Chapter 13 Instability on non-parallel flow.

13.1 Introduction and formulation

We have concentrated our discussion on the instabilities of parallel, zonal flows. There is the largest amount of literature on that subject and, in addition to its obvious meteorological applicability, it is also true that many of the strongest ocean currents exhibit their major instabilities as they flow zonally, e.g. the Gulf Stream or Kuroshio. Nevertheless, there are major regions of the ocean where the flow is non-zonal, largely meridional and non-parallel. The nature of instabilities in such situations is less well understood but we can make certain a priori observations. Consider, for example, a meridional flow, independent of horizontal coordinate (or nearly so on the scale of a deformation radius) which has a vertical shear. Because of the β effect such a flow requires a driving mechanism, i.e. an input of potential vorticity such as a wind stress curl. Once that flow is set up we can see right away that it will be baroclinically unstable since a disturbance wave with crests running east-west will have fluid trajectories across the jet, capable of releasing the available potential energy while at the same time not feeling at all the stabilizing influence of planetary β since the motion is strictly east west. This simple example can be generalized to describe the instability of such a horizontally uniform baroclinic flow which is directed in any non-zonal direction. A disturbance with crests oriented east-west will always have some component of the velocity which crosses the basic horizontal density gradient while not sensing β. The details of such a calculation can be found in Chapter 7 of GFD.

The problem becomes more interesting (and complex) when the flow is not only non-zonal but not rectilinear. For example, suppose we have a large scale, steady, meandering current. Will such a flow be unstable? Are there a priori conditions to determine whether it can be? Is there a stability threshold that must be exceeded for the instability to be manifested? The answers to questions like these are not completely in hand but there are limited examples and theoretical ideas worth examining.

Consider first a quasi-geostrophic flow, which is a continuous function of z which has the property that its horizontal boundaries are isopycnal (or isentropic for the atmosphere). The generalization to include temperature or density variations on the boundary is straightforward and can also be found in chapter 7 of GFD.

Let us suppose that the basic state streamfunction, Ψ and potential q_0 satisfy,

$$q_o = Q(\Psi, z) \tag{13.1.1}$$

On each horizontal level the potential vorticity is constant on streamlines in the basic state so that $J(\Psi, q_o) = 0$. This can occur if the flow is unforced and inviscid so that this condition is, in fact, the equation of motion describing the flow. Or, as sometimes happens, the flow is forced and dissipative but the end state *resonates* with the steady state flow given by (13.1.1). The first example might be the Fofonoff solution for the barotropic ocean circulation, (figure 13.1.1) while in the second case, numerous numerical models with slip boundary conditions can end up in a state like (13.1.1) in a final steady state. Are such flows stable?

