1.2 The shallow water equations and one-dimensional wave propagation.

Traditional discussions of hydraulic effects such as those found in engineering text books are often based on analyses of steady flows. At the same time, interpretation of these effects almost always involves waves and wave propagation. We therefore preface our discussion of steady hydraulics with a discussion of wave propagation in shallow water. Attention is restricted to flows governed by the shallow water equations in one spatial dimension:

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}+g\frac{\partial d}{\partial y}=-g\frac{\partial h}{\partial y} \qquad (1.2.1)$$

and

$$\frac{\partial d}{\partial t}+d\frac{\partial v}{\partial y}+v\frac{\partial d}{\partial y}=0\,. \qquad (1.2.2)$$

For a homogeneous fluid, g is the ordinary gravitational acceleration. In oceanic and atmospheric models, we often consider flows of uniform density that are overlain by a much thicker and inactive layer of slightly lower (but still uniform) density. In such cases, the above equations remain valid provided that g is interpreted as the *reduced gravity*: the ordinary g multiplied by the fractional difference in density between the two layers. Such a model is often referred to as having a '$1\frac{1}{2}-$layer 'stratification.

Now consider a steady flow with uniform velocity V and depth D over a horizontal bottom ($dh/dy=0$). Infinitesimal disturbances to this flow, denoted v' and η, can be introduced by setting $v=V+v'$ and $d=D+\eta$, where $v'<<V$ and $\eta<<D$. Substitution into (1.2.1) and (1.2.2) and neglect of quadratic terms in the primed variables, leads to the linear shallow water equations

$$\frac{\partial v'}{\partial t}+V\frac{\partial v'}{\partial y}+g\frac{\partial \eta}{\partial y}=0\,, \qquad (1.2.3)$$

and

$$\frac{\partial \eta}{\partial t}+V\frac{\partial \eta}{\partial y}+D\frac{\partial v'}{\partial y}=0\,. \qquad (1.2.4)$$

The most general solution to these equations is given by

$$\eta=f_+(y-c_+t)+f_-(y-c_-t) \qquad (1.2.5)$$

and

$$v'=\left(\frac{g}{D}\right)^{1/2}[f_+(y-c_+t)-f_-(y-c_-t)] \qquad (1.2.6)$$

where

$$c_+ = V + (gD)^{1/2} \tag{1.2.7}$$

and

$$c_- = V - (gD)^{1/2} \tag{1.2.9}$$

Equations (1.2.5) and (1.2.6) describe waveforms traveling at speeds c_+ and c_-. The propagation speeds are made up of two components: the unidirectional background flow velocity V and a bi-directional propagation velocity $\pm(gD)^{1/2}$, equivalent to the propagation speed in a fluid at rest. For $V>0$ the two types of disturbance travel in opposite directions when $(gD)^{1/2} > V$, and we call the background flow *subcritical* in this case. The quiet and smooth currents common in rivers away from dams or rapids are generally subcritical. Propagation in the positive y-direction occurs for both waves when $(gD)^{1/2}<V$, in which case the background flow is called *supercritical*. Flows in spillways, waterfalls, and in parts of rapids are supercritical. If $(gD)^{1/2}=V$, one wave propagates in the direction of the background flow and the other is stationary: $c_-=0$. In this case the background flow is *critical* and can support stationary disturbances. Critical flow is normally a local phenomenon which occurs near the crests of dams and spillways. Because of the long wave approximation, there is no distinction between the speeds of phase and energy propagation of these waves.

The Froude number, defined by

$$F_d^{\,2} = \frac{V^2}{gD},$$

is often used to characterize the relative importance of inertia and gravity in the dynamics of a particular flow. F_d is clearly the ratio of the advective component to the intrinsic 'propagation' component of the phase speed and is <1,=1, >1 for subcritical, critical, and supercritical flow.

