1.4 The hydraulics of steady, homogeneous flow over an obstacle.

We are now in a position to review one of the simplest examples of hydraulic behavior: that of a steady, homogeneous, channel flow passing over an obstacle or through a width contraction. The channel will continue to have rectangular cross-section with gradually varying width *w* and bottom elevation *h*. The steady shallow-water equations governing such a flow are

$$v\frac{\mathrm{d}v}{\mathrm{d}y}+g\frac{\mathrm{d}d}{\mathrm{d}y}=-g\frac{\mathrm{d}h}{\mathrm{d}y} \qquad (1.4.1)$$

and

$$d\frac{\mathrm{d}v}{\mathrm{d}y}+v\frac{\mathrm{d}d}{\mathrm{d}y}=-vdw^{-1}\frac{\mathrm{d}w}{\mathrm{d}y}. \qquad (1.4.2)$$

Before considering specific solutions it is worth noting some general properties which follow from (1.4.1) and (1.4.2). Elimination of $\frac{\mathrm{d}v}{\mathrm{d}y}$ between the two equations leads to

$$\frac{\mathrm{d}}{\mathrm{d}y}(d+h)=\frac{F_d^2\left[\frac{\mathrm{d}h}{\mathrm{d}y}-\frac{d}{w}\frac{\mathrm{d}w}{\mathrm{d}y}\right]}{F_d^2-1}, \qquad (1.4.3)$$

where $F_d^2=v^2/gd$. This expression gives the rate of change of the free surface elevation *d*+*h* along the channel in terms of the rate of change of the geometrical parameters *w* and *h*. Positive values of the numerator on the right hand side are associated with constrictions of the geometry due to increasing bottom elevation or to decreasing width. If the flow is subcritical ($F_d^2<1$) the denominator is negative and the fluid depth decreases in response to constrictions. This is the situation when flow in a reservoir approaches a dam. Supercritical flow ($F_d^2>1$) experiences increases in depth in response to constrictions, a situation that can be observed in river rapids; the free surface rises where the current passes over a stone on the river bottom. Finally, critical flow ($F_d^2=1$) with a finite free-surface slope[1] requires that the rate of contraction be zero: $\frac{\mathrm{d}h}{\mathrm{d}y}=\frac{d}{w}\frac{\mathrm{d}w}{\mathrm{d}y}$. This condition holds where $\frac{\mathrm{d}h}{\mathrm{d}y}$ and $\frac{\mathrm{d}w}{\mathrm{d}y}$ are both zero, as at the crest or sill of an obstacle in a constant-width channel, at a narrows of a constant-elevation channel, or at a section where the minimum width coincides with a sill. Critical flow can also occur where increases in bottom elevation coincide with increases in width, or vice versa, such that the rate of geometrical constriction is zero according to the above criterion. Locations of critical flow are called *critical* or *control sections.*

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[1] There *are* physically meaningful solutions having infinite free surface slopes. These 'hydraulic jumps' will be discussed in Section 1.6.

There is a simple reason explaining why critical flow can occur only at special locations. The forcing of the flow is entirely due to topographic and side-wall variations and is clearly stationary. Where the flow supports stationary free disturbances (i.e. where the flow is critical) this forcing gives rise to a resonant response. The resonance, which corresponds to the vanishing of the denominator in (1.3.3), is prevented only if the forcing, as measured by numerator, is zero. In other words, the net rate of contraction or expansion of the channel geometry must be zero where the flow is critical, else the forcing would give rise to a growing disturbance that would alter the flow.

A second consideration in the interpretation of critical flow involves the nondispersive character of the waves in question. Let us temporarily leave the shallow water model, in which all waves are considered infinitely long in comparison with the fluid depth, and consider the phase speed of a linear wave of finite length $2\pi/k$. For a wave propagating on a uniform current of depth D and velocity V, the phase speed is given by

$$c = V \pm [(g/k)\tanh(kD)]^{1/2} \simeq V \pm (gD)^{1/2}\left[1 - \frac{(kD)^2}{6} + \cdots\right]$$

and group speed by

$$\frac{\partial(ck)}{\partial k} \simeq V \pm (gD)^{1/2}\left[1 - \frac{(kD)^2}{2} + \cdots\right].$$

The latter gives the speed at which disturbance energy propagates. In the shallow-water limit ($kD \to 0$) the waves are nondispersive; phase and energy propagate at the same speed $(gD)^{1/2}$. When long waves are resonantly forced, neither the phase nor energy is able to escape. Energy accumulates and the disturbance grows to the point where nonlinear dynamics intercede. For waves with finite wavelength ($kD>0$) the phase and energy propagate at different rates. Stationary waves may be excited by topographic variations, but their growth is limited by the fact that energy is able to propagate away from the region of excitation. This process is responsible for the generation of 'lee' waves, which are stationary waves of finite length. In summary, the resonant excitation of free disturbances by topography leads to consequences most profound when the waves are nondispersive.