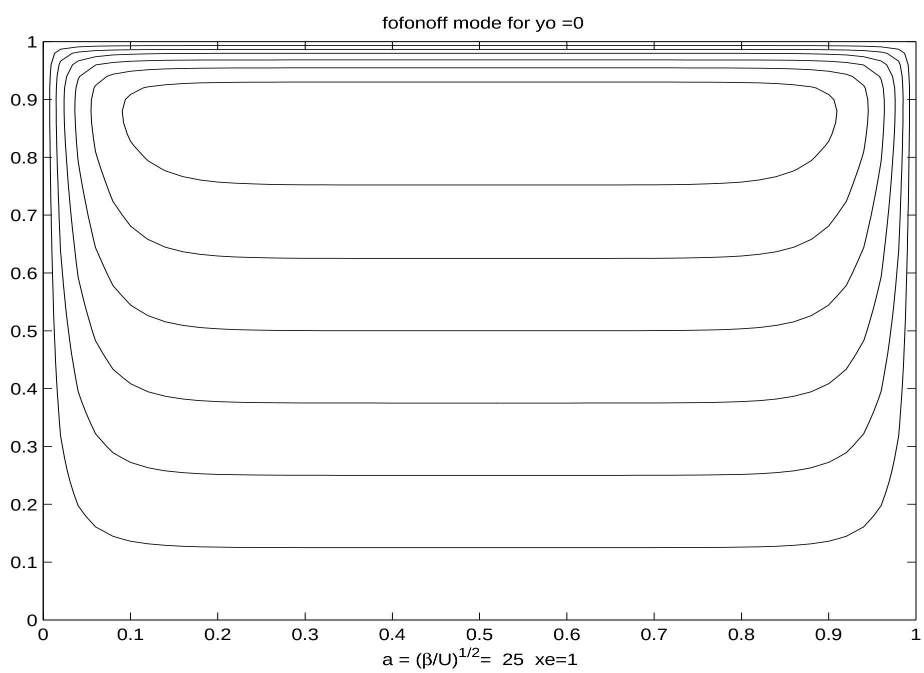

Figure 13.1.1 A Fofonoff mode .

The solution of figure 13.1.1 certainly contains strong shears in the boundary layers and one would be forgiven for believing that such a flow must support at least barotropic instability. However, an important theorem which we will not prove but only state (its proof is in Chapter 7 of GFD) says that a necessary condition for the instability for flows satisfying (13.1.1) is that

$$\frac{\partial q_o}{\partial \Psi} < 0. \tag{13.1.2}$$

somewhere in the flow. For the Fofonoff mode $q_o = a^2 \Psi, \qquad a^2 > 0$, so that the condition (13.1.2) is *not satisfied* so the flow, appearances to the contrary, must be stable. Note that for parallel zonal flows for which $\Psi = \Psi(y,z)$, we can write,

$$\frac{\partial q_o}{\partial \Psi} = \frac{\partial q_o / \partial y}{\partial \Psi / \partial y} = -\frac{\partial q_o / \partial y}{U_o} < 0 \tag{13.1.3}$$

or,

$$U_o \frac{\partial q_o}{\partial y} > 0 \tag{13.1.4}$$

This condition is equivalent to the Fjörtoft condition (4.3.15) we derived for zonal flows.

On the other hand let's consider the stationary Rossby wave. In the presence of a uniform zonal flow *U*. Steady solutions can be found in the form,

$$\Psi = -Uy + A\cos(kx + ly), \quad K^2 = k^2 + l^2 = \frac{\beta}{U} \tag{13.1.5 a,b}$$

Now for this barotropic Rossby wave the total potential vorticity is,

$$q_o = \nabla^2 \Psi + \beta y = -K^2 \left\{ A\cos(kx+ly) - \frac{\beta}{K^2} y \right\}$$

$$= -K^2 \Psi \tag{13.1.6}$$

such that $\partial q_o / \partial \Psi < 0$. The necessary condition for instability is satisfied no matter how weak the shears are in the wave i.e. no matter how small the wave amplitude, or the wave's vorticity gradient is compared to the planetary vorticity gradient. Is, in fact, the wave unstable for arbitrarily small amplitudes if (13.1.6) is true? That example is the one we shall examine more closely. To make our analysis more pertinent to real oceanographic situations we will examine the baroclinic version of the Rossby wave in the context of the two-layer model. Large scale Rossby waves are observed in the Pacific. Their scales are large compared to the deformation radius and they are baroclinic (1st baroclinic mode). Are they baroclinically unstable? Is there a stability threshold?

13.2 The two layer baroclinic model for the wave instability

If we return to the two-layer equations of chapter 10, in particular (10.1.14) and for the time being, ignore the effect of bottom topography and friction, we can rewrite the equations in terms of the baroclinic and barotropic stream functions, i.e. if we define,

