



ELSEVIER

Dynamics of Atmospheres and Oceans 22 (1995) 135–160

dynamics
of atmospheres
and oceans

The interaction of a pair of point potential vortices in uniform shear

David Walsh ^{*}, Lawrence J. Pratt

Woods Hole Oceanographic Institution, Woods Hole, MA 02543, USA

Received 5 August 1993; revised 14 February 1995; accepted 17 March 1995

Abstract

A simple point vortex model is formulated to investigate the deformation and translation of lens-like oceanic eddies, such as Mediterranean Salt Lenses, in large-scale shear. The idealized eddy is represented by a pair of quasigeostrophic ‘point potential vortices’ at different depths in a uniformly stratified fluid. The point vortices are assumed to be embedded in a flow with uniform vertical and horizontal shear, and they are advected by the background flow as they interact with one another. The model successfully reproduces many aspects of the behaviour of low-mode disturbances found in models with continuous (non-singular) representations. Depending upon the strengths of the vortices, their initial separation, and the intensity of the background shear, the vortex pair is either torn apart by the shear, or else remains coupled for all time, in which case the vortices execute a periodic motion while propagating with respect to the ambient fluid. Solutions representing steadily translating point vortex configurations are obtained for certain values of the model parameters. For a given vertical separation between the vortices and a specified background shear, there can exist up to three steadily translating solutions, each with a different horizontal separation between the vortices, and each aligned perpendicular to the background flow direction. The translation speed of these pairs is directly proportional to the difference in the strengths of the vortices. A detailed analysis of the character of the steadily translating solutions is made, and the linear stability properties of the solutions are investigated.

1. Introduction

In recent years, interest in the behaviour of oceanic mesoscale lenses and their role in the general circulation has grown dramatically. Perhaps the best docu-

^{*} Corresponding author present address: Dalhousie University, Department of Oceanography, Halifax, N.S. B3H 4J1, Canada.

mented of these are the Mediterranean Salt Lenses or Mediterranean Water Eddies ('Meddies'). Meddies are large lenses of warm and salty Mediterranean water, typically 800 m thick and 80–100 km in diameter, with strong anticyclonic internal circulations (Armi et al., 1989; Richardson et al., 1989). Long-term observations have been made of Meddies, giving a first look at the life cycles of these eddies. Armi et al. (1989) have documented the evolution of a Meddy over a 2 year period, during which time it drifted over 1000 km southward at an average speed of 1.8 cm s^{-1} . Richardson et al. (1989) have examined the evolution of three different Meddies which were seeded with SOund Fixing And Ranging (SOFAR) floats and tracked for up to 2 years as they drifted through the Canary Basin. Richardson et al. (1991) have re-examined historical data sets in an attempt to identify possible Meddy sightings, and report numerous anomalous T - S measurements which they attributed to Meddies. Given the large number of Meddies that have been found in the eastern North Atlantic and the large distance they are known to travel from their probable formation site near Cape St. Vincent (Richardson et al., 1991), it is likely that they are an important mechanism of along-isopycnal heat and salt transport at mid-depths in the region, and they may play a significant role in determining the large-scale structure of the Mediterranean Salt Tongue.

Richardson et al. (1989) reported that SOFAR floats in Meddies moved at roughly 1.4 cm s^{-1} in a south or southwestward direction relative to nearby floats outside Meddies, implying that Meddies in fact 'propagated' through the surrounding waters. The observed propagation of Meddies has not been definitively tied to a particular physical mechanism, although a number of plausible hypotheses have been put forward to explain their motion. Typically these hypotheses employ either β -plane dynamics or interactions with external flows to explain the observed translation. The results of a numerical simulation by Beckmann and Käse (1989) indicate that a predominantly southward drift at speeds of roughly 1 cm s^{-1} is possible on a β -plane. More recently, Colin de Verdiere (1992) has suggested that the lateral erosion of the cores of Meddies by thermohaline intrusions (e.g. Ruddick and Hebert, 1988) and the subsequent geostrophic adjustment and flattening of the core must be balanced by southward motion on a spherical Earth. Both Beckmann and Käse (1989) and Colin de Verdiere (1992) investigated mechanisms by which Meddies may translate in the absence of any background flow. However, the general agreement between Meddy trajectories and near-surface currents led Richardson et al. (1989) to hypothesize that the Meddies were being 'advected' by large-scale currents near the surface. This suggested that the large-scale baroclinic shear in which Meddies are embedded may play an important role in their translation. To explore a possible mechanism behind the proposed advection by near-surface currents, Hogg and Stommel (1990) (hereafter HS) formulated an idealized point vortex model which demonstrated how Meddies could be advected by currents at shallower depths if the potential vorticity associated with the Meddy were nonuniformly distributed in depth. For the vortex induction propagation mechanism they described to be effective, Hogg and Stommel's model requires that a Meddy be 'tilted' by the external shear.

Hogg and Stommel's prediction that Meddies may be tilted by external shear was confirmed by Walsh (1992), who used data from neutrally buoyant SOFAR floats at different depths within two different Meddies to show that there is indeed a measurable tilt to the rotation axis of the two Meddies. Walsh et al. (1995) have shown that the size of the observed Meddy tilts is consistent with the predictions of a simple analytical model which represents a Meddy as a 'patch' with anomalous potential vorticity in a stratified fluid, based upon their best estimates of the large-scale shear in the region. Käse and Zenk (1987) and Prater and Sanford (1994) have reported Meddies which have multiple velocity maxima at different depths in the water column, which is consistent with the vertically inhomogeneous potential vorticity field assumed in Hogg and Stommel's model formulation. Walsh et al. (1995) discussed similar results obtained from float data which suggest that rotation rates within the warm and salty cores of Meddies may vary substantially with depth in some cases.

In this work we investigate the interaction of an eddy with an external shear using a very simple point vortex analogue, which is an extension of that of HS to include continuously stratified dynamics. The application of the point vortex ansatz in the study of oceanic eddies can be traced to the two-layer model of Hogg and Stommel (1985); the use of point vortices to study stratified quasigeostrophic (QG) flow was suggested in a review paper by Flierl (1987). Hogg and Stommel coined the term 'heton' to describe a certain configuration of baroclinic, geostrophic point vortices which were effective in transporting heat, and proposed the heton as a novel means of releasing available potential energy in geostrophic systems. Since then other workers (e.g. Legg and Marshall, 1993) have employed layered point vortex models to investigate other geophysical phenomena. In a study reminiscent of HS's model of Meddy translation, Wu and Emmanuel (1993) have investigated the effect of vertical shear on hurricane movement using a two-layer QG model with a point cyclone in the lower layer and a finite patch of zero potential vorticity air containing a point source of mass in the upper layer. Their numerical experiments indicate that a vertical shear tilts the vortex pair in the downstream direction, and the resulting interaction between the vortices induces a cross-stream velocity component to the trajectory of a hurricane.

2. The model

The model flow field consists of a pair of three-dimensional 'point potential vortices' with strengths Q_1 and Q_2 in a flow with constant horizontal and vertical shear. The two vortices are advected by the background flow as they interact with each other (see Fig. 1). Following HS, it will be convenient to look upon the vortex pair as a crude representation of a continuous vortex. The separation of the point vortices represents the size and deformation of the analogous continuous vortex, and their strengths represent the integrated potential vorticities in the upper and lower halves, respectively. In their Meddy study, HS used a three-layer point vortex model which incorporated point vortices in the upper two layers, with a uniform

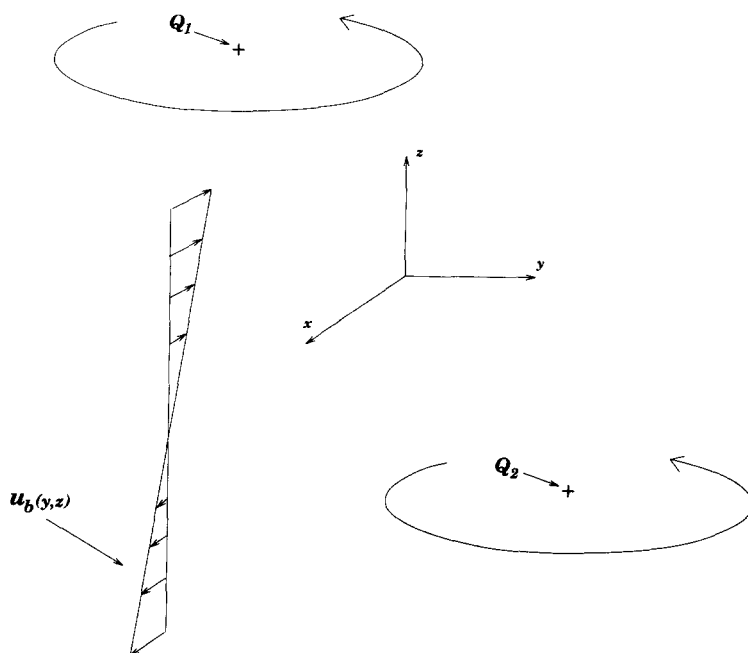


Fig. 1. Schematic diagram of two-point potential vortices of strengths Q_1 and Q_2 in a background flow with uniform shear.

flow in the uppermost layer to mimic the effect of large-scale vertical shear on Meddies. This upper layer flow introduced a thickness gradient or 'equivalent β ' which they neglected in their analysis. We shall see that no such approximation is necessary within the continuously stratified model framework.

