NUMERICALLY SIMULATING NON-GAUSSIAN SEA SURFACES

By Barry Vanhoff, Steve Elgar, and R. T. Guza

ABSTRACT: A technique to simulate non-Gaussian time series with a desired ("target") power spectrum and bispectrum is applied to ocean waves. The targets were obtained from observed bottom pressure fluctuations of shoaling, nonbreaking waves in 2–9 m water depth. The variance (i.e., frequency integrated spectrum), skewness, and asymmetry (i.e., frequency integrated bispectrum) of the simulated time series compare favorably with the observations, even for highly skewed and asymmetric near-breaking waves. The mean lengths of groups of high waves from non-Gaussian simulated time series are closer to observed values than those from Gaussian simulations. The simulations suggest that quadratic phase coupling between waves (of different frequencies) in shallow water results in longer wave groups than occur with linear, uncoupled waves having the identical power spectrum.

INTRODUCTION

In this study, a method for generating numerical realizations of quadratically nonlinear (e.g., non-Gaussian) time series with a specified ("target") power spectrum and bispectrum (Vanhoff and Elgar 1997) is applied to ocean surface waves. Low amplitude waves have linear dynamics and Gaussian statistics. Realizations of Gaussian sea-surface elevation time series with a specified power spectrum can be generated numerically by coupling Fourier amplitudes (determined by the spectrum) with independent, random phases (Rice 1954; Andrew and Borgman 1981). However, in intermediate-depth and shallow water (kh ≤ 1, where k is a representative wave number and h is the depth) quadratic interactions between even moderate energy Fourier components of the wave field result in non-Gaussian statistics that are described by the bispectrum (Hasselmann et al. 1963). The non-Gaussian properties of the sea surface are not reproduced by a linear combination of Fourier components with random phases, but can be approximated by numerical simulations that reproduce both the observed amplitudes (e.g., power spectrum) and the quadratic phase coupling between components (e.g., bispectrum).

Techniques to generate realizations of linear (Gaussian) and quadratically phase coupled (non-Gaussian) random processes are briefly reviewed in the next section. Target power spectra and bispectra are based on observations of nonbreaking ocean waves in 2–9 m water depth. Next it is shown that the variance, skewness, and asymmetry of numerically simulated sea-surface elevation time series compare favorably to the observed statistics even for nearly breaking waves. It is shown in the following section that observed mean lengths of groups of high waves (mean run length) in shallow water are predicted better by non-Gaussian than by Gaussian simulations, and that quadratic nonlinear interactions increase the mean run length. Conclusions follow.

NUMERICAL SIMULATION OF SEA-SURFACE ELEVATION TIME SERIES

A discretely sampled sea-surface elevation time series x(n) can be represented as (Rice 1954)

\[ x(n) = \sum_{k=1}^{K} C_k \cos(2\pi f_k n + \phi_k) \tag{1} \]

where 2K = number of samples; \( f_k = f_c/k \); and \( f_c \) = Nyquist frequency. The Fourier amplitude \( C_k \) at frequency \( f_k \) is

\[ C_k = \sqrt{2P(f_k)} \tag{2} \]

where \( P(f_k) \) = the power spectrum of \( x(n) \), defined by

\[ P(f) = E[|X(f)|^2] \tag{3} \]

where \( X(f_k) \) = discrete Fourier coefficients of \( x(n) \); and \( E[ ] \) = expected value, or average, operator. If the Fourier phases \( \phi_k \) are random and uniformly distributed on \([0, 2\pi)\) and \( K \gg 1 \), then \( x(n) \) has Gaussian statistics (Rice 1954). Realizations of a Gaussian sea surface with a specified power spectrum \( P(f) \) can be generated by coupling the amplitudes \( C_k \) with random phases \( \phi_k \) \([1]\).

An alternative representation to (1) is

\[ x(n) = \sum_{k=1}^{K} a_k \cos(2\pi f_k n) + b_k \sin(2\pi f_k n) \tag{4} \]

where \( a_k, b_k \) = independent Gaussian distributed random variables with zero mean and variance \( P(f_k) \). As \( K \rightarrow \infty \) statistics of sea surfaces generated with (1) are identical to those using (4) (Rice 1954). For power spectral shapes typically observed in the ocean, (1) and (4) produce sea surfaces with similar statistics for \( K \) as low as 32 (Elgar et al. 1985).

