the obstacle of height $h_m>h_c$ is introduced, exciting an upstream bore. In (b) the obstacle has been lowered to a height $h_m<h_c$ such that h_m/d_o lies in region EAF. Here the bore continues to propagate upstream and the flow over the sill remains critical. Next the obstacle is lowered to point to the *left* of curve AF, causing the bore to reverse directions and move downstream towards the obstacle (c). Eventually the bore moves past the obstacle (d) and a supercritical state is achieved.

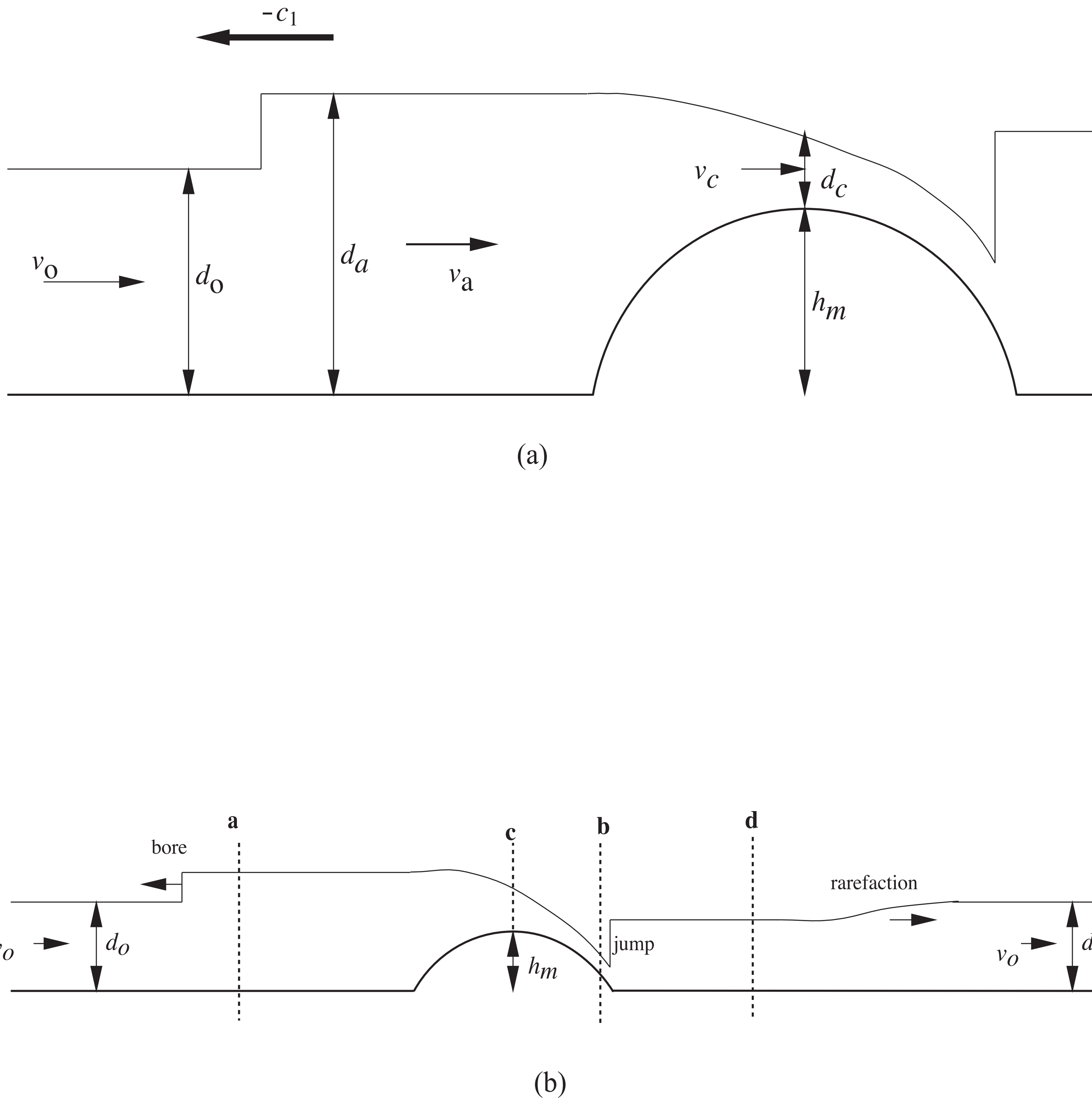
Finally, the curve AD separates flows with and without hydraulic jumps attached to the downslope of the obstacle. For initial conditions lying below AD the jump would be positioned on the down slope of the obstacle. Above AD the jump would move downstream leaving supercritical flow behind. On AD the hydraulic jump will become stationary right at the foot of the obstacle, as shown in Figure 1.7.1b. In order to find the obstacle height at which this last situation occurs one must piece together the segments of steady flow shown at sections 'a', 'b', 'c' and 'd' in the figure. There are 10 unknowns, including the depths and velocities at these four sections, the upstream bore speed, and the obstacle height. Four constraints are provided by the shock joining conditions across the bore and hydraulic jump. Also volume transport and energy (Bernoulli function) are conserved between sections 'a' and 'c' and between 'c' and 'b', providing 4 additional constraints. The final two constraints are provided by the condition of critical flow at the sill and the conservation of $R_{-}=v_o-2(gd_o)^{1/2}$ across the rarefaction wave that moves downstream of the obstacle. The algebra involved in the determination of the obstacle height from these ten relations is formidable.