With more complicated flows it may be difficult to unambiguously define *upstream* and *downstream*. Such is the case when the fluid is stratified and has positive and negative horizontal velocity at different depths. In such cases we reserve the term *subcritical* to mean that signal speeds c_+ and c_- belonging to a particular pair of waves are of opposite sign: $c_+c_-<0$. In this case information carried by this particular type of wave can travel in both directions. Supercritical flow is defined by $c_+c_->0$ and corresponds to information flow in one direction only. Finally, critical flow is defined by $c_+c_-=0$ and corresponds to arrest of one or both of the waves. Note that this definition applies to the homogeneous flow under consideration and is independent of the sign of V.

A simple example of wave generation that will be built on throughout this book is the linear dam-break problem. As shown in Figure 1.2.1a, consider two bodies of resting fluid with slightly different depths $D \pm a$, separated by a barrier located at *y*=0. At *t*=0 the barrier is removed, allowing the deeper fluid to move towards positive *y*. Assuming *a*<<*D*, the subsequent motion can be approximated by solving (1.2.3) and (1.2.4) with *V*=0 and subject to the initial conditions

$$\eta(y,0) = -a\,\mathrm{sgn}(y) = a\begin{cases}-1 \text{ (y>0)}\\ +1 \text{ (y<0)}\end{cases}$$

and

$$v'(y,0) = 0$$

As shown in Figure 1.2.1b, the solution

$$\eta(y,t) = -\tfrac{1}{2}a\left[\mathrm{sgn}(y-c_+t) + \mathrm{sgn}(y-c\_t)\right]$$

and

$$v'(y,t) = \tfrac{1}{2}a\left(\tfrac{g}{D}\right)^{1/2}\left[-\mathrm{sgn}(y-c_+t) + \mathrm{sgn}(y-c\_t)\right]$$

consists of two step-like wave fronts propagating away from *y*=0 at the speeds $c_\pm = \pm(gD)^{1/2}$. Left behind is a uniform stream with velocity $a(g/D)^{1/2}$ and with depth equal to the mean initial depth. It is apparent that, between the two wave fronts, the available potential energy associated with the initial mismatch in fluid depths has been entirely converted to kinetic energy (see Exercise 1). The total conversion of potential to kinetic energy is a feature that does not persist in the presence of rotation.

Another view of linear, long-water wave propagation, one that will be helpful in understanding the next section's material on nonlinear waves, comes from the method of characteristics. If (1.2.4) is multiplied by $(g/D)^{1/2}$ and the product is added to (1.2.3), the resulting equation can be arranged in the form:

$$\left(\frac{\partial}{\partial t} + \left(V + (gD)^{1/2}\right)\frac{\partial}{\partial y}\right)\left(v' + \left(\frac{g}{D}\right)^{1/2}\eta\right) = 0\,. \qquad (1.2.10)$$

Subtraction of the two results in

$$\left(\frac{\partial}{\partial t} + \left(V - (gD)^{1/2}\right)\frac{\partial}{\partial y}\right)\left(v' - \left(\frac{g}{D}\right)^{1/2}\eta\right) = 0\,. \qquad (1.2.11)$$

The operator in (1.2.10) can be interpreted as the time derivative seen by an observer moving at the wave speed $c_+ = V + \left(gD\right)^{1/2}$. To that observer the value of $\left(v' + \left(g/D\right)^{1/2}\eta\right)$, sometimes known as the *linearized Riemann invariant*, remains fixed. A similar interpretation holds for (1.2.11), with an observer moving at speed

$c_- = V - (gD)^{1/2}$ seeing a fixed value of $\left(v' - (g/D)^{1/2}\eta\right)$. In this context, c_- and c_+ are called *characteristic speeds*. The general solutions (1.2.5) and (1.2.6) can be deduced directly form the characteristic forms (1.2.10) and (1.2.11) of the linear shallow water equations.