Now consider the class of steady flows that arises when w is constant and the channel contains a single obstacle of height h_m, as shown in Figure 1.1.1c. Normally, computation of the flow is carried out using statements of conservation of energy:

$$\frac{v^2}{2} + gd + gh = B\,, \tag{1.4.4}$$

and conservation of volume transport:

$$vdw = Q\,, \qquad (1.4.5)$$

obtained through integration of (1.4.1) and (1.4.2) with respect to *y*. The constants *B* and *Q* represent the Bernoulli function (or 'head') and volume flow rate. The former is the energy per unit mass of a fluid parcel and is always independent of depth in our shallow-water system. For steady flow (1.4.4) indicates that *B* is independent of *y* as well.

Solutions for the fluid depth can be found by eliminating *v* between (1.4.4) and (1.4.5), with the result:

$$\frac{Q^2}{2w^2d^2} + gd = B - gh. \qquad (1.4.6)$$

The quantity *B-gh* , sometimes called the *specific energy,* is the total energy minus the potential energy provided by the bottom elevation. The specific energy represents the intrinsic energy of the flow. Changing the bottom elevation alters the specific energy, forcing the depth to adjust to new values. One approach to the steady flow problem is to imagine that *Q* and *B* are fixed conditions set far upstream of the obstacle, so that one can march along the channel, using (1.4.6) to calculate the depth at each *y*. Of course (1.4.6) is cubic and there may be more than one value of *d* for each *h,* a situation that can be clarified by plotting *h* (or, more conveniently, *B-gh*) as a function of *d*. To make such a plot as general as possible, we first nondimensionalize (1.4.6) by dividing by *gD*, where *D* is a scale chosen for convenience as $(Q/w)^{2/3}g^{-1/3}$. Then let $\tilde{d} = d/D$, $\tilde{h} = h/D$, and $\tilde{B} = B/gD$. The resulting relation

$$\frac{1}{2(\tilde{d})^2} + \tilde{d} = \tilde{B} - \tilde{h} \qquad (1.4.7)$$

is plotted in Figure 1.4.1a. To construct a solution at a particular *y*, first set the normalized energy $\tilde{B}$ and note the bottom elevation $\tilde{h}$ at that *y*. This fixes a point on the vertical axis, $\tilde{B} - \tilde{h}$. If the latter is >3/2, two possible solutions $\tilde{d}$ exist corresponding to the right- and left-hand branches of the curve. There is one solution for $\tilde{B} - \tilde{h}$ =3/2 corresponding to the minimum of the curve. Here

$$\frac{\partial}{\partial \tilde{d}}\left(\frac{1}{2\tilde{d}^2} + \tilde{d}\right) = -\frac{1}{\tilde{d}^3} + 1 = 0\,. \qquad (1.4.8)$$

and therefore $\tilde{d}$=1 or $d=(Q/w)^{2/3}g^{-1/3}=(vd)^{2/3}g^{-1/3}$ or, finally, $F_d^2 = v^2/gd = 1$. The solution at the minimum of the curve therefore corresponds to critical flow. The left hand branch of the curve is associated with smaller depths and, since the flow rate is the same, larger velocities. Therefore the left-hand branch corresponds to supercritical ($F_d^2 > 1$) flow, while the right-hand branch corresponds to subcritical ($F_d^2 > 1$) flow. Constructing a solution requires choosing between the right- and left-hand branches, and there is nothing thus far to suggest how this choice is to be made.

