1.3 Nonlinear steepening and rarefacation.

A basic grasp of the hydraulic properties of a steady flow requires that one understand the characteristics of linear disturbances that propagate on that flow. However, finite amplitude disturbances are crucial in understanding how such solutions are established and how hydraulic jumps and other types of shock waves are formed. A feature common to most nonlinear disturbances that arise in hydraulic models is that they are governed by hyperbolic partial differential equations. The defining characteristics of quasi-linear hyperbolic systems in 2 dimensions is described in detail in Appendix B, as are the methods for transforming the governing equations into standard forms. However, a heuristic definition would center on the properties that two independent types of disturbances (waves) exist and that these waves propagate through the physical domain at a finite speeds. In the example of the previous section, the disturbances were the two linear gravity waves that propagated at speed $c_{\pm} = V \pm (gD)^{1/2}$. However, standard methodology allows one to consider wave amplitudes sufficiently large to destroy the distinction between the wave and the background flow.

To begin, it is helpful to rewrite the one-dimensional shallow water equations (1.2.1) and (1.2.2) in the form

$$\frac{d_{\pm}R_{\pm}}{dt} = -g\frac{dh}{dy} \tag{1.3.1}$$

where

$$\frac{d_{\pm}}{dt} = \frac{\partial}{\partial t} + [v \pm (gd)^{1/2}]\frac{\partial}{\partial y} \tag{1.3.2}$$

and

$$R_{\pm} = v \pm 2(gd)^{1/2}. \tag{1.3.3}$$

The procedure for obtaining this new form is discussed in Exercise 1 and the reader who seeks a more general discussion can consult Appendix B or look at standard texts such as Courant and Friedricks (1976) or Whitham (1974).

To interpret (1.3.1-1.3.3) first note that the operators $\frac{d_{\pm}}{dt}$ are time derivatives seen by observers traveling with *characteristic speeds*

$$\frac{dy_{\pm}}{dt} = v \pm (gd)^{1/2}. \tag{1.3.4}$$

These speeds are nothing more than the linear wave speeds with *V* and *D* replaced by the local velocity and depth, *v* and *d*. As before, it is helpful to think of the characteristic speeds as defining individual signals that move through the fluid and that compose general wave forms. Since the characteristic speeds vary throughout the flow field, different parts of a wave form move at different rates, leading to steeping (convergence) or rarefacation (spreading) of the wave. If the bottom slope is zero (*dh/dy=0*), an observer moving at one of the characteristic speeds sees a fixed value of the Riemann invariant R_+ or R_-. The latter are nonlinear generalizations of the functions introduced in the previous section. Among other things, they serve as indicators of the presence of 'forward' and 'backward' wave forms. If, for example, R_- is uniform in *y* then the flow field contains no 'backward' wave forms (i.e., those propagating at speed v-$(gd)^{1/2}$). The forward propagating waves in such a field are sometime called *simple waves*. A simple physical interpretation of Reimann invariants in terms of energy or momentum has proved to be elusive.

The characteristic speeds have real and unequal values for all flows in which the depth is non-zero and this implies that (1.2.1) and (1.2.2) are hyperbolic. The importance of this property is that solutions to the initial-value problems can be constructed using the method of characteristics. Suppose that one is given the initial conditions $v(y,0)$ and $d(y,0)$ for all *y* and asked to compute the evolution of the flow for $t>0$. The initial values of the Riemann invariants are given by $R_+ = v(y,0) + 2[gd(y,0)]^{1/2}$ and $R_- = v(y,0) - 2[gd(y,0)]^{1/2}$ and, provided that $dh/dy=0$, these values are conserved along *characteristic curves* (or 'characteristics'), paths traced out in the (y,t)-plane by moving away from $t=0$ at the appropriate characteristic speed . Unlike the case for the linear waves considered in the previous section, the slopes of the characteristic curves depend on the local values of the velocity and depth and may no longer be constant.