$$\psi_b = h_1 \psi_1 + h_2 \psi_2,$$

$$\psi_T = \psi_1 - \psi_2, \qquad \text{(13.2.1 a,b,c)}$$

$$h_n = \frac{D_n}{D_1 + D_2}$$

then the two layer equations can be rewritten,

$$\frac{\partial}{\partial t}\nabla^2\psi_b + J(\psi_b,q_b) + \beta\psi_{b_x} + h_1 h_2 J(\psi_T,q_T) = 0,$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_T - F\psi_T\right] + \beta\psi_{T_x} + (h_1 - h_2)J(\psi_T,q_T) + J(\psi_b,q_T) + J(\psi_T,q_b) = 0, \qquad (13.2.3.\ a.b.c)$$

$$q_b = \nabla^2\psi_b, \qquad q_T = \nabla^2\psi_T - F\psi_T, \quad F = F_1 + F_2 = \frac{f_o{}^2 L^2 (D_1 + D_2)}{g' D_1 D_2}$$

Recall that in this nondimensional system time has been scaled with the advective time L/U while the parameter β is the ratio of the planetary vorticity gradient to the relative vorticity gradient, namely $\beta = \beta_{\text{dim}} L^2 / U$ where U is the scale of the horizontal velocity in the basic state. We are particularly interested in the system when, for the large scale Rossby wave, β is a <u>large</u> parameter. That is when we would expect the wave to be the most stable.

The solution for the basic wave baroclinic wave is,

$$\psi_T = \Psi_T = Ae^{i(kx+ly-\omega t)} + * \qquad (13.2.4\ a,b)$$

$$\omega = -\frac{\beta k}{k^2 + l^2 + F}$$

For large β this yields a rapid oscillation on the advective time scale used to obtain the nondimensional equations (13.2.3 a,b). This is the time scale on which the (nearly) linear Rossby waves will propagate. On the other hand, if the wave is unstable we anticipate its growth rate will be on the advective time scale. If β is large these two time scales are well separated and we should take advantage of that. We do so by explicitly introducing a new "fast" time,

$$t_* = \beta t \tag{13.2.5}$$

and we consider each variable a function of both t_* and t such that the local time derivative in (13.2.3) is transformed to ,

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial t_*} \tag{13.2.6}$$

Inserting this into the equations and dividing by β, we obtain ,

$$\left(\frac{\partial}{\partial t_*} + \frac{1}{\beta}\frac{\partial}{\partial t}\right)\nabla^2 \psi_b + \frac{1}{\beta} J(\psi_b, q_b) + \psi_{b_x} + \frac{1}{\beta} h_1 h_2 J(\psi_T, q_T) = 0,$$

$$\left(\frac{\partial}{\partial t_*} + \frac{1}{\beta}\frac{\partial}{\partial t}\right)\left[\nabla^2 \psi_T - F\psi_T\right] + \psi_{T_x} + (h_1 - h_2)\frac{1}{\beta} J(\psi_T, q_T) + \tag{13.2.7}$$

$$\frac{1}{\beta} J(\psi_b, q_T) + \frac{1}{\beta} J(\psi_T, q_b) = 0,$$

Note that if β were small the basic wave would just be a large scale flow of the cosine type Kuo examined (since β is zero there would be no difference between a zonal and meridional flow) and instability would be assured. That is why the large β case is of greatest conceptual interest.

We now expand each function in an asymptotic series in the small parameter β^{-1}, e.g.

$$\psi_b = \psi_b^{(o)} + \beta^{-1}\psi_b^{(1)} + O(\beta^{-2}) + \ldots$$

$$\psi_T = \psi_T^{(o)} + \beta^{-1}\psi_T^{(1)} + O(\beta^{-2}) + \ldots \tag{13.2.8}$$

At lowest order we obtain the linear Rossby wave equations on the fast time variable.

$$\frac{\partial}{\partial t_*}\nabla^2\psi_b{}^{(o)} + \frac{\partial \psi_b{}^{(o)}}{\partial x} = 0,$$

$$\frac{\partial}{\partial t_*}\left[\nabla^2\psi_T{}^{(o)} - F\psi_T{}^{(o)}\right] + \frac{\partial \psi_T{}^{(o)}}{\partial x} = 0 \qquad (13.2.9\text{ a,b})$$