Ignoring for the moment the dilemma of being forced to choose between two possible solutions, we arbitrarily begin on the subcritical branch of the solution curve. To construct a solution over a particular obstacle, begin at the section ($y=y_1$) upstream of the obstacle, where $\tilde{h}=0$. To find the depth $\tilde{d}$ at this section, go to Fig. 1.4.1a and read off the value $\tilde{d}(y_1)$ corresponding to $\tilde{B}-\tilde{h}(y_1)=\tilde{B}$. Next, move forward along the channel to where the bottom elevation $\tilde{h}$ begins to increase. The new solution occurs at a lower value of $\tilde{B}-\tilde{h}$, requiring one to trace the solution curve to the lower left, as suggested by the arrows drawn above the curve. The depth therefore decreases as we move up the upstream face of the obstacle. We can continue in this way until we reach the obstacle's sill $y=y_s$ and $\tilde{h}=\tilde{h}_m$. If the sill elevation $\tilde{h}_m$ is sufficiently small that $\tilde{B}-\tilde{h}_m>3/2$ then the minimum of the solution curve is not reached and the depth $\tilde{d}(y_s)$ will exceed the critical depth $\tilde{d}=1$. Continuing further downstream causes one to retrace the solution curve in Figure 1.4.1a as indicated by the arrows drawn underneath the curve. When the obstacle is passed ($y=y_2$), the depth returns to its upstream value and the solution is subcritical at all points between. The free surface will appear as shown in Figure 1.4.1b. Note that the solution is symmetrical in the sense that equal bottom elevations upstream and downstream of the sill see the same fluid depth. If the left-hand branch of the solution curve had been traced for the same topographic variations, a symmetrical supercritical solution with $\tilde{d}$ *increasing* over the obstacle would have resulted. We will refer to these solutions as pure subcritical or pure supercritical flow.

Next suppose that $\tilde{B}-\tilde{h}(y_s)=3/2$ so that the minimum of the solution curve is just reached at the sill. If the approach to the sill had been along the subcritical solution branch, there are two choices in constructing the downstream solution. First, one might retrace the subcritical solution branch as in the above example. Second, one might continue onto the supercritical branch and obtain an asymmetrical solution with the fluid depth *decreasing* in the downstream direction. The second situation is depicted in Figure 1.4.2. As it turns out, the first of these scenarios results in a solution with a discontinuity in the free surface slope at the sill and can be ruled out. The proof is explored in in Exercise 1 below. The preferred solution is thus the one with subcritical flow upstream, supercritical flow downstream, and critical flow at the sill. This type of flow, which resembles flow over a dam or spillway, is often described as being *hydraulically controlled* for reasons that will soon become apparent. For now, we simply note that small disturbances generated downstream of the sill are unable to propagate upstream. The subcritical flow upstream of the sill is therefore immune to weak forcing imposed downstream of the sill.

It is also natural to ask what happens when $\tilde{B}-\tilde{h}(y_s)<3/2$, in which case no solution exists at the obstacle's crest. This situation occurs when the energy $\tilde{B}$ is insufficient to allow the fluid to surmount the obstacle. For example, one might start with the hydraulically controlled flow described above and raise the elevation of the sill a small amount, creating a small region about the sill for which no steady solution would exist. Under these conditions a time-dependent adjustment must take place leading to a new upstream flow with a larger $\tilde{B}$ $(=B(gQ/w)^{-2/3})$. As we shall discuss in Section 1.6, the value of $\tilde{B}$ is altered the minimal amount required to allow a smooth steady state. The new steady state is therefore hydraulically controlled. Note that change can be effected by increasing the Bernoulli function B or by decreasing the transport Q.

So far, we have constructed various solutions by fixing the energy parameter $\tilde{B}$ and varying the sill height of the topography. For a different perspective, consider the family of solutions obtained for a fixed topography by varying $\tilde{B}$. Figure 1.4.3 shows the free surface profiles of the solutions over an obstacle of unit dimensionless height. Each value of $\tilde{B}$ shown is associated with two solutions, one having supercritical and one subcritical flow upstream of the obstacle. For $\tilde{B}$>2.5 the curves are the symmetrical, purely sub- or supercritical solutions discussed before. For $\tilde{B}$=2.5 the two solutions intersect each other at the sill; one of these is the hydraulically controlled solution discussed above and the other, its mirror image, is supercritical upstream and subcritical downstream of the sill. For $\tilde{B}$<2.5 the solutions cease to be continuous across the sill.

The asymmetrical solution that is supercritical upstream and subcritical downstream is difficult to reproduce in the laboratory, most likely due to an instability that occurs near the sill. A heuristic demonstration of the instability can be made through consideration of a small-amplitude, long-wave disturbance introduced about the sill (Figure 1.4.4). This disturbance may be synthesized using the two linear wave modes of the system, which propagate at speeds $v-(gd)^{1/2}$ and $v+(gd)^{1/2}$. Since the slower wave propagates in the downstream direction *upstream* of the sill and in the upstream direction *downstream* of the sill, any energy carried by this wave will become continuously focused and amplified at the sill.