Figure Captions

1.7.1 The various transients generated by the introduction of an obstacle into a uniform stream when h_o exceeds the critical value h_c for upstream influence.

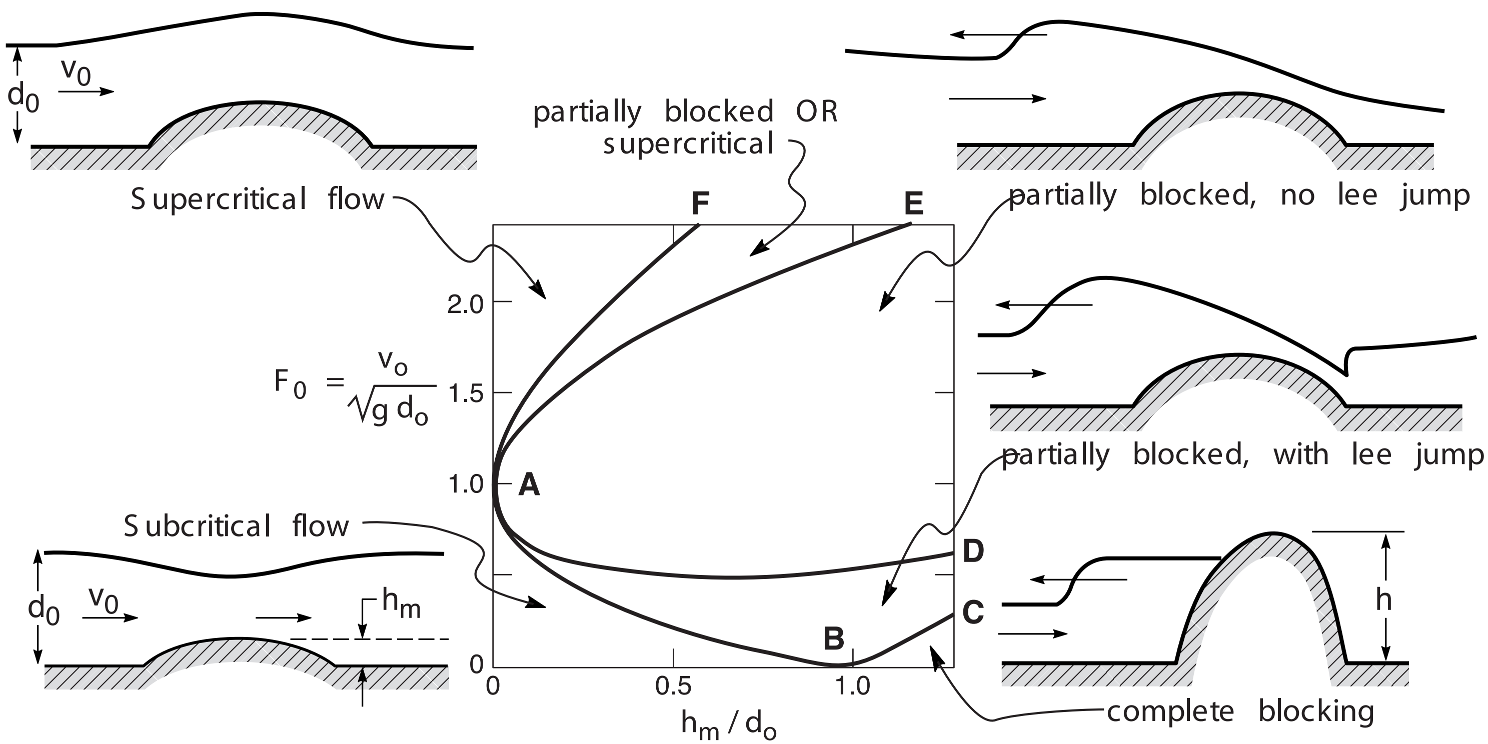
1.7.2 The various asymptotic regimes of the Long-type initial-value experiment in terms of the initial conditions.

1.7.3 Evolution of a shallow stream when an obstacle of height h_m is introduced in a moving stream of depth d_o, such that the initial conditions lie to the right of curve AE in Figure 1.7.2. The obstacle height is then lowered so that h_m/d_o lies in region EAF (frame *b*). Later h_m/d_o is decreased so as to lie to the left of curve AF (*c* and *d*). The scale of the vertical axis z/d_o varied from frame to frame, but the intersection of the surface with the left edge always lies at h_m/d_o=1.