Nonrandom phase relationships between triads of Fourier components (with frequencies \( f_1, f_2, f_{12} \)) of a non-Gaussian random process characterized by quadratic nonlinearities are described statistically by the bispectrum \( B(f_1, f_2) \) (Hasselmann et al. 1963). For a discretely sampled process (Haurwitz 1965; Kim and Powers 1979)

\[ B(f_1, f_2) = E[X(f_1)X(f_2)X^*(f_1 + f_2)] \tag{5} \]

If the three Fourier components on the right-hand side of (5) are independent of each other (e.g., their phases are random as in a Gaussian process), \( B(f_1, f_2) = 0 \). Owing to symmetry relations, \( B(f_1, f_2) \) is completely defined by its values in a triangle in \((f_1, f_2)\)-space with vertices at \((0, 0), (f_2, f_0), (f_1, 0)\) (Hasselmann et al. 1963). See Nikias and Raghuveer (1987) and Elgar and Chandran (1993) for reviews of bispectra.

Time series with nonzero bispectra cannot be simulated accurately using (1) or (4). However, a quadratically phase coupled time series [with Fourier coefficients \( \hat{X}(f) \), and specified power spectrum and bispectrum] can be generated by passing a Gaussian process [produced by (1) or (4)] through the quadratic filter (Vanhoff and Elgar 1997).
where $G(f_i) = \mathcal{G}^*(-f_i)$ are Fourier coefficients of a Gaussian realization with target power spectrum $P(f_i)$ and $Q(f_i, f_m) = \text{the second-order Volterra kernel (Schenetz 1980), given in terms of } B(f_i, f_m) \text{ in (9)].}

The power spectrum \( [3] \) of a time series simulated using (6) is

$$
\hat{P}(f_i) = \mathbb{E}[|G(f_i)|^2] + \sum_{f_m \neq f_i} |Q(f_i, f_m - f_i)|^2 \mathbb{E}[|G(f_i)|^2] \mathbb{E}[|G(f_m - f_i)|^2] 
$$

$$
\hat{P}(f_i) = P(f_i) + \sum_{f_m \neq f_i} |Q(f_i, f_m - f_i)|^2 P(f_m)P(f_m - f_i); \quad -f_n \leq f_m \leq f_n
$$

Cross-product terms proportional to $\mathbb{E}[|G(f_i)|^2|G^*(f_i)G^*(f_m - f_i)]$ and its conjugate do not appear in (7) because these are identically zero for the Gaussian $G(f)$, By constraining $Q$ to have the same symmetry properties as the bispectrum, the simulated bispectrum [using (6) in (5)] becomes (Vanhoff and Elgar 1997)

$$
\hat{B}(f_i, f_m) = 2Q^*(f_i, f_m)P(f_i)P(f_m) + P(f_i)P(f_i + f_m)
$$

$$
+ 8 \sum_{f_n \neq f_i} Q(f_i, f_m - f_i)Q(-f_m, -f_n + f_m)
$$

$$
\times Q^*(f_i - f_m, f_n - f_m)P(f_i - f_m)P(f_i - f_m + f_n); \quad -f_n \leq f_m \leq f_n
$$

Equating the target bispectrum \( [5] \), to the simulated bispectrum \( \hat{B} \) [8]), assuming the last term in (8) is small (discussed in the following), and solving for $Q$ yields

$$
Q(f_i, f_m) = \frac{B^*(f_i, f_m)}{2[P(f_i)P(f_m) + P(f_i)P(f_i + f_m) + P(f_m)P(f_i + f_m)]}
$$

Summarizing, to generate a realization of a random process with power spectrum $P$ and bispectrum $B$, a Gaussian time series with power spectrum $P$ [generated using (1) or (4)] is passed through a quadratic filter [6]) specified by $P$ and $B$ [9])). By repeating the process with new sets of random phases, independent realizations of a non-Gaussian time series are produced.

The simulated power spectrum and bispectrum are not generally identical to their respective targets. For a time series with nonzero bispectrum, the terms within the summation in (7) are positive definite, and thus simulated power spectral levels $\hat{P}(f_i)$ exceed the target values $P(f_i)$. These errors can be eliminated by modifying (7), but the fidelity of the simulated bispectrum is degraded.