Now picture a steady flow that is critical at the sill, and consider the effect of increasing h_m by a small amount. The original flow no longer has sufficient energy to surmount the obstacle. At the sill, the temporal increase in h_m results in the resonant excitation of a disturbance. The disturbance remains stationary only as long as its amplitude remains small. Eventually it becomes large enough to break free and move upstream, establishing a new value of $\tilde{B}$. This process is known as *upstream influence* and will be illustrated further in Sections 1.6 and 1.7. Upstream influence and *hydraulic control* both refer to the regulation of the upstream conditions.

If it is known in advance that the flow is hydraulically controlled, one can derive a transport relation or 'weir formula' to measure the volume transport. Weir formulas make use of the criticality of the flow at the sill to express the transport in terms of the fluid depth at some convenient section. This circumvents the need to directly measure the velocity, which is normally more difficult than measuring the depth. As an example, consider a reservoir of width w_s that drains across a sill of width w_s. The condition of criticality $v_c=(gd_c)^{1/2}$ at the sill can be used to write the volume transport $Q=vdw$

as $g^{1/2}d_c^{3/2}w_s$, so that Q is determined by measurement of fluid depth at the sill. It is often more convenient to determine Q by a depth measurement in the reservoir itself, say at y=y_1 in Figure 1.4.2b. Equating energy and volume transport at the reservoir and sill sections leads to

$$\frac{v_1^2}{2}+gd_1=\frac{v_c^2}{2}+gd_c+gh_m\,, \tag{1.4.9}$$

and

$$v_1d_1w_1=v_cd_cw_s. \tag{1.4.10}$$

Eliminatation of v_1 and use of the critical condition leads to

$$\frac{3}{2}\left(\frac{gQ}{w_s}\right)^{2/3}-\frac{Q^2}{2d_1^2w_1^2}=g(d_1-h_m)=g\Delta z\,, \tag{1.4.11}$$

where Δz is the difference in elevation between the free surface at y=y_1 and the sill. Measuring Δz and d_1 allows Q to be determined from the above formula. In many cases, y_1 can be chosen as a location where the depth d_1 or width w_1 is sufficiently large that the second term on the left-hand side can be neglected. In this case the simplified relation

$$Q=\left(\tfrac{2}{3}\right)^{3/2}w_sg^{1/2}\Delta z^{3/2}\,. \tag{1.4.12}$$

can be used as an approximation. For the reduced gravity analog of the current model, weir formulas would permit calculation of volume transport based on the interface elevation upstream of the critical section.

Although equation (1.4.11) was motivated by the practical necessity of measuring volume flux, it also provides particular view of the concept of hydraulic control. In a controlled state, there is a fixed relationship between the parameters governing the flow, in this case Q and Δz, and the geometrical parameters describing the control section, in this case the sill height h_m. For non-controlled solutions no such relationship exists, implying that one has more freedom to manipulate these flows. We will elaborate on this point further. In addition, it is easy to show that critical flow is associated with a number of variational properties of steady flows. For fixed Q and h, the energy B of the flow is minimized over all possible values of d, which can be seen from Figure 1.4.3 or from taking $\partial B/\partial d=0$ in (1.4.6). Similarly, it can be shown that for fixed B and h, Q is maximized over all d. Hydraulically controlled solutions thus minimize the energy available at a given volume transport, which is consistent with the idea that the fluid is barely able to surmount the obstacle. In addition, these solutions tend to maximize the transport available at a given energy level.

Exercises

1) Using l'Hôpital's rule in connection with (1.4.3), derive an expression for the slope of the free surface at a sill under critical flow conditions. You may assume that the channel width is constant. From the form of the result, show that critical flow can occur over a sill ($\mathrm{d}^2h/\mathrm{d}y^2 < 0$) but not a trough ($\mathrm{d}^2h/\mathrm{d}y^2 > 0$). Also show that a solution passing through a critical state at a sill generally cannot be subcritical (or supercritical) both upstream and downstream of the sill without incurring a discontinuity in the slope of the free surface.

2) Construct a nondimensional solution curve akin to that of Figures 1.4.1 or 1.4.2 for the case of a channel of constant *h* but variable *w*.