-c_1
v_c
d_c
v_o
d_o
d_a
v_a
h_m
(a)
a
c
b
d
bore
rarefaction
jump
v_o
d_o
h_m
v_o
d_o
(b)

Figure 1.7.1



Supercritical flow
partially blocked OR supercritical
partially blocked, no lee jump
partially blocked, with lee jump
Subcritical flow
complete blocking
d_0
v_0
h_m
h
F
E
A
D
C
B
$F_0 = \frac{v_o}{\sqrt{g\,d_o}}$
h_m / d_o
2.0
1.5
1.0
0
0.5

Figure 1.7.2

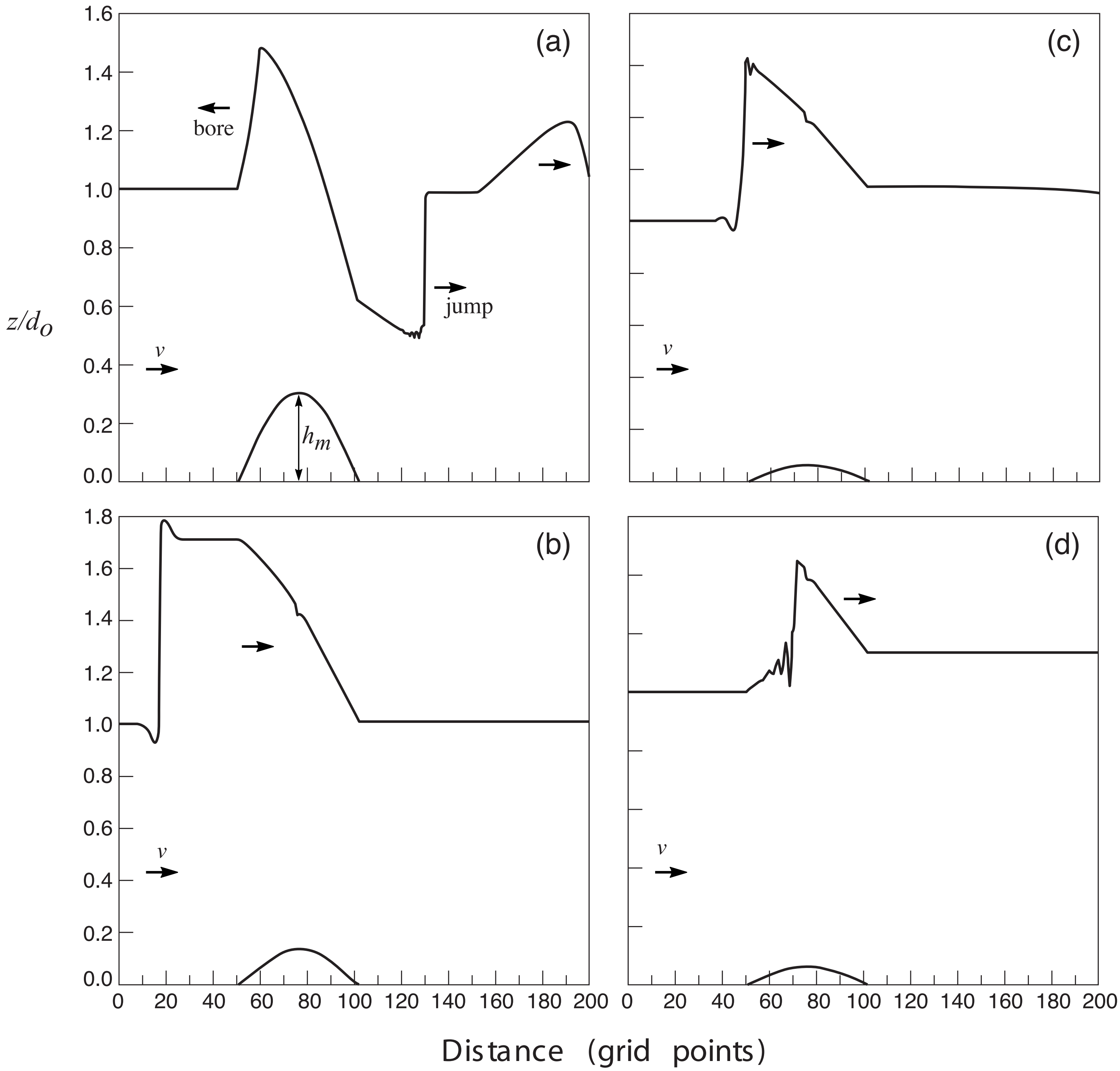

(a)
(b)
(c)
(d)
bore
jump
v
h_m
z/d_O
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
1.8
0 20 40 60 80 100 120 140 160 180 200
Distance (grid points)

Figure 1.7.3