Like HS, we will examine the influence of vertical shear on the translating vortex pairs. However, we will also examine the effect of horizontal shear on the vortex pairs, and will discuss the stability of the steadily translating configurations, neither of which was considered by HS. The model we will use is sufficiently simple that exact nonlinear solutions can be obtained. Solutions representing steadily translating point vortex pairs are found, and in many cases multiple equilibrium solutions exist. It is well known that in the absence of an external flow, purely antisymmetric vortex pairs (i.e. two vortices with equal and opposite potential vorticities) will translate steadily, carrying a region of trapped fluid with them as they move. What is not so well known is that in the presence of external shear, asymmetric propagating solutions can also be found. It will be shown that for a given vertical separation and background shear, there can exist up to three steadily translating solutions, each with a different horizontal separation between the vortices, and each aligned perpendicular to the background flow direction. For each of these solutions, the sum ($Q_1 + Q_2$) of the vortex strengths is required to have a certain value, but the individual strengths of the vortices may vary. The translation speed of the pair varies linearly with $(Q_1 - Q_2)$.

Let us consider a pair of point vortices located at $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$ in a background flow with constant potential vorticity $q = q_b$. The expression relating the geostrophic streamfunction ψ to the potential vorticity q is (see, e.g. Pedlosky, 1987)

$$\psi_{xx} + \psi_{yy} + \psi_{zz} = q \quad (1)$$

$$\psi \rightarrow \psi_b \quad (r \rightarrow \infty)$$

where by assumption the potential vorticity field is of the form

$$q = q_b + 4\pi Q_1 \delta(\mathbf{r} - \mathbf{r}_1) + 4\pi Q_2 \delta(\mathbf{r} - \mathbf{r}_2) \quad (2)$$

It is assumed that the Brunt–Väisälä frequency N is constant, and the z -coordinate has been scaled such that $z \rightarrow (fL)/(ND)z$, giving the Poisson equation (1). It may be readily verified that a solution to the set (1), (2) is

$$\psi = \psi_b - \frac{Q_1}{|\mathbf{r} - \mathbf{r}_1|} - \frac{Q_2}{|\mathbf{r} - \mathbf{r}_2|} \quad (3)$$

$$\psi_b = -\alpha yz + \frac{q_b}{2} y^2$$

The velocities (u, v) are related to the streamfunction in the usual fashion:

$$u = -\frac{\partial \psi}{\partial y} \quad (4)$$

$$v = +\frac{\partial \psi}{\partial x}$$

and it follows from the form of ψ_b that the background flow has uniform horizontal and vertical shear:

$$u_b = \alpha z - q_b y \quad (5)$$

Of course, (3b) is not the most general form for ψ_b , but it will be sufficient for our purposes, as it allows an examination of the effect of both horizontal and vertical shear on the point vortex pairs.

The influence of rotation in the point vortex fields, which is hidden in the present scaling, can be clarified by writing down the velocity for a single point vortex centred at $x = y = z = 0$. In dimensional (starred) coordinates it can be shown that the velocity, which is completely azimuthal, varies in proportion to $r_*/(r_*^2 + z_*^2 + N^2/f^2)^{3/2}$, where r_* is the horizontal radial distance from the z_* -axis. For strong rotation ($N/f \ll 1$) the velocity field is columnar, with weak decay in the vertical; for weak rotation ($N/f \gg 1$) the velocity field is pancake-like, with relatively strong vertical decay. A typical value of N/f in the Canary Basin (where Mediterranean Salt Lenses are most often found) is 25. In the Appendix the wavenumber dependence of the point vortex streamfunction is investigated, and it will be shown that for large vertical wavenumbers ($k \gg fr/N$) the streamfunction has an exponential character, whereas for small vertical wavenumbers it behaves logarithmically.

Expressed in Lagrangian terms, the statement of potential vorticity conservation takes the form

$$\frac{dq}{dt} = 0 \quad (6)$$

$$\frac{dx_n}{dt} = u(\mathbf{r}_n)$$

$$\frac{dy_n}{dt} = v(\mathbf{r}_n)$$

$$\frac{dz_n}{dt} = 0$$

where $n = 1, 2$. Taking the horizontal derivatives of (3), using (6), and evaluating the resulting expressions at (x_1, y_1, z_1) and (x_2, y_2, z_2) gives a set of autonomous, nonlinear equations governing the motions of the vortices:

$$\frac{dx_1}{dt} = \frac{-Q_2(y_1 - y_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} + u_{b,1} \quad (7)$$

$$\frac{dx_2}{dt} = \frac{-Q_1(y_2 - y_1)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} + u_{b,2}$$

$$\frac{dy_1}{dt} = \frac{Q_2(x_1 - x_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}$$

$$\frac{dy_2}{dt} = \frac{Q_1(x_2 - x_1)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}$$

$$\frac{dz_1}{dt} = \frac{dz_2}{dt} = 0$$

These expressions show that the velocity of each vortex is equal to the sum of the velocity resulting from advection by the other vortex and that owing to advection by the background flow. In Fig. 2 the velocity field for a single point vortex is contoured. Velocities decay monotonically in all directions from the vortex centre, where they are infinite. It should be noted, however, that for any $z \neq 0$, the velocity is maximum some distance away from the vertical axis, as shown by the diagonal dashed lines.

Figs. 3(a)–3(i) show the trajectories (viewed from above) of two point vortices with strengths $Q_1 = 3/2$, $Q_2 = 1/2$, for various initial configurations. These were obtained by integrating the Eqs. (7) numerically. In each case the trajectory of the stronger vortex is shown by a continuous line, that of the weaker vortex by a

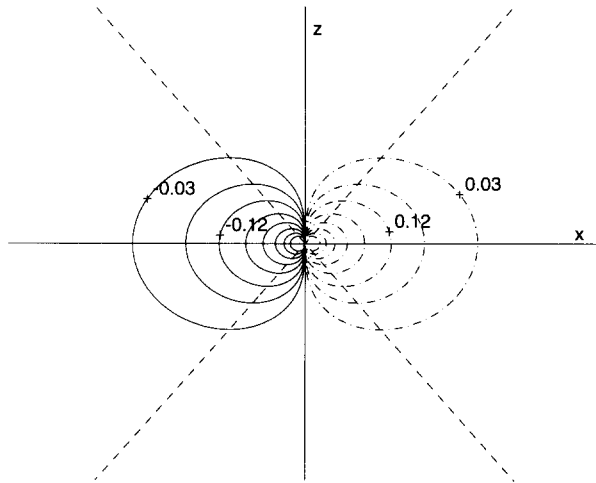


Fig. 2. Contour plot of velocity field for a single point vortex. Velocities are infinite at the origin, and decay monotonically to zero in all directions. The dashed lines connect the points at which the velocity is maximum as a function of perpendicular distance from the z -axis.

dashed line, and that of their centre of potential vorticity $(Q_1 \mathbf{r}_1 + Q_2 \mathbf{r}_2)/(Q_1 + Q_2)$ by a dotted line. The vortices are initially at $x_1 = x_2 = 0$, $z_1 = 1$, $z_2 = -1$, and $y_1 = -y_2$, as shown by the small circles in the plots. Positioning the vortices in this fashion ensures that $u_{b1} = -u_{b2}$ at $t = 0$, so there is no net advection of the pair by the background flow. Fig. 3(a) shows that in the absence of external shear ($\alpha = q_b = 0$) the vortices describe circular orbits about their common centre of vorticity. The remaining plots show translating vortex pairs in shear. In Fig. 3(b) a very small external vertical shear ($\alpha = 0.01$) is introduced, with the result that the orbits no longer close on themselves, and there is a slow drift to the right. Figs. 3(c)–3(e) show the vortices in horizontal shear ($\alpha = 0.0$, $q_b = -0.05$); Figs. 3(f)–3(i) show them in vertical shear ($\alpha = 0.05$, $q_b = 0.0$). In each of these sequences, the external shear is held fixed and the initial y separation of the vortices is varied. In Fig. 3(c) the initial y separation $y_2 - y_1$ is -2.78 , in Fig. 3(d) the separation is -2.0 , in Fig. 3(e) it is -1.0 , in Fig. 3(f) it is -3.70 , in Fig. 3(g) it is -0.448 , in Fig. 3(h) it is 0.0 , and in Fig. 3(i) the initial separation is $+1.0$.