3) Consider a 100m-deep reservoir that is drained by spillage over a dam of height 99m. Both the dam and reservoir have width *w*=100 meters. Approximate
(a) The volume flow rate from the reservoir.
(b) The depth and velocity at the sill of the dam.
(c) If you used an approximation to answer (a) estimate the error.

4) Suppose that the channel has a triangular cross-section. The width *w* at any *z* is given by

$$w(z) = 2\alpha(z - h(y))$$

where *h*(*y*) is the elevation of the bottom apex. The along channel velocity *v* and surface elevation are independent of *x*.

(a) Taking *h*=constant, find the speed of long surface gravity waves in the channel.
(b) For steady flow, formulate a solution curve like that of Figure 1.4.1a or 1.4.2a showing how the fluid depth varies with *h*.
(c) Show that the condition obtained at the extremum of the curve is the critical condition found in part (a).
(d) Write down the weir formula for the case in which the fluid originates from an infinitely deep reservoir and spills over a sill.

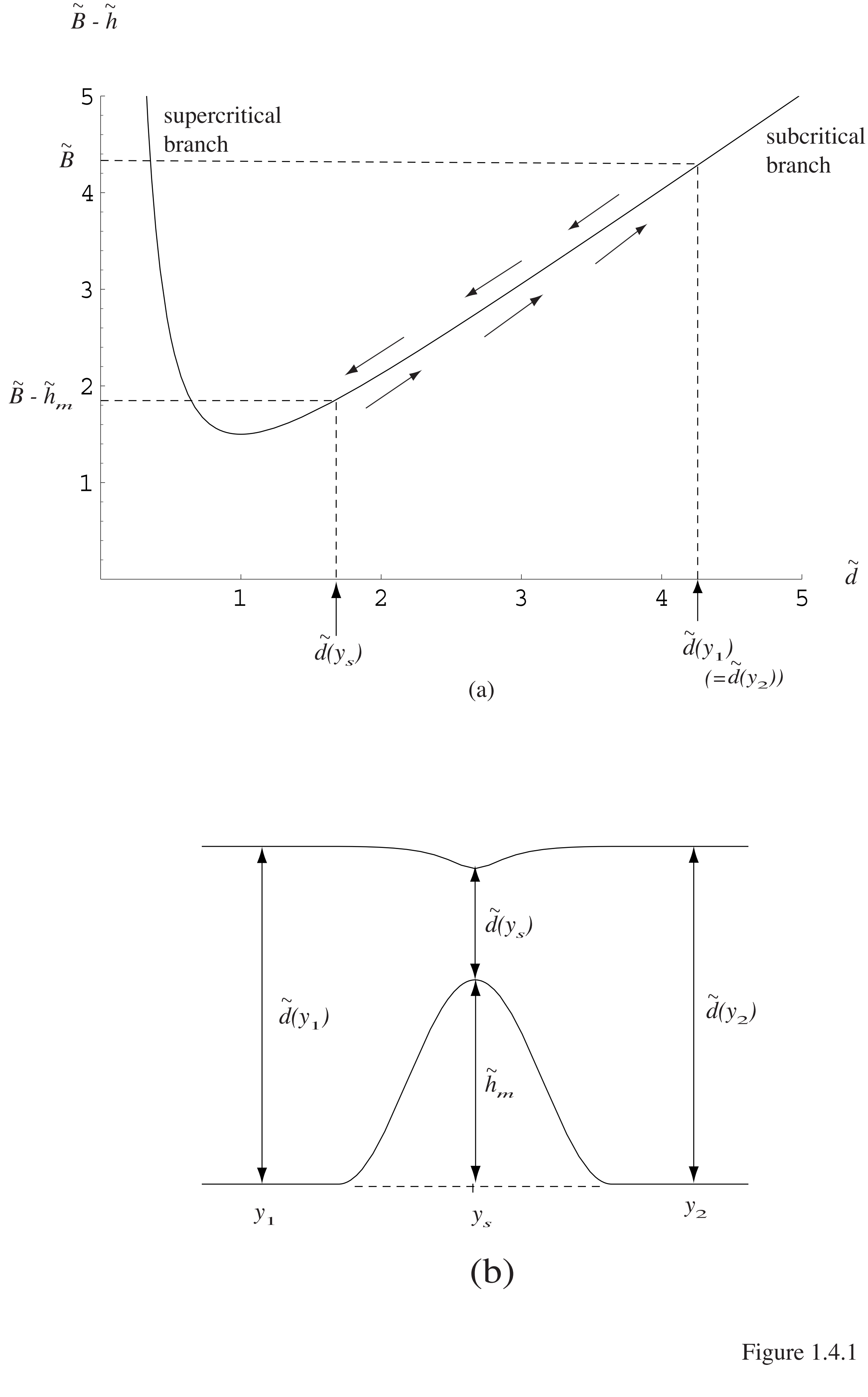
Figure Captions

Figure 1.4.1. The top frame shows a curve of $\tilde{B} - \tilde{h}$ as a function of $\tilde{d}$, as given by equation 1.4.7. The arrows on the subcritical branch of the curve show how one would trace along that curve in order to get the solution shown in the lower frame.

Figure 1.4.2. Similar to 1.4.1 except that a hydraulically controlled solution is traced following the arrows.

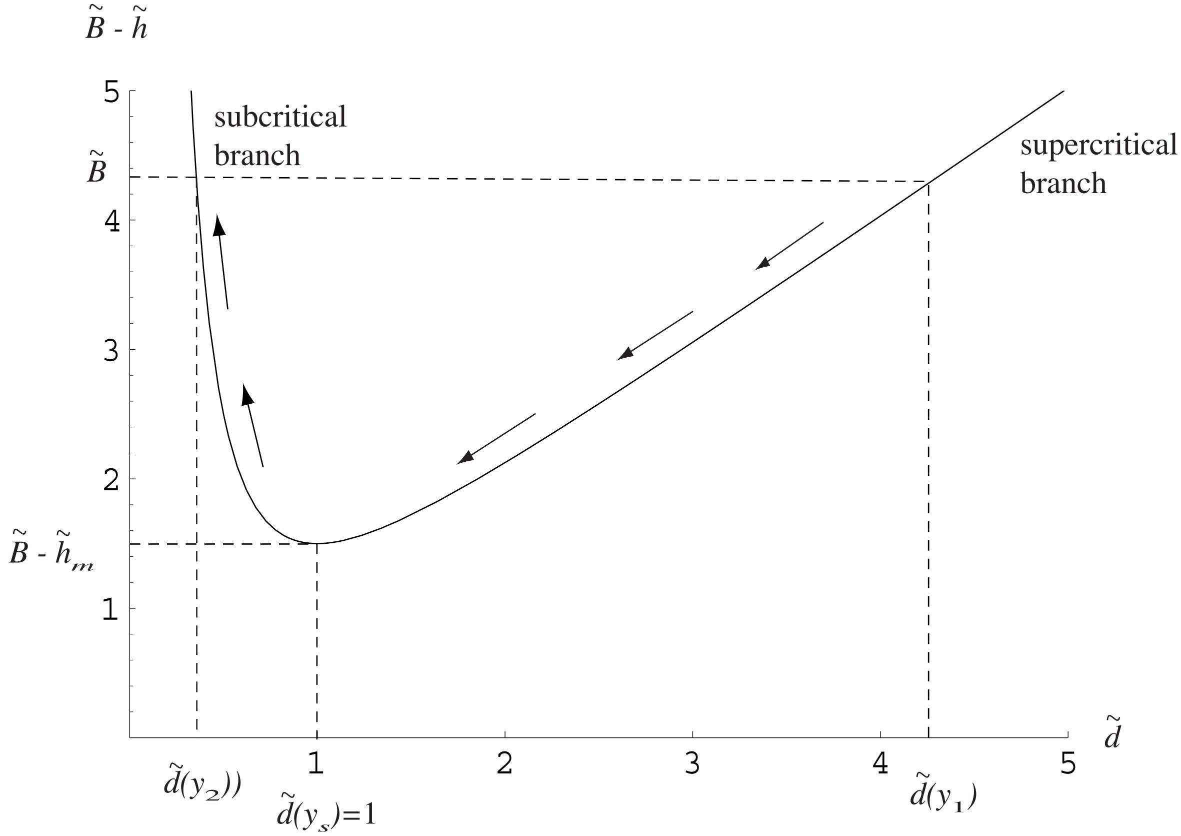
Figure 1.4.3 A family of solutions to (1.4.7) for a fixed obstacle and various values of $\tilde{B}$.

Figure 1.4.4. A supercritical-to-subcritical and the direction of propagation for the slower long wave. Disturbance energy contained in this mode would converge at the sill leading to an instability.