We can now lay out a procedure for solving any initial value problem involving smooth initial values of $v(y,0)$ and $d(y,0)$. Let $y_+(y_o,t)$ and $y_-(y_o,t)$ define the characteristic curves originating from a location $y=y_o$ on the *t*-axis of (y,t) space, also known as the *characteristic plane* (Figure 1.3.1a). The slopes of these curves are given (1.3.4) and can be calculated along *t*-axis from the initial conditions. The method of characteristics is initiated by projecting the curves forwards in time over a short interval Δt and assuming that the curves retain their initial slopes. The value of R_- or R_+ will be conserved, at least to a first approximation, along these straight-line projections. At a location such as *p* in the figure, a provisional value of R_- is determined by the curve $y_-(y_o,t)$ that passes through it. The value of R_+ is known from the $y_+(y_o,t)$ line that pass through and intersects the first curve at *p*. Provisional values of *d* and *v* can be computed from

$$v = \tfrac{1}{2}(R_+ + R_-) \tag{1.3.5}$$

$$d = [(R_+ - R_-)/4]^2 / g. \tag{1.3.6}$$

The characteristic speeds that follow from these values will generally be different than the initial estimates, implying that the characteristic curves are not straight. In this case an iterative procedure can be used to make the solution converge. Once satisfactory values of *v* and *d* have been found at $t=\Delta t$, the solution may be advanced further in time through reiteration. The method will continue to work as long as the '-' or '+' curves do not begin to intersect each other. Should the latter occur, multiple values of R_+ or R_-

would apply at the same point and the solution would be overdetermined. This situation is associated with the formation of shocks, meaning discontinuities in v and/or d, a process that will be explored later. Note that when the channel bottom contains topography, $R_{\pm}$ are no longer conserved and must be computed by through integration of (1.3.1) along characteristic curves. However the characteristic curves may still be interpreted as paths along which information travels.

Elementary examples of nonlinear evolution can be constructed through the consideration of a *simple wave*, as generated from an initial condition with uniform R_- or R_+. Consider the initial condition shown in Figure 1.3.1a, with shallow water to the right and deeper water to the left. Suppose further that the shallower region has uniform depth d_o and is motionless ($v=0$). We then choose the initial velocity to the left of the shallow region such that R_- is uniform. The value of R_- can be found by evaluating $v\text{-}2(gd)^{1/2}$ in the shallow, quiescent region, leading to $R_- = -2(gd_o)^{1/2}$. R_- must have this value for all y and therefore for all y and t reached by '-' characteristics, provided they do not intersect. An immediate consequence is that v and d become linked by the relation

$$v = 2(gd)^{1/2} - 2(gd_o)^{1/2},$$

which follows from the definition of R_-. The definition of R_+ then leads to

$$R_+ = 4(gd(y,t))^{1/2} - 2(gd_o)^{1/2},$$

and thus $d(y,t)$ itself is conserved along each '+' characteristic curve. Since both R_+ and d are conserved, v must also be conserved along each such curve and the characteristic speed must be constant and equal to its initial speed:

$$\begin{aligned} c_+ &= v(y,t) + (gd(y,t))^{1/2} \\ &= 3(gd(y,0))^{1/2} - 2(gd_o)^{1/2}. \end{aligned} \tag{1.3.7}$$

The slope $1/c_+$ of each '+' curve is constant, though different curves have different slopes. For the disturbance shown in Figure 1.3.1a, characteristics emanating from the deeper part of the disturbance are tilted more steeply than those emanating from the shallower portion. The tilt of each curve is an indication of how rapidly the signal corresponding to a particular part of the disturbance travels. The signal itself can be identified as a particular value of d. Here the deeper (left-hand) portion will propagate to the right more rapidly than the shallower, right-hand portion. The slope $|\partial d / \partial y|$ of the free surface will therefore increase in what is called *nonlinear steepening*. We leave it as an exercise for the reader to show that a disturbance of the type shown in Figure 1.3.1b would spread or *rarefy* (provided R_- remains uniform). In other words, the left-hand (shallower) d-values would propagate more slowly than those to the right.