The mechanism behind the propagation of the vortex pairs is simple, and can be seen in its purest form in the propagation of a purely antisymmetric pair ($Q_1 = 1$, $Q_2 = -1$) in a quiescent fluid. This situation is shown schematically in Fig. 4(a). For such a pair, the circulations of the vortices have opposite signs, so the advection of the second vortex by the first is in the same direction (and of the same magnitude) as the advection of the first vortex by the second — this leads to a net translation of the pair. The propagation mechanism is similar when the vortex pair is not purely antisymmetric, but in this case an external shear is needed to counterbalance the influence of the symmetric component of the potential vorticity

field, giving the vortices a preferred orientation (in a time-average sense) with respect to the external flow. For example, let us consider a vortex pair in a vertically sheared background flow, positioned such that $x_1 = x_2$, $y_1 = -y_2$, and $z_1 = -z_2$, as shown in Fig. 4(b). The vortices are both cyclonic, but the shallower one is stronger. The velocity of each vortex is equal to the sum of the background velocity (which is equal and opposite for the two vortices) and the velocity induced

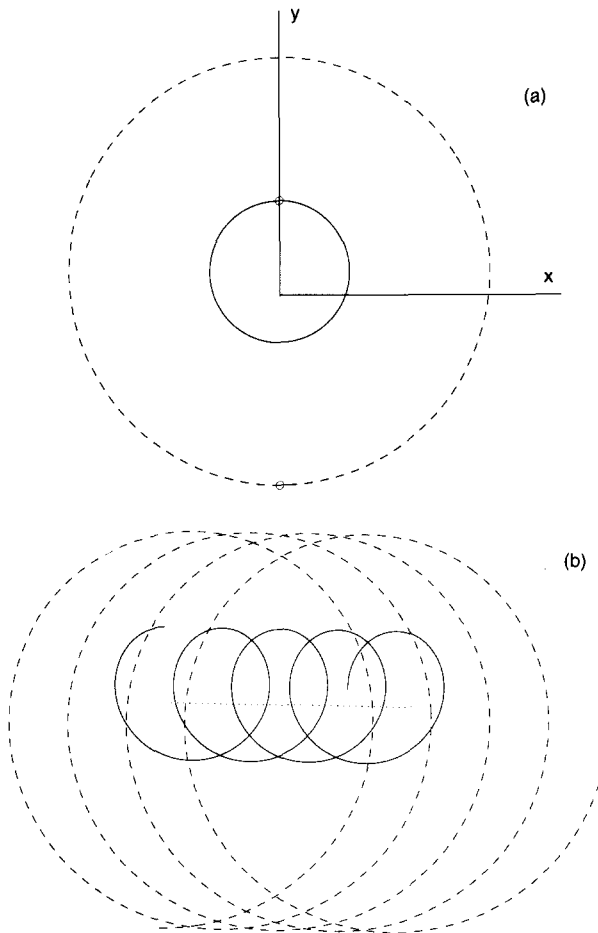


Fig. 3. Point vortex trajectories obtained by integrating (7) in time. The vortices are of different strengths ($Q_1 = 3/2$, $Q_2 = 1/2$), and are initially situated such that $x_1 = x_2 = 0$, $y_1 = -y_2$ and $z_1 = -z_2$. Plot (a) shows two point vortices circling one another in a quiescent fluid; (b) demonstrates the effect of adding a weak external vertical shear ($\alpha = 0.01$). Plots (c)–(e) show the two vortices in horizontal shear ($\alpha = 0.0$, $q_b = -0.05$), for various initial y separations, and (f)–(i) show their trajectories in a vertically sheared flow ($\alpha = 0.05$, $q_b = 0.0$).

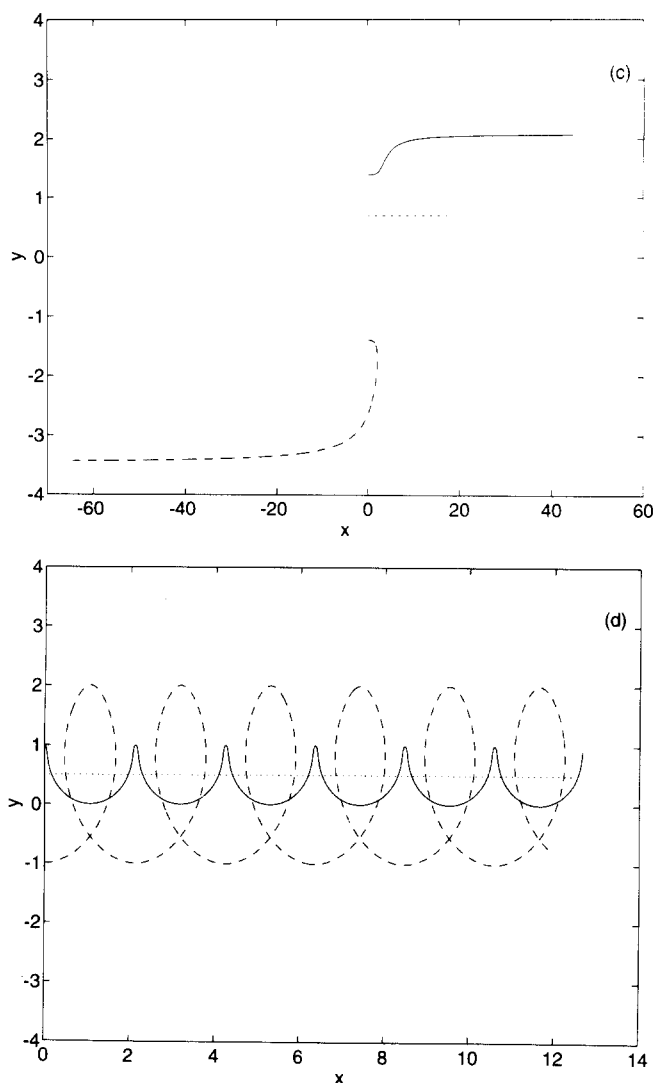


Fig. 3 (continued).

by the other vortex. The mutually induced velocities are also of opposite sign for the two vortices, but have different magnitudes. This situation is shown schematically in Fig. 4(b). If the total vorticity of the pair is sufficiently large, then it is possible for the sum of the background and induced velocity to be the same for each of the vortices, leading to a uniform translation of the pair.

3. Translation of a vortex pair

In examining the translation of the vortex pairs, it will be convenient to define a ‘centre of potential vorticity’:

$$\tilde{x} \equiv \frac{Q_1 x_1 + Q_2 x_2}{Q_1 + Q_2} \quad (8)$$

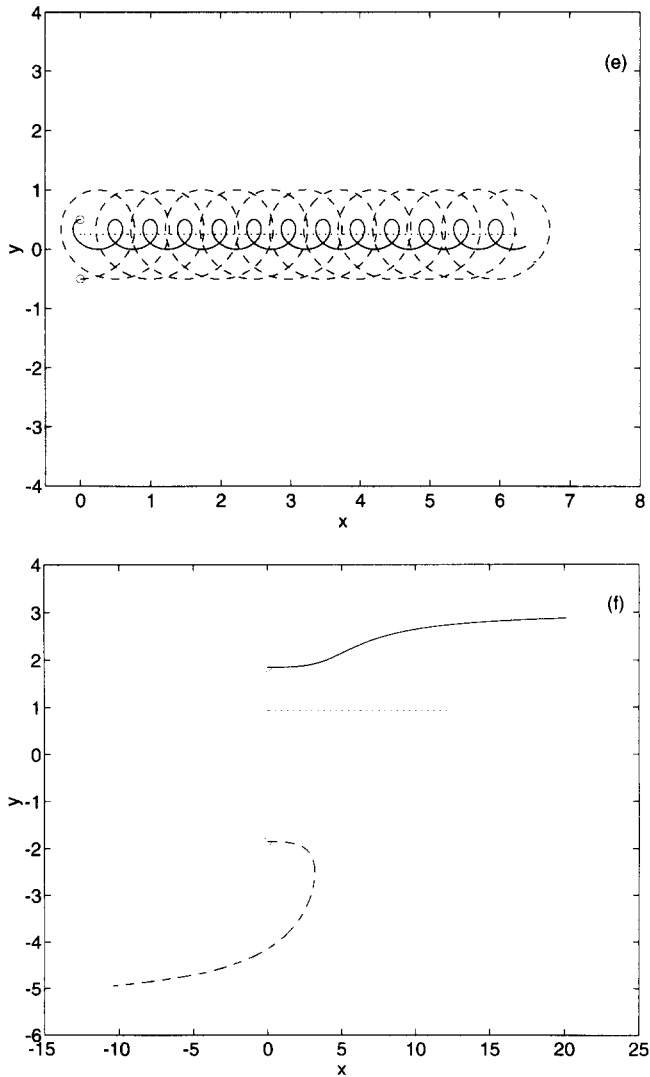


Fig. 3 (continued).