The Riemann invariants can be used measure the distribution of 'forward' and 'backward' propagating waves in a time-dependent flow field. Consider a single, forward wave (with speed c^+) of the form $\eta = \sin[y - c^+ t]$. In view of (1.2.6) the corresponding perturbation velocity is given by $v' = (g/D)^{1/2}\sin[y - c^+ t]$. The value of the 'forward' Riemann invariant $\left(v' + (g/D)^{1/2}\eta\right)$ over this waveform varies from $2(g/D)^{1/2}$ at a wave crest to $-2(g/D)^{1/2}$ at a trough, whereas the value of $\left(v' - (g/D)^{1/2}\eta\right)$ is uniformly zero over the same interval. The reverse is true for a 'backward' wave (with speed c^-). One could use this property to decompose a more complicated wave field into backwards and forwards components (see Exercise 3); forward waves project entirely onto the forward Riemann invariant and vice versa.

Now consider an initial value problem for which v' and η are specified for all y at t=0. In determining a solution for t>0 it is useful to think about the propagation of this information forward in time. Consider the space $-\infty<y<\infty$ and $t\geq 0$, also known as the *characteristic plane*. An observer moving at the speed $c_+ = V + (gD)^{1/2}$ travels through this space along one of the *characteristic curves* (or *characteristics*) indicated by a '+' in Figure 1.2.2a. The value of $\left(v' + (g/D)^{1/2}\eta\right)$ is conserved along such curves. A similar result hold for the characteristic curves labeled '-', along which $\left(v' - (g/D)^{1/2}\eta\right)$ is conserved. The characteristic curves therefore represent paths along which information travels.

As an example of the use of the method of characteristics, reconsider the dam break problem. The initial conditions are sketched below the characteristic plane in Figure 1.2.2a. We begin by considering a '+' characteristic curve originating at a point a along the positive portion of the y-axis at t=0. Here the initial conditions are $v' = 0$ and η=-a. The value of the Riemann invariant that is carried forward in time along the curve ab is therefore given by

$$v' + (g/D)^{1/2}\eta = -(g/D)^{1/2}a \text{ (along } e'f) \qquad (1.2.12)$$

This is true of all the solid curves originating from the positive portion of the y-axis. The individual values of v' and η are not yet determined; to do so we must consider the values of $\left(v' - (g/D)^{1/2}\eta\right)$, which are carried along the dashed characteristics. Along the curve $a'b$ the initial conditions give

$$v' - (g/D)^{1/2}\eta = (g/D)^{1/2} a \text{ (along } e'f\text{)} . \qquad (1.2.13)$$

At the intersection point *f* we have from the last two equations v'=0 and η=-a. This result will hold at all points within region I of the characteristic plane, as indicated in Figure 1.2.2b. This region of the flow has not yet been reached by the forward propagating wave front that is generated by the step in surface elevation. The reader may wish to verify that a similar result holds in Region II, which lies to the left of the wave front advancing to the left and where the values v'=0 and η=a are equal to the initial values.

Each point in Region III of the characteristic plane is intersected by dashed characteristic curves emanating from the positive *y*-axis and by solid curved emanating from the negative *y*-axis. The corresponding Riemann invariants are given by

$$v' + (g/D)^{1/2}\eta = (g/D)^{1/2} a \text{ (in region III)}$$

and

$$v' - (g/D)^{1/2}\eta = (g/D)^{1/2} a \text{ (in region III)}$$

and therefore η=0 and $v' = (g/D)^{1/2} a$. Thus the passage of the wave fronts leaves behind a stead flow with velocity $(g/D)^{1/2} a$. The paths of the wave fronts themselves are the characteristic curves that form the boundaries between the three regions.

Exercises

1) *Energy conversion in the linear dam-break problem.* Multiply (1.2.3) by Dv' and (1.2.4) by $g\eta$ and add the results to obtain the energy equation

$$\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial y}\right)\left(\frac{g\eta^2}{2} + \frac{Dv'^2}{2}\right) = -gD\frac{\partial(v'\eta)}{\partial y} .$$

For the solution to linear dam break problem (Figure 1.2.1b) integrate the above equation (with V=0) with respect to *y* over any fixed interval I= ($-y_o<y<y_o$). Then integrate the resulting relation with respect to *t* from 0 to ∞. Show from the final result that the available potential energy in *I* is converted entirely into kinetic energy. This finding is

consistent with the fact that the energy radiated away from *I* by the gravity waves (as measured by $\int_0^\infty \left[(v'\eta)_{y_o} - (v'\eta)_{-y_o} \right] dt$) is zero.