For a process with a single phase-coupled triad $(f_i, f_m, f_i + f_m)$, the error in $\hat{P}(f_i + f_m)$ depends on $|Q(f_i, f_m)|^2$ and the magnitude of $P(f_i)$ and $P(f_m)$. In the worst case of relatively low $P(f_i + f_m)$

$$
\hat{P}(f_i + f_m) \approx P(f_i + f_m) [1 + b^2(f_i, f_m)]
$$

where $b(f_i, f_m) = \text{biocoherence (Haubrich 1965, Kim and Powers 1979)}$

$$
b(f_i, f_m) = \frac{|B(f_i, f_m)|}{\sqrt{P(f_i)P(f_m)P(f_i + f_m)}}
$$

The biocoherence indicates the relative amount of quadratic phase coupling among the components of a triad. For no phase coupling, $b(f_i, f_m) = 0$ and $\hat{P}(f_i + f_m) = P(f_i + f_m)$. On the other hand, if all the energy at $f_i + f_m$ is phase coupled to that at $f_i$ and $f_m$, then $b(f_i, f_m) = 1$ and $\hat{P}(f_i + f_m) \approx 2P(f_i + f_m)$. In general, the error in the simulated power spectrum is a sum of errors from many partially phase-coupled triads (Vanhoff and Elgar 1997).

The error in the simulated bispectrum $\hat{B}(f_i, f_m)$, equal to the summation term on the right-hand side of (8), has contributions proportional to the sum of products of biocoherences of triads containing at least one of $f_i$, $f_m$, or $f_i + f_m$ (Vanhoff and Elgar 1997). For the case of a single phase-coupled triad, the error in $\hat{B}$ is identically zero. With many partially phase-coupled triads the error in the simulated bispectrum can be large or small, depending on whether contributions from different terms cancel. For ocean wave data considered below, the errors in $\hat{P}$ and $\hat{B}$ are not large.

**VERIFICATION OF SIMULATION METHODOLOGY**

To test the applicability of the methodology to nonlinear ocean waves, statistics of simulated sea surfaces were compared with those from observations made in shallow water (depths between 2 and 9 m) near Santa Barbara, Calif., and near Duck, N.C. Spectra and bispectra of the observed time series were estimated from $s = 192$ s records of bottom pressure (sampled at 2 Hz) converted to sea-surface elevation using linear finite depth theory. Each record was detrended to remove tidal effects, and ensemble averaging was used to form smoothed spectral estimates with 128 degrees of freedom and a frequency resolution of 0.0078 Hz. All time series considered (203 total) satisfy the condition that the linear energy flux (integrated over the frequency range 0.04–0.3 Hz) was within 15% of the flux concurrently measured in 8–9 m depth (i.e., breaking-induced dissipation was not significant). In the most shoreward records retained, waves were near breaking with steep forward faces and strong phase coupling between waves of different frequencies (Elgar and Guza 1985a,b; Elgar et al. 1997). Significant wave heights $H_{\text{sig}}$ (four times the sea-surface elevation standard deviation) ranged from 20 to 180 cm, the frequency corresponding to the centroid of the power spectrum was between 0.071 and 0.185 Hz, and spectral widths (a non-dimensional parameter related to the slope of the power spectrum (Longuet-Higgins 1975)) were between 0.037 and 0.264.

For each observed (target) power spectrum and bispectrum, 100 simulated time series were produced. The power spectra of the simulated time series are similar to the desired target (observed) values, with deviations mostly at high frequencies, as shown in Fig. 1(a) for a narrowband spectrum $H_{\text{sig}} = 1$ m; $h = 3.7$ m; $b(0.07, 0.07) = 0.8$; $b(0.07, 0.14) = 0.7$; and $NL = 0.70$, where $NL$ is defined in the following). Near the power-spectral primary peak frequency $f = 0.07$ Hz, Fig. 1(a) the simulated spectrum is nearly identical to the target spectrum. At frequencies corresponding to harmonics ($f = 0.14$ and $f = 0.21$ Hz) of the primary peak, the simulated spectral values are roughly 50% higher than observed, consistent with (10).

As expected, the maximum differences between simulated and target power spectra are smaller for a broad-band wave field with smaller maximum biocoherence $H_{\text{sig}} = 0.6$ m; $h = 1.5$ m; the maximum $b(f_i, f_m) = 0.4$; and $NL = 0.61$ [Fig. 2(a)]. The bulk nonlinearity of the wave field is quantified here by $NL = \sqrt{S^2 + A^2}$; where $S$ and $A$ are sea-surface elevation skewness and asymmetry, third moments defined respectively as the mean cube of the time series and the mean cube of the Hilbert transform (a 90° phase shift) of the time series, each normalized by the 3/2-power of the variance (Elgar and Guza 1985b; Elgar 1987). The errors in total variance of the simulated time series (i.e., the integral of the power spectrum over frequency or, equivalently, the second moment) relative to observed var-

---

**JOURNAL OF WATERWAY, PORT, COASTAL, AND OCEAN ENGINEERING / MARCH/APRIL 1997 / 69**
ues for all data sets are shown in Fig. 3. For small NL, simulated and observed total variance are nearly identical, and for the largest values of NL the difference is less than 20%.