As a solution, we will consider a triad of waves, and at this order they are independent linear solutions. For reasons that will be clear shortly the triad will be made up of 1) the baroclinic wave whose stability we are interested to investigate,2) a second baroclinic wave and 3) a barotropic wave. The need for the barotropic wave will also become clearer below but it is related to what we have discovered in the discussion of the Eady eigenfunctions, i.e. that there is a barotropic portion of the unstable wave amplitude. For our triad then,

$$\psi_b{}^{(o)} = A_1(t)e^{i(k_1x + l_1y - \omega_1 t_*)} + *$$

$$\psi_T{}^{(o)} = A(t)e^{i(kx - \omega t_*)} + * \qquad (13.2.10)$$

$$+A_o e^{i(k_o x + l_o y - \omega_o t_*)} + *$$

The amplitudes of each wave are as yet unknown functions of the slow advective time. We have taken the basic wave to be independent of y (although this is not essential) and each wave satisfies its appropriate dispersion relation for the frequency on the fast time scale, i.e.

$$\omega_1 = -\frac{k_1}{k_1^2 + l_1^2}, \quad (barotropic)$$

$$\omega = -\frac{k}{k^2 + F}, \qquad (baroclinic \;\; basic \;\; wave) \qquad (13.2.11\text{ a,b,c})$$

$$\omega_o = -\frac{k_o}{k_o^2 + l_o^2 + F} \quad (baroclinic \;\; wave)$$

At the next order in the expansion,

$$\frac{\partial}{\partial t_*}\nabla^2\psi_b^{(1)} + \frac{\partial \psi_b^{(1)}}{\partial x} = -\frac{\partial}{\partial t}\nabla^2\psi_b^{(o)} - J(\psi_b^{(o)}, q_b^{(o)}) - h_1 h_2 J(\psi_T^{(o)}, q_T^{(o)}),$$

$$(13.2.12\text{ a,b})$$

$$\frac{\partial}{\partial t_*}\left[\nabla^2\psi_T^{(1)} - F\psi_T^{(1)}\right] + \frac{\partial \psi_T^{(1)}}{\partial x} = -\frac{\partial}{\partial t}\left[\nabla^2\psi_T^{(o)} - F\psi_T^{(o)}\right]$$

$$-J(\psi_b^{(o)}, q_T^{(o)}) - J(\psi_T^{(o)}, q_b^{(o)}) - (h_1 - h_2)J(\psi_T^{(o)}, q_T^{(o)})$$

So, at this order the small correction to the streamfunction is forced by the nonlinear quadratic interactions of the three waves in the triad. The interaction of any two of the waves will yield a wave with the wavenumber and frequency of the sum and difference of the two interacting waves. That is to say suppose the basic wave and the other baroclinic wave interact via the nonlinearity of the last term in (13.2.12a). That quadratic interaction will yield a forcing term on the right hands side that has the form of a plane wave with wavenumber and frequency, $k_o + k, \quad l_o, \quad \omega_o + \omega$.

The first term on the right hand side of (13.2.12 a,b) has the wavenumber and frequency of the operator on the left hand side and if it were the only such term on the right hand side it would produce a resonance with that operator and a solution growing like t_* unless the forcing term is zero, that is, unless,

$$\frac{dA_1}{dt} = 0 \tag{13.2.13}$$

The same would clearly hold true for the other two amplitudes as well. The waves would not change their amplitude on the advective time scale, i.e. the basic wave would exhibit no instability and the only effect of the other nonlinear forcing would be to produce a small, $O(\beta^{-1})$ correction to the wave field. Not of much interest. If, however, it should happen that, in (13.2.12 a) the two baroclinic waves should interact such that the sum♦ of their wavenumbers was equal to the wavenumber of the barotropic wave and the sum of their frequencies was equal to the frequency of the barotropic wave's frequency, then the forcing on the right hand side would produce a resonance with the left hand side and a linear growth on the fast time scale for the solution of the $\psi_b^{(1)}$ solution. This would render the expansion invalid on when $\beta^{-1} t_* = t = O(1)$, i.e. on the time scale for which we expect to see an instability. To preserve our expansion for at least that long we balance those secular nonlinear terms with the time derivative term on the right hand side. This will yield an evolution equation for $A_1(t)$ on the advective time scale. Similarly, the interaction between the barotropic mode and either of the two baroclinic modes will yield a resonance in (13.2.12b) unless a similar balance is struck.