$$\tilde{y} \equiv \frac{Q_1 y_1 + Q_2 y_2}{Q_1 + Q_2}$$

$$\tilde{z} \equiv \frac{Q_1 z_1 + Q_2 z_2}{Q_1 + Q_2}$$

Using (7) and (8), one readily obtains evolution equations for $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$:

$$\frac{d\tilde{x}}{dt} = \frac{Q_1 u_{b,1} + Q_2 u_{b,2}}{Q_1 + Q_2} \equiv \tilde{u}_b \quad (9)$$

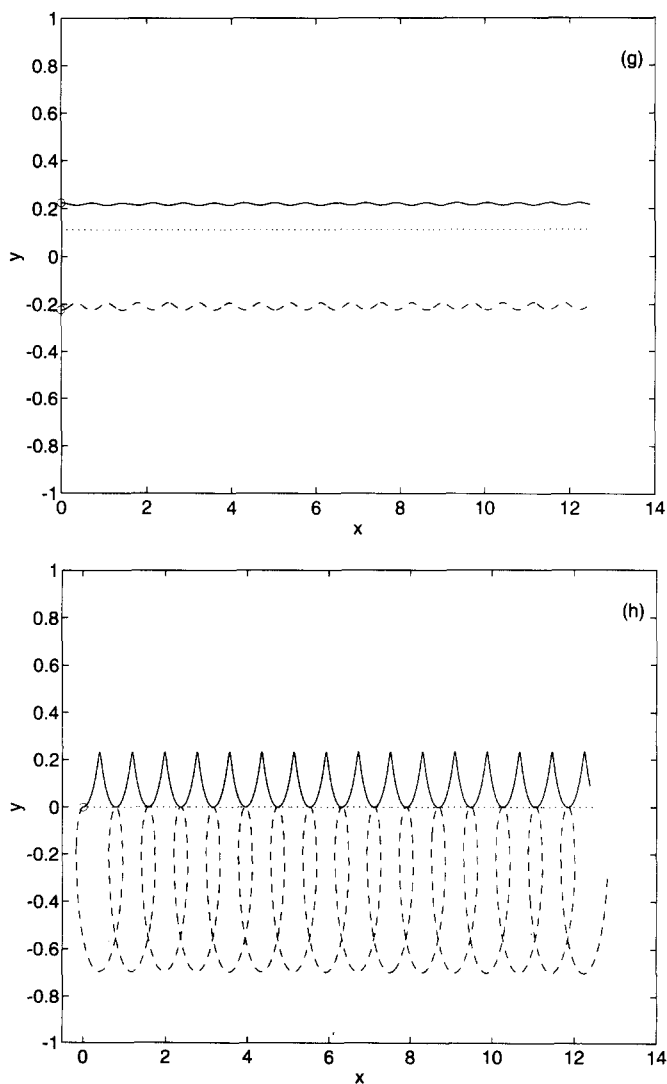


Fig. 3 (continued).

$$\frac{d\tilde{y}}{dt} = 0$$

$$\frac{d\tilde{z}}{dt} = 0$$

After making use of the definition u_b , the right-hand side of (9a) can be written in the form

$$\tilde{u}_b = \frac{1}{2} \Delta \alpha (z_1 - z_2) - q_b \tilde{y} \quad (10)$$

where the notation

$$\Delta \equiv \frac{Q_1 - Q_2}{Q_1 + Q_2} \quad (11)$$

has been introduced. The parameter Δ measures the relative magnitudes of the antisymmetric component $(Q_1 - Q_2)/2$ and the symmetric component $(Q_1 + Q_2)/2$ of the potential vorticity field. Now, it follows from (6d) that both z_1 and z_2 are constant, and (9b) shows that $\tilde{y}$ is also constant. Therefore $\tilde{u}_b$ is constant, and the centre of vorticity of the pair moves at the constant rate

$$u_0 \equiv \tilde{u}_b = \frac{1}{2} \Delta \alpha (z_1 - z_2) - q_b \tilde{y} \quad (12)$$

It should be noted that this result is true for all solutions — even if the vortices go around one another in a complicated time-dependent fashion, their centre of potential vorticity moves in a straight line with constant speed. This can be clearly

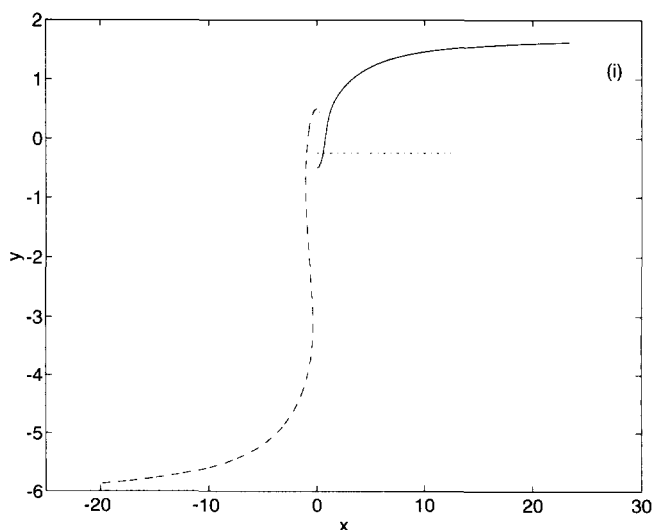


Fig. 3 (continued).

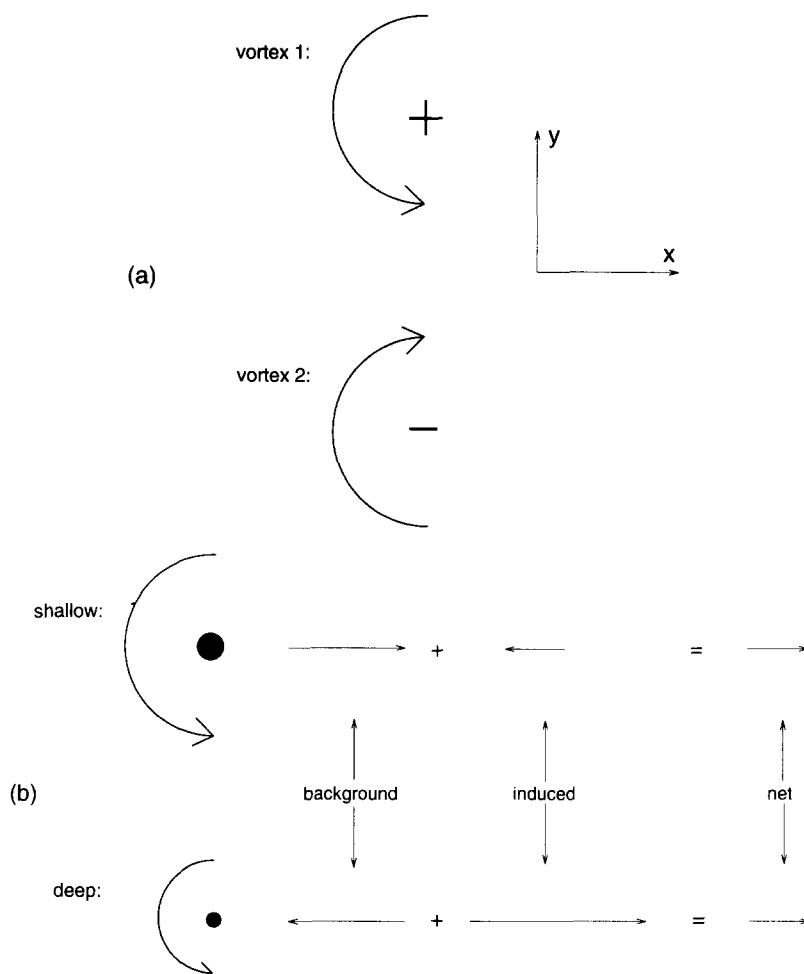


Fig. 4. The mechanism by which oppositely signed point vortices propagate is shown in (a). Vortex 1 is advected by the anticyclonic flow field of Vortex 2, causing it to move to the right. At the same time, Vortex 2 is advected to the right by Vortex 1, causing the pair as a whole to move to the right. (b) shows the situation for two vortices of the same sign in a background shear. Steady translation can occur when the sum of the velocity owing to the background flow and the velocity induced by the other vortex are equal for the two vortices.

seen in Fig. 3, where the dotted line shows the path of the centre of potential vorticity. The behaviour of the centre of potential vorticity is generally far simpler than that of the geometric centre of the pair, as the centre of potential vorticity moves at a constant rate even when the motion is unsteady. The velocity of the geometric centre is, in general, unsteady. Finally, using the definitions of $\tilde{y}$ and $\tilde{z}$ it is a simple matter to show that

$$u_0 = u_b(\tilde{y}, \tilde{z}) \quad (13)$$

which demonstrates that the speed of the pair is given by the speed of the external flow at the centre of potential vorticity. This result is a direct consequence of the linear form of the background flow field.