Steepening waves eventually experience singularities, as shown by the converging characteristic curves in Figure 1.3.1a. As more rapid signals overtake slower ones, the free surface slope increases without bound and eventually multiple values of *d* occupy the same *y*, as indicated by intersections in the characteristic curves. When this occurs, the shallow water approximation must be abandoned. Depending on the rate of steepening, nonhydrostatic effects may be overcome, allowing a breaking wave to form, or may arrest the steepening altogether. A more complete description of the sequence of events that can take place can be found in Baines (1995).

A further illustration of the power of the method of characteristics is provided by a nonlinear version of the dam break problem explored in the previous section. We now consider the motion following the destruction of a barrier separating a resting fluid of depth *D* from a region with no fluid (Figure 1.3.2). The initial conditions are

$$d(y,0)=\begin{cases}D\ (\mathrm{y}>0)\\ 0\ \ (\mathrm{y}<0)\end{cases} \tag{1.3.8}$$

and

$$v(y,0)=0. \tag{1.3.9}$$

As posed, the solution to this problem is non-unique. Different solutions can be found depending upon how one deals with the discontinuity in initial depth at y=0. A reasonable way to resolve this difficulty is to replace this discontinuity with a smooth, but abrupt, transition over $0<y<y_T$, as shown in the figure. We must specify the initial values of *d* and *v* within this short region and the corresponding characteristic speeds and Riemann invariants can be used to compute the evolution. Different specifications lead to different outcomes and this is the source of the non-uniqueness. The calculation of the evolution becomes quite simple if *d* and *v* are chosen such that either R_+ or R_- are uniform in the abrupt region and has the same value $2(gD)^{1/2}$ or $-2(gD)^{1/2}$ as in the region $y<0$. Then one of the Riemann invariants will be initially uniform throughout the fluid, allowing application of the simplifications described above for 'simple' waves. The limit $y_T\rightarrow 0$ may be taken later in order to approach the original step geometry.

Following this idea further, suppose that R_- is initially uniform in the transitional interval $0<y<y_T$. Its value must therefore be $-2(gD)^{1/2}$ in order to match that of the uniform region $y<0$. It follows that the definition of R_- in the transition region that

$$v(y,0)-2(gd(y,0))^{1/2}=-2(gD)^{1/2} \qquad (0<y<y_T).$$

However, $d<D$ in $0<y<y_T$, implying that $v(y,0)<0$. That is, the fluid in the vicinity of the barrier will initially move *to the left* after the barrier is removed. Obviously, the assumption of uniform R_- is not one that leads to a physically realistic evolution. On the other hand, the choice R_+=uniform leads to

$$v(y,0)+2(gd(y,0))^{1/2}=2(gD)^{1/2} \qquad (0<y<y_T) \tag{1.3.10}$$

so that $v(y,0)>0$ in the vicinity of y=0, as expected.

It is now easy to make a sketch of the characteristic curves for all y and t, as is done in Figure 1.3.2. Since R_+ is uniform, all of the '-' characteristics are straight. Their slope is determined by the initial value of the characteristic speed:

$$c_-(y,0) = \begin{cases} -(gD)^{1/2} & (y<0) \\ v(y,0)-(gd(y,0))^{1/2} = 2(gD)^{1/2} - 3(gd(y,0))^{1/2} & (0<y<y_T) \end{cases}.$$