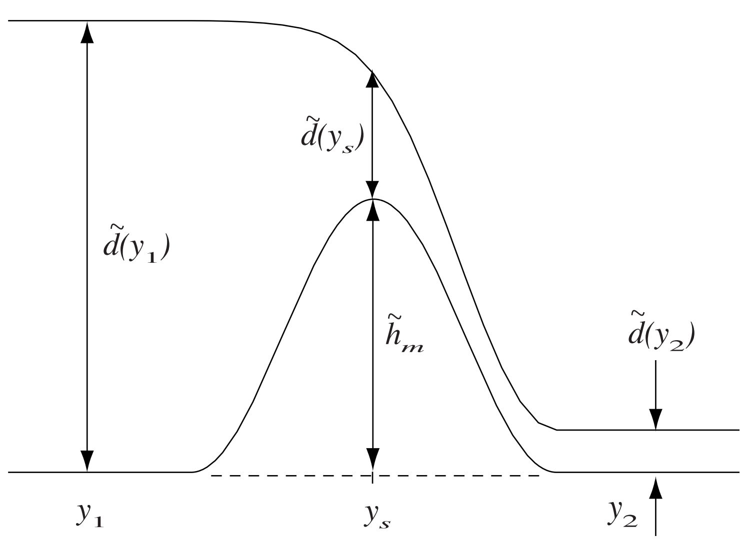


$\tilde{B}$ - $\tilde{h}$
supercritical
branch
subcritical
branch
$\tilde{B}$
$\tilde{B}$ - $\tilde{h}_m$
$\tilde{d}$
5
4
3
2
1
1
2
3
4
5
$\tilde{d}(y_s)$
$\tilde{d}(y_1)$
$(=\tilde{d}(y_2))$
(a)
$\tilde{d}(y_s)$
$\tilde{d}(y_1)$
$\tilde{d}(y_2)$
$\tilde{h}_m$
y_1
y_s
y_2
(b)

Figure 1.4.1



(a)



(b)