4. Relative motions of the vortices

To examine the relative motions of the two vortices it is convenient to cast the set (7) in a new form, which explicitly decouples the translation of the vortices from their relative motions. For convenience, the notation $X(t) \equiv x_1 - x_2$, $Y(t) \equiv y_1 - y_2$, $Z \equiv z_1 - z_2$, $R(t) \equiv (X^2 + Y^2 + Z^2)^{1/2}$ is introduced, after which subtracting (7b) from (7a), and (7d) from (7c) gives

$$\begin{aligned}\frac{dX}{dt} &= -\frac{(Q_1 + Q_2)Y}{R^3} + \alpha Z - q_b Y \\ \frac{dY}{dt} &= +\frac{(Q_1 + Q_2)X}{R^3} \\ \frac{dZ}{dt} &= 0\end{aligned}\tag{14}$$

These equations describe the evolution of the components of the displacement vector between the vortices. It should be noted that only the symmetric component of q appears in (14); the antisymmetric component $(Q_1 - Q_2)/2$ is important in determining the bulk translation of the pair, but does not affect their relative motions.

Before considering time-dependent solutions to (14), it is instructive to examine the possible steady solutions. Setting $dX/dt = dY/dt = 0$ gives

$$X_s = 0\tag{15}$$

which shows that all steady configurations are perpendicular to the external flow direction, and

$$\frac{Q_1 + Q_2}{(Y_s^2 + Z_s^2)^{3/2}} Y_s = \alpha Z_s - q_b Y_s\tag{16}$$

where the subscript s is used to denote a steady solution to (14). Eq. (16) is simply the mathematical statement of the fact that for a steady solution to exist, the tendency of the background flow to separate the vortices must be exactly counter-balanced by the mutual advection of the vortices. It should be noted that the solutions to (16) do not uniquely specify a point vortex configuration. As an example, let us suppose that for given values of α , q_b , $(Q_1 + Q_2)$, and Z_s some Y_s is found which satisfies (16). It is easy to see that there are in fact an infinity of such solutions, each with the same total vorticity $(Q_1 + Q_2)$. However, each of these solutions translates at a different rate, as the individual strengths of the vortices may vary.

Before discussing the steady solutions in their most general form, it is important to mention the special case $Q_1 + Q_2 = 0$, in which it is easily verified that the steady solution to (16) is given by $Y_s = \alpha Z_s / q_b$. However, when weakly perturbed, any such solutions will be pulled apart at a linear rate by the external shear (there is no ‘restoring force’). This suggests that pure dipoles are not robust structures, as a non-zero symmetric potential vorticity component is required to keep them together in shear — even when the shear is extremely small. As a result of this, it will be assumed that $(Q_1 + Q_2)$ is nonzero in the discussion that follows.

The steady balance (16) can be conveniently expressed in terms of the tilt angle Θ , defined by

$$\Theta = \tan^{-1}(Y_s/Z_s) \quad (17)$$

which is a measure of the angle that a line connecting the vortices makes with respect to the vertical. Assuming that Z_s is nonzero, then $\Theta = 0$ if the vortices are vertically aligned, whereas if $\Theta = \pm\pi/2$ the separation distance Y_s is infinite. In the special case in which $Z_s = 0$ it can be shown (Walsh, 1992) that all steadily translating solutions are linearly unstable, and therefore we will assume from now on that Z_s is nonzero. In terms of Θ , (16) takes the form

$$\sin \Theta \cos^2 \Theta = \hat{\alpha} - \hat{q}_b \tan \Theta \quad (18)$$

where $\hat{\alpha}$ and $\hat{q}_b$ are defined by

$$\begin{aligned} \hat{\alpha} &= \frac{\alpha Z^3}{Q_1 + Q_2} \\ \hat{q}_b &= \frac{q_b Z^3}{Q_1 + Q_2} \end{aligned} \quad (19)$$

The roots of (18) characterize the possible steady solutions as functions of the parameters $\hat{\alpha}$ and $\hat{q}_b$. Depending on the values of $\hat{\alpha}$ and $\hat{q}_b$, (18) may have no roots, or as many as three.

Owing to the relative complexity of Eq. (18), solutions were obtained numerically. Fig. 5(a) shows the roots as a function of the horizontal shear $\hat{q}_b$ for various values of $\hat{\alpha}$. A dashed line denotes an unstable solution, whereas a continuous line represents a stable root. The details of the stability calculation will be discussed shortly. First, let us consider the case in which $\hat{\alpha}$ is zero (i.e. no external vertical shear). In this case, as long as $\hat{q}_b$ is less than -1 there is just one (unstable) root at $\Theta = 0$, representing a vertically aligned pair. As $\hat{q}_b$ increases through -1 a ‘pitchfork’ bifurcation occurs, in which the unstable root gives rise to a stable root plus two unstable roots, which represent tilted configurations. When $\hat{q}_b$ becomes positive the two unstable roots disappear, leaving one stable root at $\Theta = 0$. The second curve shows the case in which $\hat{\alpha} = -0.25$. We see that the introduction of a small vertical shear has destroyed the symmetry that is evident when $\hat{\alpha} = 0.0$.

In Fig. 5(b) the roots of Eq. (18) are plotted as functions of the vertical shear $\hat{\alpha}$ for selected values of $\hat{q}_b$. First, let us consider the case $\hat{q}_b = 0$, which is shown by

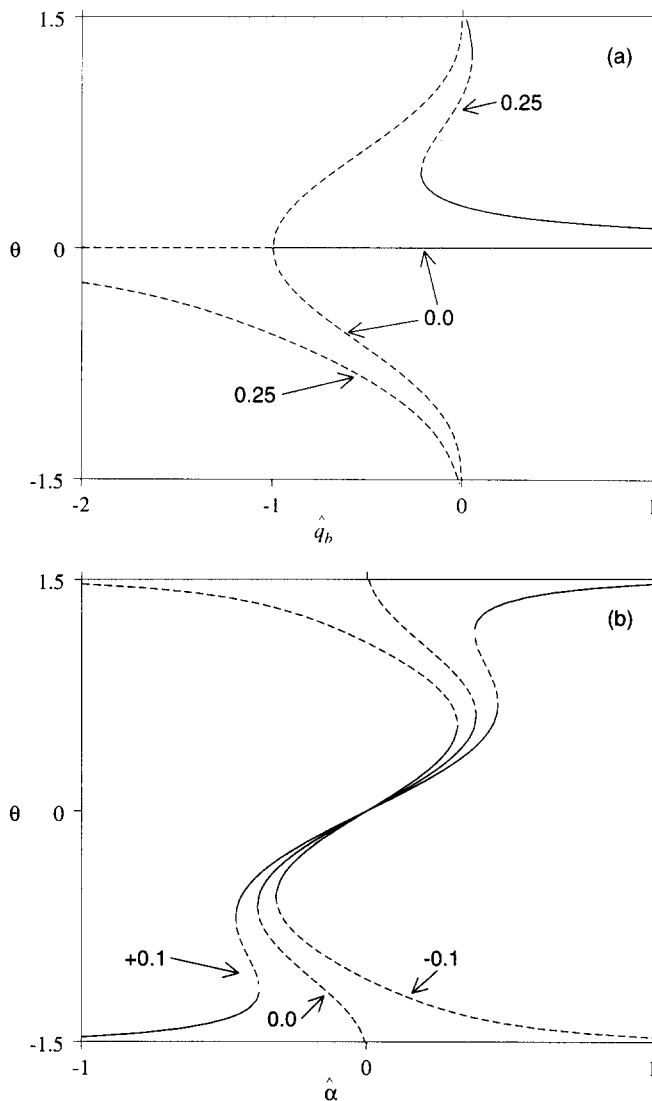


Fig. 5. Roots of Eq. (18), representing steady solutions in external shear. (a) shows the roots as a function of the horizontal shear $\hat{q}_b$ for various values of $\hat{\alpha}$; (b) shows the dependence on the vertical shear $\hat{\alpha}$ for various values of $\hat{q}_b$. Continuous lines represent stable solutions; unstable solutions are shown by dashed lines.

the middle curve. If $|\hat{\alpha}|$ is not too large there are in general two solutions, whereas if $|\hat{\alpha}|$ exceeds a critical value there are no solutions, as the shear is too large for the vortices to remain coupled. In the vertical shear case, the possibility of more than one equilibrium solution is a direct consequence of the non-monotonic

character of the point vortex flow field: the velocity reaches a maximum at a finite distance from the centre on any horizontal plane above or below the vortex. Next, let us consider the situation in which $\hat{q}_b$ is nonzero and positive ($\hat{q}_b = 0.1$). It should be noted that for $\hat{\alpha} = \hat{q}_b = 0$ there are roots at $\Theta = \pm \pi/2$. However, the addition of a very small external horizontal shear shifts these roots to $\hat{\alpha} \rightarrow \pm \infty$, implying that these roots are not structurally stable. The cause of this behaviour is simple: the roots at $\Theta = \pm \pi/2$ represent configurations in which the separation Y_s is infinite, so that the vortices cannot interact. Thus, the addition of even a very small horizontal shear can only be balanced by the addition of a compensating vertical shear.