Thus, *if* the three wavenumbers and frequencies of the quasi-linear waves satisfy the following *resonance conditions* the waves will form a *resonant triad*. A bit of algebra in which the resonant terms are thus balanced and removed from the forcing leads to the following amplitude equations,

$$\left(K^2 + F\right)\frac{dA}{dt} = -A_1 A_o^* \left[\vec{K}_1 \times \vec{K}_o\right]\left[K_o^2 + F - K_1^2\right]$$

$$\left(K_o^2 + F\right)\frac{dA_o}{dt} = -A_1 A^* \left[\vec{K}_o \times \vec{K}_1\right]\left[K^2 + F - K_1^2\right] \tag{13.2.14 a,b,c}$$

$$\left(K_1^2\right)\frac{dA_1}{dt} = h_1 h_2 A_o A \left[\vec{K}_1 \times \vec{K}_o\right]\left[K_o^2 - K^2\right]$$

♦ With the proper attention to signs this could be the sum or difference of the wavenumbers and frequencies. There is no change in the final result.

Here, $\vec{K}, \vec{K}_o, \vec{K}_1$ are the wave vectors for the basic wave, the second baroclinic wave and the barotropic wave respectively, the same symbols without the arrows are the magnitude of the vectors and

$$\vec{K}_1 = \vec{K}_o + \vec{K} \qquad (13.2.15)$$

as shown in the figure 13.2.1.

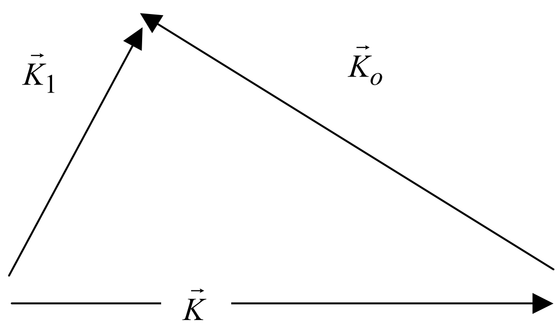

Figure 13.2.1 The resonant triad of two baroclinic waves and a barotropic wave.

We have also used the following elementary result $\vec{K}_1 \times \vec{K}_o = \vec{K} \times \vec{K}_o$.

Before discussing the solutions of the triad equations let's pause to consider the resonance conditions in more detail. We have to be sure such resonant triads can actually exist. The resonant conditions are three algebraic equations, i.e.

$$
\begin{aligned}
&k + k_o = k_1, \\
&l + l_o = l_1, \qquad (13.2.16\ \mathrm{a,b,c}) \\
&\frac{k}{k^2 + l^2 + F} + \frac{k_o}{k_o{}^2 + l_o{}^2 + F} = \frac{k_1}{k_1{}^2 + l_1{}^2}
\end{aligned}
$$

The first two equations just algebraically state the condition shown in figure 13.2.1. The third equation is the condition that the frequencies of the two baroclinic waves add to yield the barotropic frequency. Note that the wavenumber components can be negative. In the case of greatest interest we will consider the basic wave to be propagating in a purely westward direction so that *l* is zero. This implies that $l_1 = l_o$. For a given value of *k* can find k_1 in terms of *k* and k_o from (13.2.16a) . Then, (13.2.16c) becomes an algebraic equation that yields l_o in terms of k_o. We therefore, for a given *k* get a family of possible triads and for each member of that family we can test the stability of the basic wave to the disturbance provided by the other two components of the triad. Note that it is a simple matter to prove, using the triad equations and the triad resonance conditions that,

$$\left[K^2 + F\right]|A|^2 + \left[K_o{}^2 + F\right]|A_o|^2 + \left[K^2\right]|A_1|^2 = E = const. \qquad (13.2.17)$$

so that the total energy in the triad is conserved on the long time scale. If energy is released to the perturbations consisting of the other two members of the triad it must come from the energy stored in the original basic wave.