Figure 1.4.2

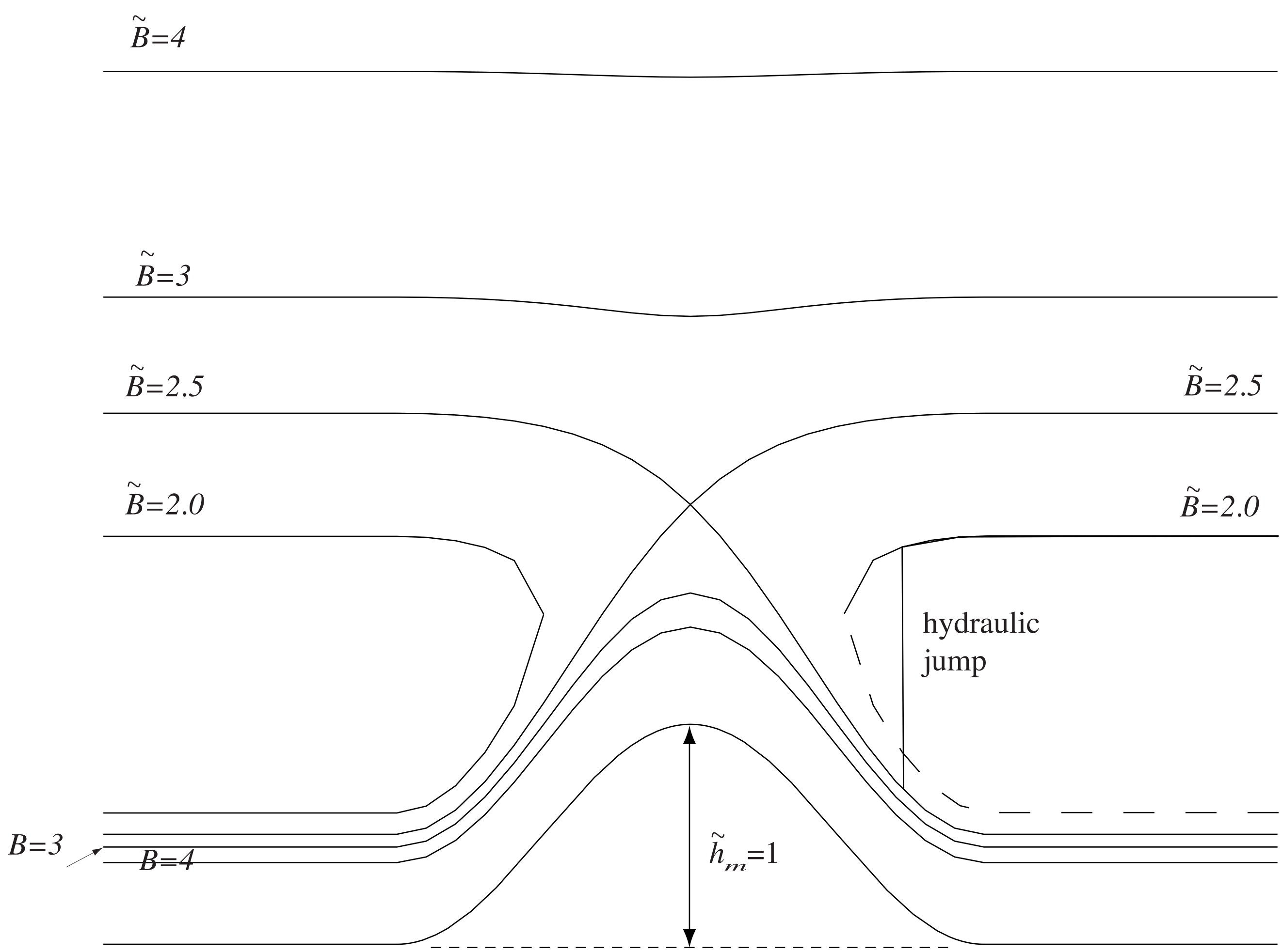

$\tilde{B}$=4
$\tilde{B}$=3
$\tilde{B}$=2.5
$\tilde{B}$=2.5
$\tilde{B}$=2.0
$\tilde{B}$=2.0
hydraulic
jump
B=3
B=4
$\tilde{h}_m$=1

Figure 1.4.3

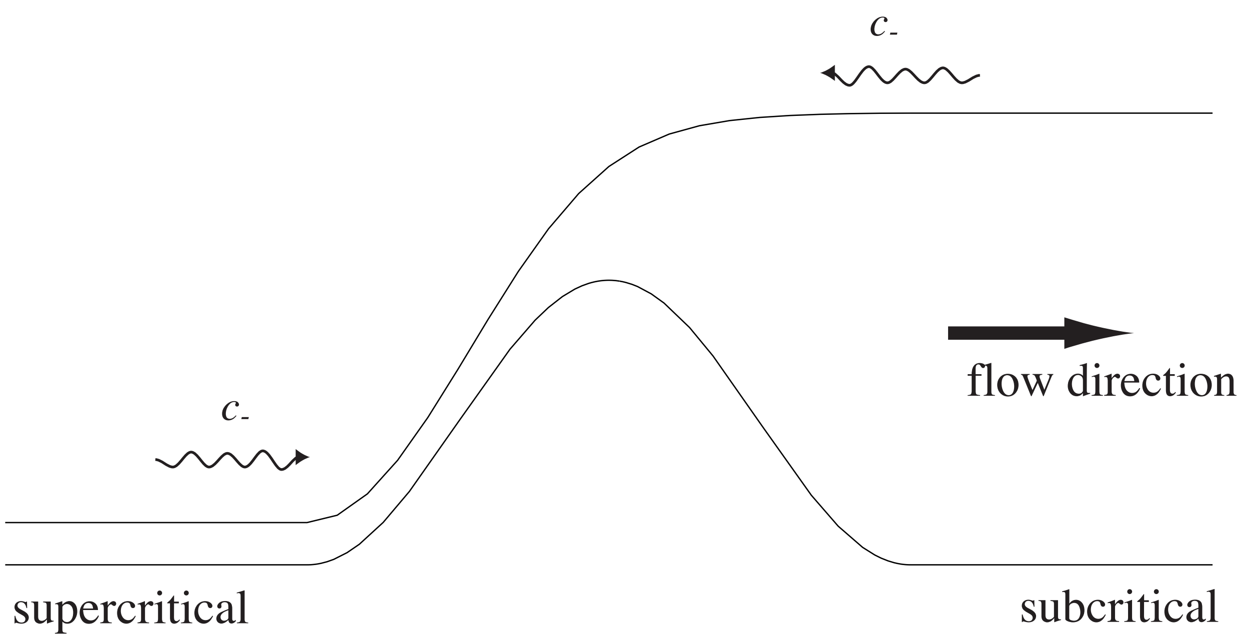


Figure 1.4.4