5. Stability analysis

The linear stability properties of the steady solutions will now be investigated. This is done by superimposing small perturbations on the solutions, and then deducing whether the perturbations grow by solving linearized stability equations. Putting $X = X_s + X'$, $Y = Y_s + Y'$, $Z = Z_s + Z'$ in (14), using the steady-state relation (16), and assuming that the perturbation quantities are small (i.e. $X'^2 + Y'^2 + Z'^2 \ll X_s^2 + Y_s^2 + Z_s^2$) gives linearized evolution equations for X' and Y' :

$$\frac{d^2 X'}{dt^2} + \mu^2 X' = 0 \quad (20)$$

$$\frac{d^2 Y'}{dt^2} + \mu^2 Y' = \kappa Z'$$

where

$$\mu^2 \equiv \frac{(Q_1 + Q_2)^2}{R_s^6} \left(\frac{\hat{\alpha}}{\sin \Theta \cos^2 \Theta} - 3 \sin^2 \Theta \right) \quad (21)$$

$$\kappa \equiv \frac{(Q_1 + Q_2)^2}{R_s^6} \left(\frac{\hat{\alpha}}{\cos^3 \Theta} + 3 \sin \Theta \cos \Theta \right)$$

and Eqs. (17) and (18) were used to write μ^2 and κ in terms of the angle Θ . The forcing term $\kappa Z'$ gives rise to a time-independent correction $Y' = \kappa Z' / \mu^2$, which can be ignored for the purposes of this stability analysis. We therefore set $Z' = 0$ with no loss of generality. From the definition of μ^2 , we see that the steady-state solutions are linearly stable if the quantity

$$\mathcal{D} \equiv \frac{\hat{\alpha}}{\sin \Theta \cos^2 \Theta} - 3 \sin^2 \Theta \quad (22)$$

is greater than zero, as is clear from the form of (20) that X' and Y' will be oscillatory if $\mathcal{D} > 0$, and will grow exponentially in time if $\mathcal{D} < 0$. The stability of the steady roots shown in Fig. 5 was determined by computing the value of $\mathcal{D}$ for

each point in the plot. When $\hat{\alpha} = 0$, so that the background flow has only horizontal shear, (22) shows that all tilted vortex configurations are unstable, which is consistent with the results shown in Fig. 5(a).

The instability mechanism can be understood more clearly by rewriting μ^2 in terms of the basic-state flow field. This can be accomplished using (14a):

$$\frac{dX}{dt} = U(X, Y, Z) \quad (23)$$

$$U(X, Y, Z) \equiv -\frac{(Q_1 + Q_2)Y}{R^3} + U_b$$

Differentiating with respect to time gives

$$\frac{d^2X}{dt^2} = \frac{\partial U}{\partial X} \frac{dX}{dt} + \frac{\partial U}{\partial Y} \frac{dY}{dt} \quad (24)$$

We now linearize about the steady state by putting $U = U_s + U'$, $X = X_s + X'$, $Y = Y_s + Y'$, $Z = Z_s + Z'$, where U_s is defined by

$$U_s = -\frac{(Q_1 + Q_2)Y}{(Y^2 + Z^2)^{3/2}} + U_b \quad (25)$$

and it follows that the linearized form of (24) is

$$\frac{d^2X'}{dt^2} \approx \frac{\partial U_s}{\partial Y} \frac{dY'}{dt} \quad (26)$$

Using the linearized form of (14b) it follows that

$$\frac{d^2X'}{dt^2} \approx \frac{Q_1 + Q_2}{R_s^3} \frac{\partial U_s}{\partial Y} X' \quad (27)$$

where $\partial U_s / \partial Y$ is to be evaluated at the point (Y_s, Z_s) , where $U_s = 0$. Comparing this with (20a) shows that μ^2 can be written in the form

$$\mu^2 = -\frac{Q_1 + Q_2}{R_s^3} \frac{\partial U_s}{\partial Y} \quad (28)$$

and it follows that for fixed Q_1 and Q_2 , the sign of $\partial U_s / \partial Y$ determines the stability of the solutions. It is easy to show that (28), together with the steady-state condition $U_s = 0$ leads to the expression for μ^2 found in Eq. (21).

The above discussion indicates that the stability properties of the steady solutions may be determined graphically. The graphical solution to (16) is shown in Fig. 6. The tendency of mutual interactions of the vortices to produce relative motions is shown by the continuous line labelled $U_v (U_v \equiv -(Q_1 + Q_2)Y_s/R_s^3)$; the differential advection by the background flow ($U_b \equiv \alpha Z - q_b Y$) is represented by the dashed line. Steady solutions lie at the intersections between the two curves.

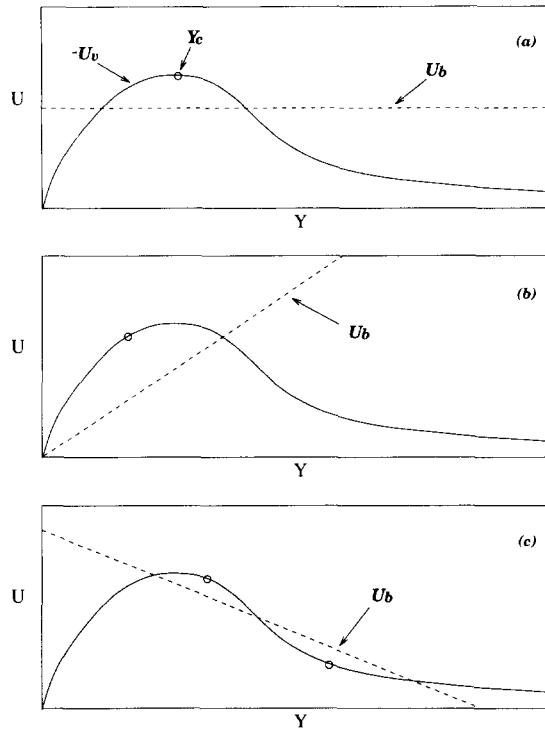


Fig. 6. Graphical solution of (16) for various combinations of $\hat{q}_b$ and $\hat{\alpha}$. The roots are represented by the intersections of the curves marked U_v and U_b . The stability of these roots depends upon the relative slopes of these curves near the intersection points (see text).

The stability of these solutions is determined by the relative slopes of the curves U_v and U_b near the intersection points. In particular, the solutions are unstable if

$$\frac{\partial U_b}{\partial Y} > -\frac{\partial U_v}{\partial Y} \quad (29)$$

where the derivatives are evaluated at the fixed point. Three cases have been illustrated: the first is the case in which $\hat{q}_b = 0$ (pure vertical shear), the second is that in which $\hat{\alpha} = 0$ (pure horizontal shear), and in the third case $\hat{\alpha}$ is negative and $\hat{q}_b$ is positive. The stability of the solutions can be inferred directly: if the slope of U_v is greater (more positive) than that of U_b , the solution is linearly stable; otherwise, it is unstable. The small circles represent the points where the slopes are equal. In Fig. 6(a), the inequality in (29) is satisfied everywhere to the right of the point Y_c indicating that the right-most equilibrium point is unstable. In Fig. 6(b) two roots are shown: the one at $Y=0$ is stable, whereas the second is unstable, according to (29). In this case there is an additional unstable root, which is the mirror image of the first unstable root. Fig. 6(c) shows the situation in which

there is both horizontal and vertical shear. It is clear that in this case there are three roots, and the inequality (29) shows that only the middle root is unstable.

Using the ideas discussed in the last two paragraphs, the instability mechanism can be illustrated in simple physical terms. If we begin with a particular steady solution and separate the vortices slightly, the vortices will advect each other at a slightly different rate, and will also experience a different differential advection by the background flow (denoted by U_b). If U_b increases more rapidly than the mutual advection, the pair will be rotated in the direction of the external shear. This partial alignment with the background shear will lead to a further separation of the vortices. This scenario therefore represents the unstable case. Conversely, if U_b increases less rapidly than the mutual advection, the perturbed pair will rotate into the external shear. The external flow will then tend to push the vortices back together — this scenario represents the stable case.