Simulated [Figs. 1(d,e) and 2(d,e)] bispectra are similar to observed values (Figs. 1(b,c) and 2(b,c)). Third moments (the real and imaginary parts of the bispectrum integrated over bi-

**FIG. 3.** Error [(Simulated − Observed)/Observed] in Total Variance (Second Moment) versus Observed NL

**FIG. 4.** Error [(Simulated − Observed)/Observed] in Real and Imaginary Third Moment versus Observed NL

**FIG. 5.** Simulated versus Observed Skewness and Asymmetry
APPLICATION TO WAVE GROUPS

In deep and intermediate-depth water, the statistics of groups, or runs, of waves exceeding a particular height are usually consistent with a Gaussian sea surface and are simulated accurately with random phases ([1]) or random Fourier coefficients ([6]) (Andrew and Rongman 1981; Goda 1983; Elgar et al. 1984, 1985; Battjes and van Vledder 1984; Longuet-Higgins 1984; Thomas et al. 1986; Medina and Hudspeth 1990; Liu et al. 1993; and many others). However, in shallow water, deviations from a Gaussian sea surface are often significant and the mean length of groups (mean run length) of waves greater than the significant wave height increases, as shown in Fig. 6 (spectra and bispectra for this wave field in 3.7 m depth are shown in Fig. 1). The observed mean run lengths are larger than predicted by Gaussian simulations, and are more consistent with non-Gaussian simulations.

Mean run lengths from Gaussian and non-Gaussian simulations are compared to observed values for all data sets in Fig. 7. The scatter is large owing to the limited statistical stability of the individual observed run lengths (Elgar et al. 1984). For small nonlinearity NL both simulation methods predict accurately the observed mean run lengths (Fig. 8, where individual data points are accumulated into NL bins 0.2 wide and are plotted at the bin midpoint). As NL increases, errors in both simulations increase, but errors in the Gaussian simulated run lengths are twice as large as errors in the non-Gaussian simulations.

Mean run lengths are well known to increase with increasing spectral width (Goda 1983; and many others), but mean run lengths also increase with increasing nonlinearity, as shown in Fig. 9 (where individual data points are accumulated into NL bins 0.2 wide). After removing the effect of spectral width, the partial correlation coefficient (0.56) between the observed mean run lengths and NL is significant at the 99% level (Jenkins and Watts 1968).

To examine further the effect of nonlinearity on mean run length, non-Gaussian sea surfaces with identical spectral widths, but with different NL, were simulated. For a particular observed spectra and bispectra, the range of NL was obtained by scaling the target bispectrum by a factor 0 ≤ α ≤ 1.5. The simulation methodology was altered slightly to eliminate errors in the power spectrum owing to the second term on the right-hand side of (7), thus producing simulated time series with constant spectral width and variable NL. The Fourier coefficients of the simulated time series are given by

\[ X(f_s) = \hat{G}(f_s) + \sum_{f_{ws}=f_{wu}}^{f_w} \alpha Q(f_s, f_t - f_s) \hat{G}(f_t) \hat{G}(f_t - f_s) \]

where \( \hat{G}(f_s) \) = Fourier coefficients of the Gaussian simulation with power spectrum \( \hat{P}_g(f_s) \) chosen using nonlinear minimization (Gill and Murray 1978) such that the simulated power spectrum \( \hat{P}(f_s) \) [i.e., (13)] is identically equal to the target power spectrum \( P(f_s) \). Although errors in the simulated power spectrum (and in the total variance, Fig. 3) are eliminated, additional errors are introduced into the simulated bispectrum.
Simulations suggest that quadratic nonlinearity causes an observed shoreward increase in the mean run lengths of shoaling waves.

ACKNOWLEDGMENTS

This research was supported by the Office of Naval Research (Coastal Dynamics and AASERT graduate student support), the Naval Research Laboratory, and the National Science Foundation (CoOP program). The field data were obtained in collaboration with E. B. Thornton (Santa Barbara) and E. Gallagher, T. H. C. Herbers, and B. Raubenheimer (Duck).

APPENDIX. REFERENCES