Since $d(y,0)$ decreases monotonically from D to 0 as y increases from 0 to y_T, c_- increases from $-(gD)^{1/2}$ to $2(gD)^{1/2}$ over the transitional interval. The '-' characteristic curves originating from this interval therefore fan out as shown in Figure 1.3.2. Since d and v are constant along these curves, the developing flow consists of a rarefaction wave. The leading edge *(d=0)* of this wave moves to the right at speed $2(gD)^{1/2}$ whereas the rear edge moves to the left at speed $(gD)^{1/2}$. The leading edge speed is also the fluid velocity at the leading edge. One of the fanning characteristic curves has $c_-=0$ and therefore points directly upwards in the Figure. In the limit $y_T\to 0$ this curve lies at $y=0$, the position of the barrier. Thus, the flow at $y=0$ immediately becomes steady and critical after the barrier is destroyed. The flow at all other y approaches this same critical state as $t\to\infty$. The depth d_∞ and velocity v_∞ of this final state are determined by the condition of criticality $[v_\infty = (gd_\infty)^{1/2}]$ and by the uniformity of $R_+=v_\infty + 2(gd_\infty)^{1/2} = 2(gD)^{1/2}$, leading to $v_\infty = \frac{2}{3}(gD)^{1/2}$ and $d_\infty = (\frac{2}{3})^2 D$. The volume transport per unit width of channel is therefore given by

$$v_\infty d_\infty = \left(\frac{2}{3}\right)^3 g^{1/2} D^{3/2} \qquad (1.3.11)$$

If the initial depth in $y>0$ is finite then the advancing edge of the flow forms a shock. Calculation of the solution for this case requires more advanced development of the theory of shocks and this is left for later in this chapter. The interested reader may consult Stoker (1957) for the full solution.

Exercises

1) *Derivation of Riemann Invariants.* Obtain the homogeneous form of (1.3.1) from the shallow water equations (1.2.1) and (1.2.2) by the following procedure:

a) Try to write the homogeneous versions of (1.2.1) and (1.2.2) in the *characteristic form*

$$\left(\frac{\partial}{\partial t} + c_\pm \frac{\partial}{\partial y}\right) v + \alpha_\pm(v,d)\left(\frac{\partial}{\partial t} + c_\pm \frac{\partial}{\partial y}\right) d = 0$$

by multiplying (1.2.2) by a factor $\alpha_{\pm}(v,d)$, adding the result to (1.2.1), and calculating $\alpha_{\pm}(v,d)$ and $c_{\pm}(v,d)$ such that the above form is achieved.

b) Use this result to find the functions $R_{\pm}(v,d)$ satisfying $\left(\frac{\partial}{\partial t}+c_{\pm}\frac{\partial}{\partial y}\right)R_{\pm}=0$.

2) Linearize the Riemann invariants $R_{\pm}$ about a uniform background flow $v=V$ and $d=D$. How do the resulting expressions relate to the traveling wave functions $f_{+}(y-c_{+}t)$ and $f_{-}(y-c_{-}t)$ defined in Section 1.2?

3) Consider the initial condition $v=0$ and

$$d(y,0)=\begin{matrix} d_o & (|y|>\mathrm{L}) \\ d_o+a(1-\frac{|y|}{L}) & (|y|\leq \mathrm{L}) \end{matrix}.$$

Altough this initial condition does not formally give a 'simple wave' solution, a simple-wave character emerges in parts of the domain after a finite time has elapsed. Use this behavior to discuss the qualitative features of the nonlinear evolution of this disturbance and compare with the linear result (Exercise 2 of Section 1.2).

4) For the example shown in Figure 1.3.1a, at what time does wave breaking (shock formation) *first* occur? [Hint: do not necessarily be satisfied with the obvious answer.]

5) Consider the following twist on the classical dam break problem with initial conditions (1.3.8) and (1.3.9). Suppose that at $t=0$ the barrier is not destroyed but instead is made to recede from the reservoir at a constant speed $c_o<2(gD)^{1/2}$. Use the method of characteristics to sketch the solution.

Figure Captions

1.3.1 Characteristic curves for two initial value problems, one with deeper water to the right (a) and the second with deeper water to the left (b).

1.3.2 The full dam break problem as visualized with a gradual initial change in depth, rather than a discontinuity, near $x=0$. The characteristic curves are shown in the upper frame and the rarefying surface disturbance and intrusion are shown in the lower frame.