6. Phase plane description

The set (14) represents a nondivergent flow in (X, Y, Z) space. The character of the flow can be understood by plotting solution trajectories. As a result of the simple form of the third equation, the flow is two dimensional, and has only a parametric dependence upon Z . It is therefore sufficient to consider trajectories in the (X, Y) plane. Fixed points in (X, Y) space represent steadily translating configurations, whereas closed trajectories represent solutions which are periodic in a translating reference frame. Saddle points represent unstable steady solutions; centres represent stable steady solutions. We shall find that all solutions are periodic if the external shear is not too large, and are aperiodic otherwise.

Let us consider first background flows with purely horizontal shear. Fig. 7(a) shows the phase plane trajectories for the case in which $z_1 = 1$, $z_2 = -1$, and $Q_1 + Q_2 = 2$, consistent with the point vortex trajectories shown in Fig. 3. The qualitative behaviour of the solutions depends on the sign of q_b . When q_b is negative ($q_b = -0.05$) there are three fixed points in the (X, Y) plane. The first is a centre at $X = Y = 0$, corresponding to a stable solution. The closed trajectories surrounding the elliptic point representing periodic steady-state solutions. In addition, there are two hyperbolic points located on the Y -axis, representing unstable solutions. Both of these solutions represent configurations in which the vortices are displaced at right angles to the external flow. The symbols in the plots represent the initial configurations for the numerical runs shown in Fig. 3. The ‘○’ located near the saddle point in Fig. 7(a) represents the initial condition for the run in Fig. 3(c), and it shows that the run was very close to an unstable steady solution. The ‘+’ symbol represents the run shown in Fig. 3(d), and ‘×’ marks the starting point for the run in Fig. 3(e).

When q_b is positive ($q_b = 0.05$) there is just one fixed point (as shown in Fig. 7(b)) — the same stable fixed point at the origin discussed above. In this case, however, there are no other fixed points, as the background shear is of the wrong sense to balance the motions of the vortices. It is worth noting that in this case all

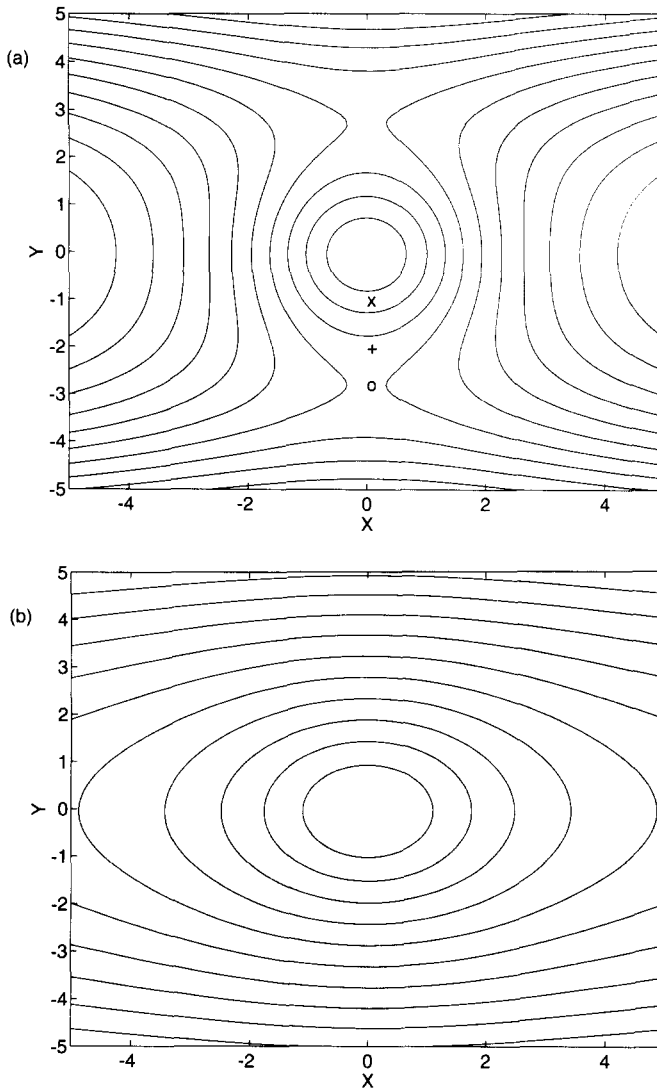


Fig. 7. Phase plane behaviour of point vortex pairs in the (X, Y) plane. (a) shows the case in which the vortices are in horizontal shear ($q_b = -0.05$); in (b) the vortices are again in horizontal shear, but with $q_b > 0$ ($q_b = 0.05$), whereas in (c) they are in vertical shear ($\alpha = 0.05$). The fixed points represent steadily translating point vortex configurations. The 'O' in (a) marks the initial condition for the run in Fig. 3(c), the '+' represents the run in 3(d), and the 'x' represents the run shown in 3(e). The 'x' in (c) shows the initial condition for the trajectory in Fig. 3(f), '+', 'O', and '*' represent the trajectories shown in Figs. 3(g), 3(h), and 3(i), respectively.

trajectories are closed, so the vortices cannot be carried arbitrarily far apart by the flow, and therefore they remain at least weakly coupled for all time. This implies that any coupled vortex configuration of this kind (not necessarily steady) is more robust when $q_b > 0$ than when $q_b < 0$.

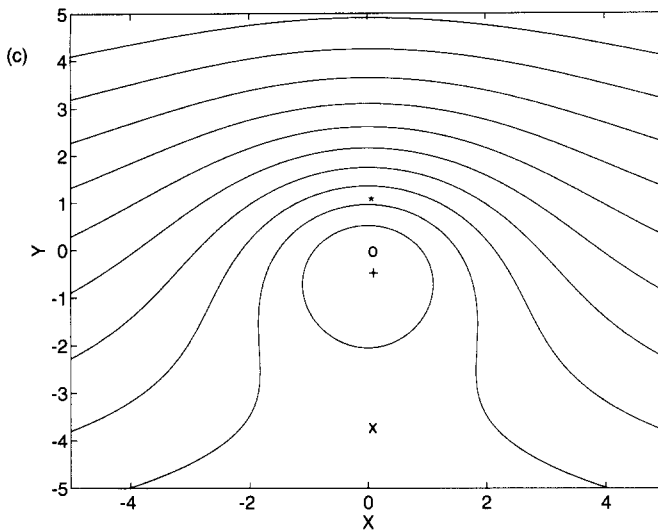


Fig. 7 (continued).

If the background flow is vertically sheared ($\alpha = 0.05$) the situation is different, as shown in Fig. 7(c) (again with $z_1 = 1$, $z_2 = -1$, and $Q_1 + Q_2 = 2$). In this case there are two fixed points — one stable, one unstable. Both represent pairs which are 'tilted' by the background flow. There are no solutions representing vertically aligned vortex pairs, as were found in the horizontally sheared case. The saddle point near the bottom of the figure represents a strongly tilted unstable configuration; the centre near the middle of the plot represents a weakly tilted stable configuration. The '+' represents the initial condition for the run in Fig. 3(g), which was clearly close to the stable fixed point. The 'x' represents the run in Fig. 3(f) which was near the unstable fixed point. The 'O' and the '*' show the initial conditions for the runs in Figs. 3(h) and 3(i), respectively. As $\hat{\alpha}$ decreases, the region of closed trajectories surrounding the stable fixed point becomes smaller until eventually no bound-states are possible.

7. Discussion

A variety of solutions representing pairs of QG point potential vortices in uniform shear have been discussed. These translating point vortex solutions are the three-dimensional generalizations of the layered solutions of HS. Unlike HS, we have also considered the influence of horizontal shear, and a linear stability analysis of the family of steady solutions is presented. The mutual interactions between the vortices, together with the influence of the background flow on the pair, allow for a wide variety of possible behaviours. As expected, modon-like propagation can occur when the vortices have opposite signs. However, the vortex

pairs may also translate with respect to the ambient fluid when the vortices have the same sign, provided that there is a background shear, and the vortices have different strengths. The propagation mechanism is the same in either case — the antisymmetric component of the potential vorticity field causes the pair to propagate. The role of the background shear in the propagation is to give the pair an average transverse tilt, and the vortex induction mechanism then causes them to propagate in a direction parallel to that of the external flow.

Solutions periodic in a translating frame of reference were found for background flows with vertical and/or horizontal shear. Stable, steadily translating solutions exist in vertically sheared background flows, analogous to those discussed by HS. In horizontally sheared flows, all steadily propagating point vortex configurations were found to be unstable. The only possible stable equilibrium in this case was found to occur when the vortices are vertically aligned, so that the pair does not translate. For a given vertical separation between the vortices, and specified vertical and horizontal shear, there can be up to three steadily translating point vortex configurations, each with a different horizontal separation between the vortices, and each aligned perpendicular to the background flow. These solutions translate at a rate proportional to $(Q_1 - Q_2)$, which measures the asymmetry of the potential vorticity field. Changing $(Q_1 - Q_2)$ alters the translation speed of the vortex pair, but does not otherwise affect their behaviour as seen from a reference frame translating with speed u_0 . The difference between propagation of point vortex pairs in a quiescent fluid and in shear is that, in the latter case, a certain minimum symmetric potential vorticity component is needed to keep the vortices aligned with respect to the external flow. Pure dipoles are not viable in shear, as they cannot resist the tendency of the shear to tear them apart.