2) Consider the initial condition *v=0* and

$$d(y,0) = \begin{matrix} d_o & (|y| > \mathrm{L}) \\ d_o + a(1 - \frac{|y|}{L}) & (|y| \le \mathrm{L}) \end{matrix}.$$

Discuss the evolution of this disturbance according to linear theory.

3) Using Riemann invariants, decompose the following flow field into 'forward' and 'backward' waves:

$$\eta(y,t) = -\sin[y]\cos[t] \text{ and } v' = (g/D)^{1/2}\cos[y]\sin[t].$$

4) *Linear wave speeds in the presence of vertical shear.* Consider the wave problem for a free surface flow with uniform depth *D* and velocity *V*(*z*). Define *v*=*V*(*z*)+*v*′(*y*,*z*,*t*) and show that the linearized *y*-momentum and continuity equations are

$$\frac{\partial v'}{\partial t} + V\frac{\partial v'}{\partial y} + w\frac{\partial v'}{\partial z} + g\frac{\partial \eta}{\partial y} = 0 \text{ and } \frac{\partial v'}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

(a) Assuming the waveforms $(v', w, \eta) = \mathrm{Re}(\tilde{v}(z), \tilde{w}(z), \tilde{\eta})e^{il(y-ct)}$, obtain the relation

$$(c - V)\frac{\partial w}{\partial z} + w\frac{dV}{dz} + igl\tilde{\eta} = 0.$$

Divide this equation by $(c\text{-}V)^2$ and integrate the result from the bottom (here *z*=0) to the free surface (*z=D* by the linear approximation) to obtain

$$\left[\frac{\tilde{w}}{c - V}\right]_{z=D} - \left[\frac{\tilde{w}}{c - V}\right]_{z=0} = -igl\tilde{\eta}\int_0^D \frac{dz}{(c - V)^2}.$$

(b) Apply the kinematic boundary conditions at the bottom and free surface to obtain the result.

$$g\int_0^d \frac{dz}{(V - c)^2} = 1$$

Show that the case *V*=constant results in $c=V\pm(gD)^{1/2}$. For nonconstant *V* observe that real values of *c* must lie outside the range of variation of *V*.

(c) Finally, show that if the variations of V are weak: $V = V_o + \varepsilon\hat{V}(z)$ with ε<<1 and $\int_0^D \hat{V}dz = 0$, that

$$(V_o - c)^2 = gD + \frac{3\varepsilon^2}{D}\int_0^D \hat{V}^2 dz,$$

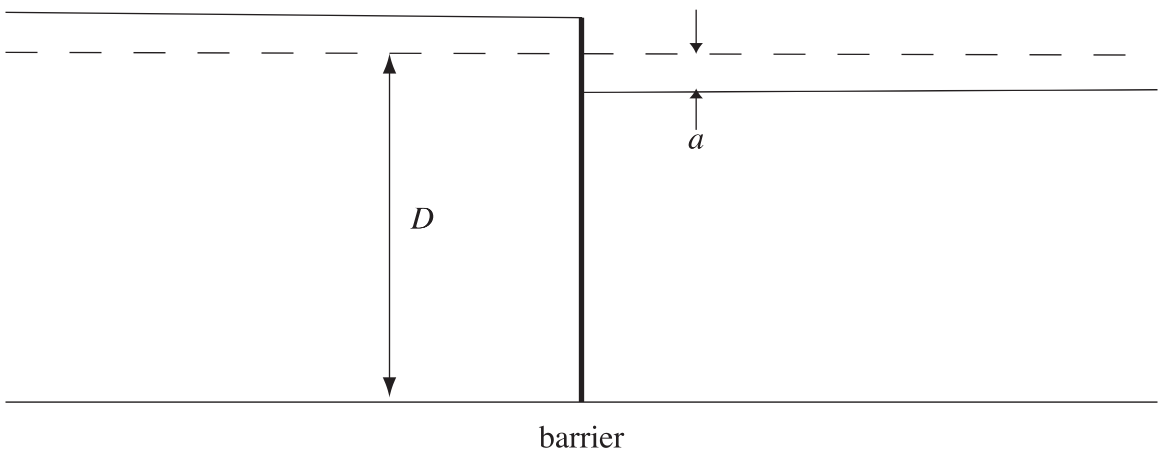
and therefore a section at which $V_o=(gD)^{1/2}$ allows upstream propagation.