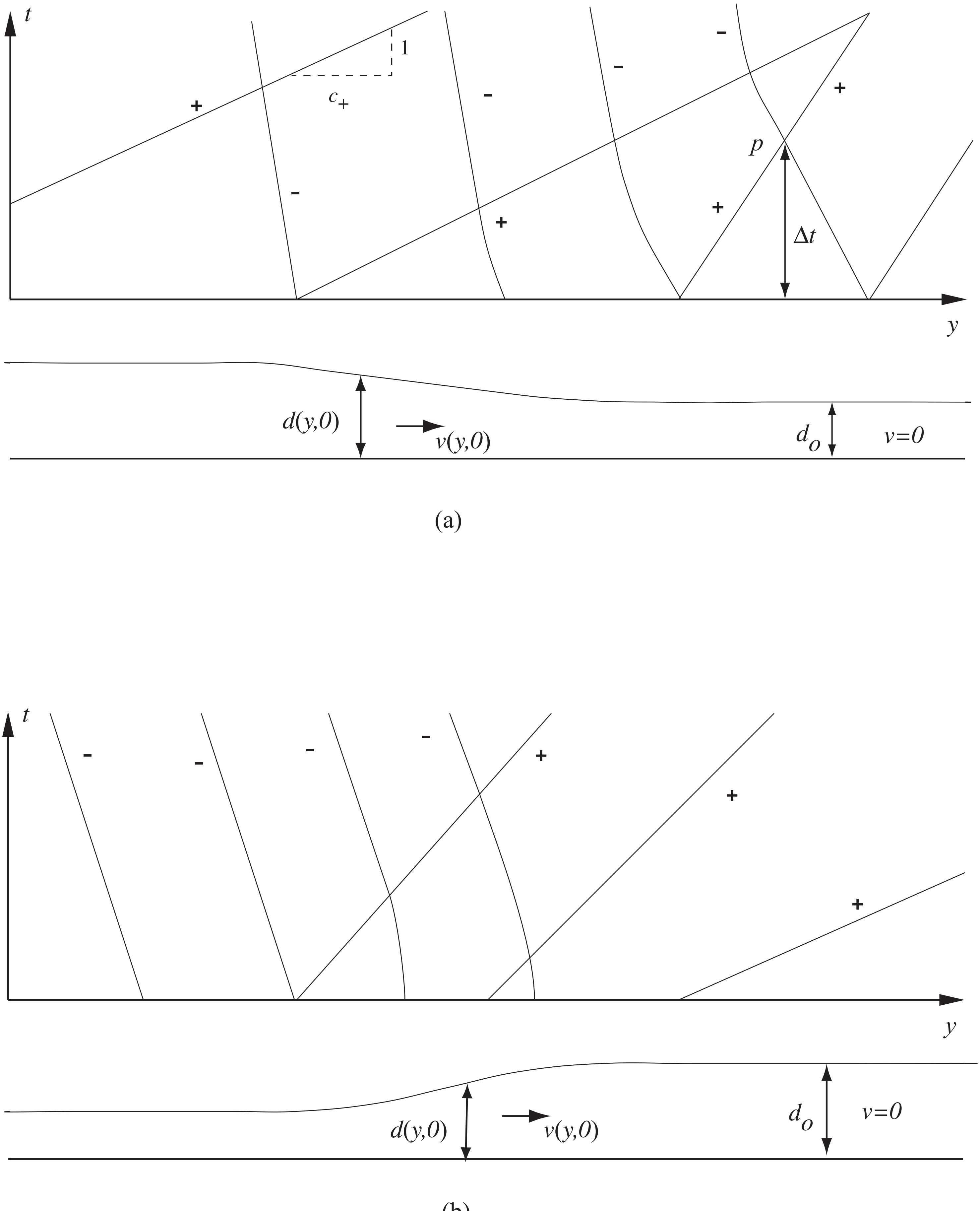

Fig 1.3.1