It is worth noting that there is no inherent limit on the asymmetry Δ (and hence on the speed) of the pair. This is different from the result reported by Flierl (1988), who noted that a class of solutions representing weakly perturbed columnar geostrophic vortices broke down if the depth variation of potential vorticity within the vortex core became too large. Flierl speculated that highly baroclinic solutions may have a more complicated structure involving patches of anomalous vorticity which close off in the vertical. Walsh (1992, 1995) observed a similar solution breakdown in a model of a lens-shaped geostrophic eddy in large-scale shear. If Flierl's speculation is correct, then highly baroclinic lenses may necessarily have a multiple-core structure similar to that of the 'double Meddy' documented by Prater and Sanford (1994) (and reminiscent of the point vortex representation we employ here).

Despite the inevitable differences between the point vortex and continuous representations, it is felt that these point vortex results provide considerable insight into the behaviour of oceanic lenses in large-scale shear. In particular, the solutions qualitatively capture the transverse deflection of the rotation axis of Mediterranean Salt Lenses which has been measured by Walsh (1992) and Walsh et al. (1995), using data from neutrally buoyant SOFAR floats deployed in Salt Lenses. The present model also reproduces the vortex-induction mechanism originally put forth by HS as a possible explanation for the observed translation of

Meddies. Walsh (1992, 1995) has shown that this mechanism is readily generalized to more realistic models of lens-like eddies which employ continuous representations of the potential vorticity field. It is very encouraging that this simple point vortex representation can reproduce many of the important physical mechanisms inherent in more sophisticated models.

Acknowledgements

The authors wish to acknowledge the support of the National Science Foundation and the Office of Naval Research. This research was funded by NSF Grants OCE-891 6446, OCE-911 5359 and OCE-87-00601, and by ONR Grant N 00014-89-J-1182.

Appendix: Wavenumber dependence of point vortex fields

In this Appendix the character of the short- and long-range influences in the continuously stratified point vortex model will be examined. Let us consider an impulsive potential vorticity distribution of the form

$$\psi_{xx} + \psi_{yy} + \left(\frac{f^2}{N^2} \psi_z \right)_z = \delta(x - x_0, y - y_0, z - z_0) \quad (\text{A1})$$

$$\psi \rightarrow 0, \quad \text{as } r \rightarrow \infty$$

If we assume for simplicity that N^2 is constant, then Fourier transforming (A1) in z gives

$$\hat{\psi}_{xx} + \hat{\psi}_{yy} - k^2 \frac{f^2}{N^2} \hat{\psi} = \delta(x - x_0, y - y_0) \exp(ikz_0) \quad (\text{A2})$$

where we use the standard notation

$$\hat{\psi} = \int_{-\infty}^{\infty} \psi e^{ikz} dz$$

The solution to (A2) is

$$\hat{\psi} = -\frac{\exp(ikz_0)}{2\pi} K_0 \left(\frac{fk}{N} r \right), \quad r \equiv [(x - x_0)^2 + (y - y_0)^2]^{1/2} \quad (\text{A3})$$

where K_0 is a modified Bessel function. We see that the streamfunction associated with a given vertical wavenumber has the same functional form as the baroclinic

modes in layer models (e.g. Hogg and Stommel, 1990). The modified Bessel function K_0 has the following asymptotic behaviour (e.g. Arfken, 1966):

$$K_0(\Phi) \approx \left(\frac{\pi}{2\Phi}\right)^{1/2} e^{-\Phi}, \quad \Phi \rightarrow +\infty \quad (\text{A4})$$

$$K_0(\Phi) \approx -\ln(\Phi), \quad \Phi \rightarrow 0^+$$

Thus $\hat{\psi}$ has the asymptotic behaviour

$$\hat{\psi} \approx -\frac{\exp(ikz_0)}{2\pi} \left(\frac{\pi N}{2fkr}\right)^{1/2} e^{-fkr/N}, \quad r \gg \frac{N}{fk} \quad (\text{A5})$$

$$\hat{\psi} \approx \frac{\exp(ikz_0)}{2\pi} \ln(fkr/N), \quad r \ll \frac{N}{fk}$$

For small vertical wavenumbers ($fkr/N \rightarrow 0$), the streamfunction is approximately logarithmic, whereas high vertical wavenumbers decay exponentially in the far-field. Thus for a disturbance with vertical wavenumber k , there is a ‘short-range’ influence which is logarithmic in character (which obtains when the range r is much less than the baroclinic deformation radius $k^{-1}N/f$), and a ‘long-range’ influence which decreases exponentially with distance (which applies when r is much greater than $k^{-1}N/f$). By inverse Fourier transforming (A3), one obtains the following expression for the streamfunction ψ :

$$\psi = -\frac{N}{4\pi f} \left[(x-x_0)^2 + (y-y_0)^2 + \frac{N^2}{f^2} (z-z_0)^2 \right]^{-1/2} \quad (\text{A6})$$

Thus, although the pressure field for a mode with a vertical wavenumber k varies as $K_0(fkr/N)$, superposing all vertical wavenumbers gives a field with an algebraic dependence on x , y , and z , as in (A6).

References

- Arfken, G., 1966. *Mathematical Methods for Physicists*. Academic Press, New York.
- Armi, L., Hebert, D., Oakey, N., Price, J., Richardson, P.L., Rossby, T. and Ruddick, B., 1989. Two years in the life of a Mediterranean Salt Lens. *J. Phys. Oceanogr.*, 19: 354–370.
- Beckmann, A. and Käse, R.H., 1989. Numerical simulation of the movement of a Mediterranean Salt Lens. *Geophys. Res. Lett.*, 16: 65–68.
- Colin de Verdiere, A., 1992. On the southward motion of Mediterranean Salt Lenses. *J. Phys. Oceanogr.*, 22: 413–420.
- Flierl, G.R., 1987. Isolated eddy models in geophysics. *Annu. Rev. Fluid Mech.*, 19: 493–530.
- Flierl, G.R., 1988. On the instability of geostrophic vortices. *J. Fluid Mech.*, 197: 349–388.
- Hogg, N.G. and Stommel, H.M., 1985. The heton, an elementary interaction between discrete baroclinic geostrophic vortices, and its implications concerning eddy heat flow. *Proc. R. Soc. Lond., Ser. A*, 397: 1–20.
- Hogg, N.G. and Stommel, H.M., 1990. How currents in the upper thermocline could advect Meddies deeper down. *Deep-Sea Res.*, 37: 613–623.

- Käse, R.H. and Zenk, W., 1987. Reconstructed Mediterranean salt lens trajectories. *J. Phys. Oceanogr.*, 17: 158–163.
- Legg, S. and Marshall, J., 1993. A heton model of the spreading phase of open-ocean deep convection. *J. Phys. Oceanogr.*, 23: 1040–1056.
- Pedlosky, J., 1987. *Geophysical Fluid Dynamics*. Springer, New York, 710 pp.
- Prater, M.D. and Sanford, T.B., 1994. A Meddy off Cape St. Vincent. Part I: Description. *J. Phys. Oceanogr.*, 24: 1572–1586.
- Richardson, P.L., Walsh, D., Armi, L., Schröder, M. and Price, J.F., 1989. Tracking three meddies with SOFAR floats. *J. Phys. Oceanogr.*, 19: 371–383.
- Richardson, P.L., McCartney, M.S. and Maillard, C., 1991. A search for Meddies in historical data. *Dyn. Atmos. Oceans*, 15: 241–265.
- Ruddick, B.R. and Hebert, D., 1988. The mixing of meddy “Sharon”. In: J.C. Nihoul and B.M. Jamart (Editors), *Small Scale Turbulence and Mixing in the Ocean*. Elsevier Oceanography Series, Vol. 46, Elsevier, New York, pp. 249–262.
- Walsh, D., 1992. A model of a Mediterranean Salt Lens in external shear. Ph.D. Thesis, Woods Hole Oceanographic Institution–Massachusetts Institute of Technology Joint Program in Oceanography, Woods Hole, MA, Cambridge, MA, 168 pp.
- Walsh, D., 1995. A model of a mesoscale lens in external shear; part I: linear calculations. *J. Phys. Oceanogr.*, 25 (5): 735–746.
- Walsh, D., Richardson, P.L. and Lynch, J., 1995. Observations of tilting Meddies. *J. Phys. Oceanogr.*, submitted.
- Wu, C. and Emmanuel, K., 1993. Interaction of a baroclinic vortex with background shear: application to hurricane movement. *J. Atmos. Sci.*, 50: 62–76.