Further discussion of the implications of these results can be found in Garrett and Gerdes (2003). The derivation of the wave speeds appear in Freeman and Johnson (1970).

Figure Captions

1.2.1 The linear dam break problem.

1.2.2 Characteristic curves and regions of influence for the linear dam break.



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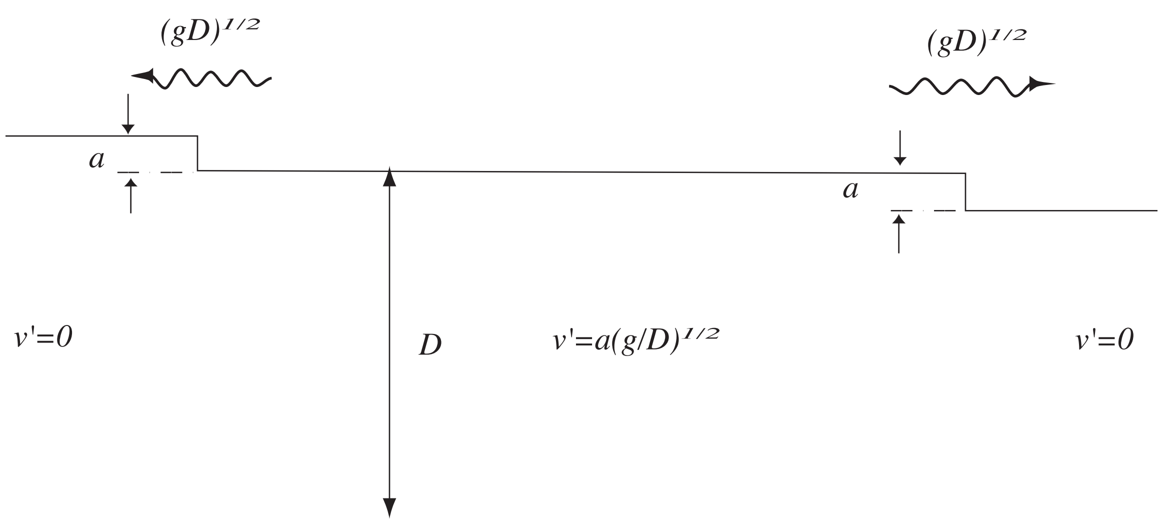
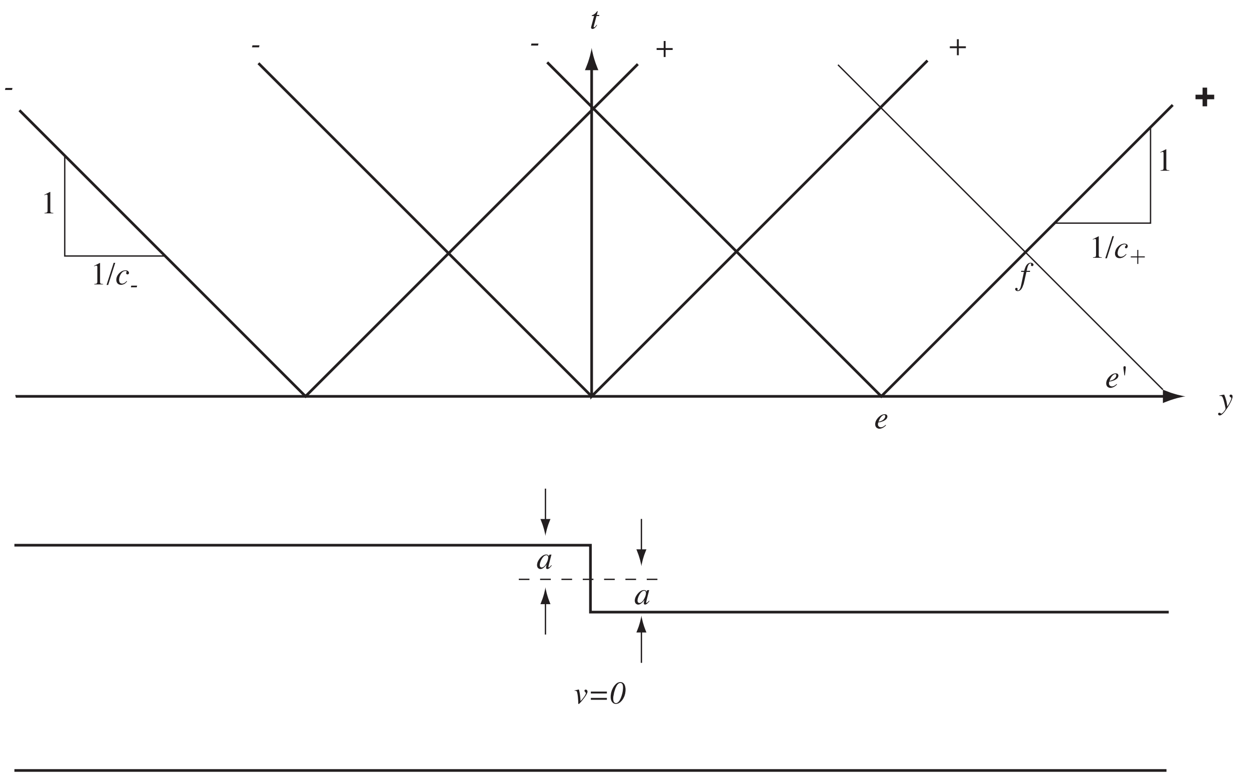