To examine the possible instability of the basic wave we can write the wave amplitudes of the triad as the basic wave plus an amplitude perturbation, namely,

$$A = \overline{A} + a,$$

$$A_o = \quad a_o, \qquad (13.2.18\ a,b,c)$$

$$A_1 = \quad a_1 .$$

where the small *a*'s are in fact small perturbations of the basic wave amplitude $A = \overline{A}$. When this is inserted into (13.2.14), the first equation has a right hand side that is $O(a^2)$ while linearization of the right hand side of the other two equations leads to the <u>linear</u> system,

$$\left(K^2+F\right)\frac{da}{dt}=-a_1 a_o^*\left[\vec{K}_1\times\vec{K}_o\right]\left[K_o^2+F-K_1^2\right]=O(a^2)$$

$$\left(K_o^2+F\right)\frac{da_o}{dt}=a_1\overline{A}^*\left[\vec{K}_o\times\vec{K}_1\right]\left[K^2+F-K_1^2\right] \qquad (13.2.19a,b,c)$$

$$\left(K_1^2\right)\frac{da_1}{dt}=-h_1 h_2 a_o\overline{A}\left[\vec{K}_1\times\vec{K}_o\right]\left[K_o^2-K^2\right]$$

Working with the last two equations yields ,

$$\left[K_o^2+F\right]a_{o_{tt}}=a_o h_1 h_2\left|\overline{A}\right|^2(\vec{K}_1\times\vec{K}_o)^2\left[K_o^2-K^2\right]\left[K^2+F-K_1^2\right]/K_1^2 \qquad (13.2.20)$$

Solutions of the form $a_o=a_o(0)e^{\lambda t}$ yield the growth rate , λ, i.e.

$$\lambda^2=\frac{h_1 h_2\left|\overline{A}\right|^2(\vec{K}_1\times\vec{K}_o)^2\left[K_o^2-K^2\right]\left[K^2+F-K_1^2\right]}{K_1^2\left[K_o^2+F\right]} \qquad (13.2.21)$$

A positive value of the right hand side of (13.2.21), i.e. a real growth rate and an instability requires that,

$$K_o^2+F>K^2+F>K_1^2 \qquad (13.2.22)$$

Which can be interpreted as saying that for the basic wave to be unstable to two other members of the triad its *total* wave number, including the two-layer equivalent of the vertical wavenumber, must be intermediate to the wavenumbers of the other two members of the triad. It must lose energy to both larger and smaller scales.

We have been talking about the triad and the triad instability in the subjunctive. Can we in fact find unstable triads that satisfy the resonance conditions (13.2.16 a,b,c) *and* yield real growth rates. Recall that we are particularly interested in cases where F is large, i.e. when the wavelength of the basic wave is large compared to a deformation radius. Figure 13.2.2 shows the value of l_o which completes the resonant triad as a function of k_o for a given value of k (solid line). The dashed line shows the growth rate. The meridional wavenumber has been scaled with $F^{1/2}$ i.e. with the deformation radius. For large F the meridional wavenumber which completes the triad is large compared to the O(1) x-wavenumbers. The scale of the disturbance in y is much smaller than in x and this we anticipated from the Eady or two layer version of the problem if β is rendered weak by the wave orientation. With a large y wavenumber and a small x wavenumber the perturbation acts very much like a f-plane perturbation on a meridional shear flow with no beta effect. Indeed for very large F it is easy to show that with the y wavenumber of the order of the deformation radius and the x wavenumbers of O(1) the equation for the growth rate simplifies t