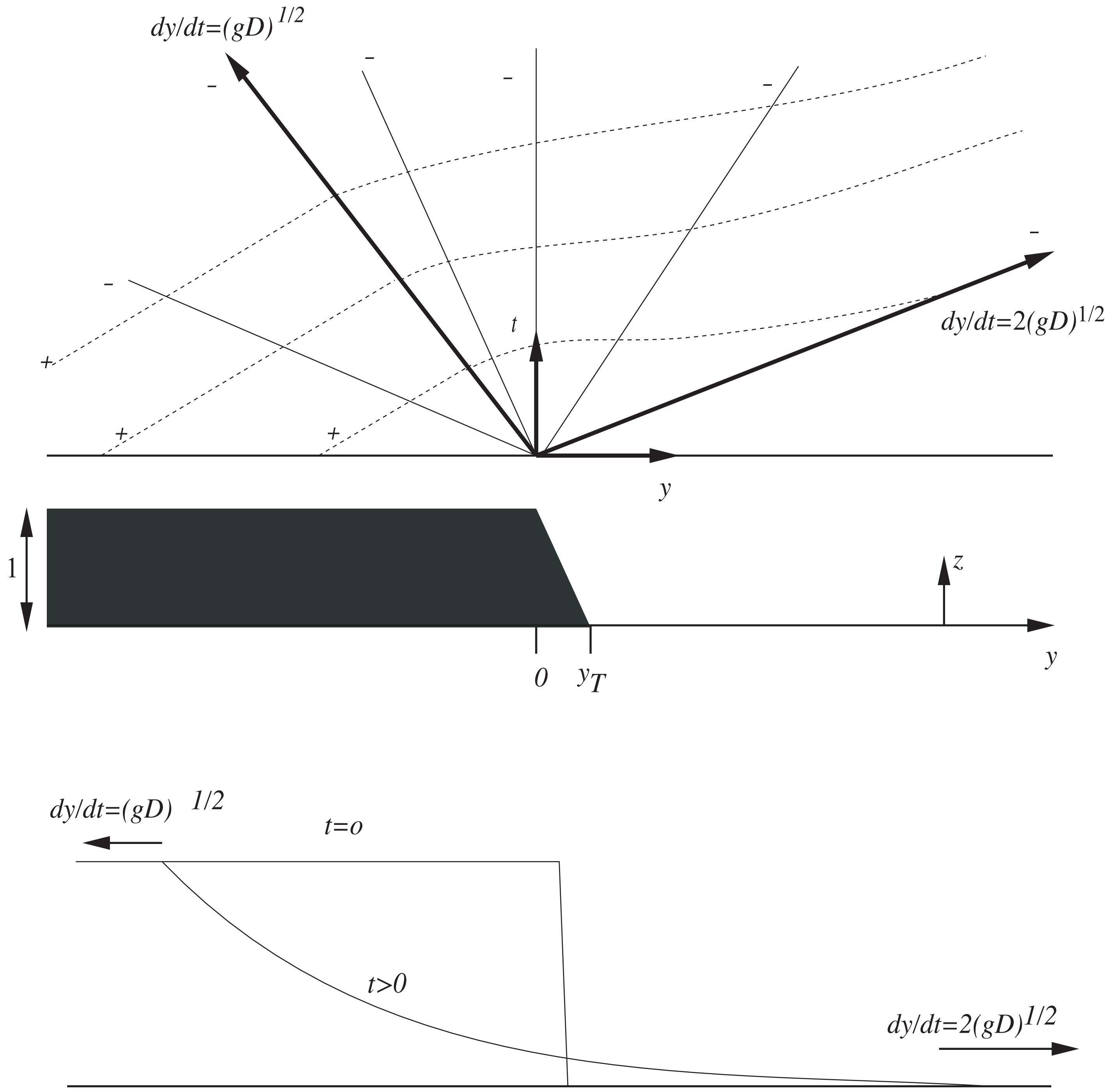

Figure 1.3.2