(b)

Figure 1.2.2



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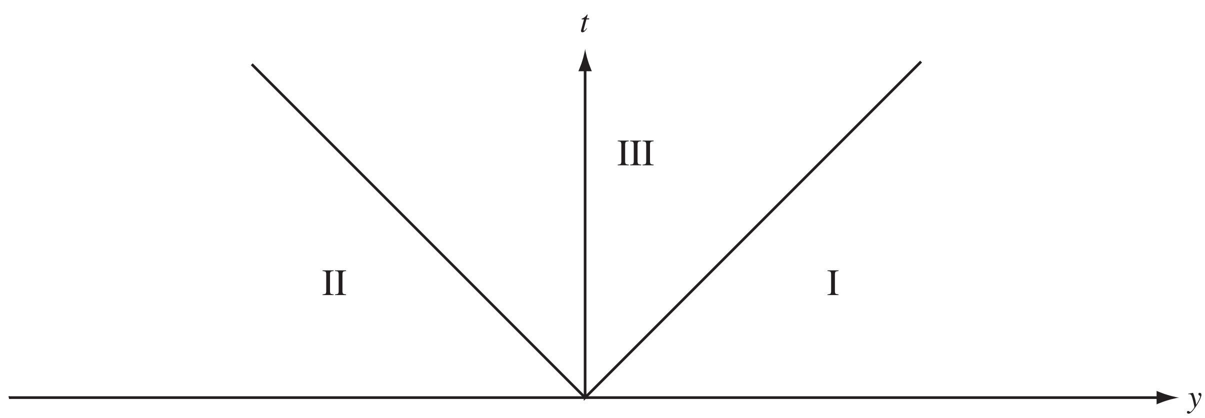

(b)

Figure 1.2.2