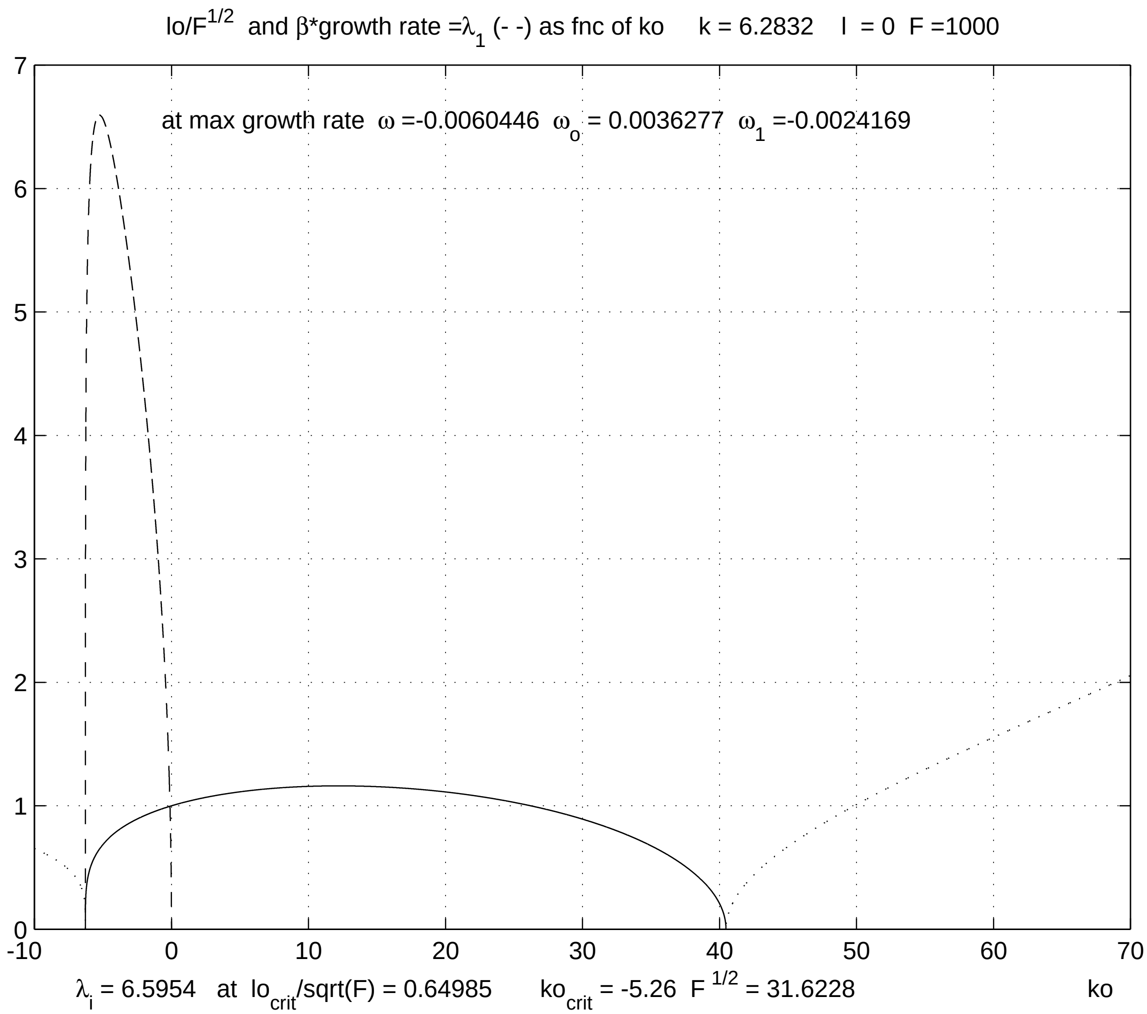

Figure 13.2.2 The meridional wavenumber and the growth rate (dashed curve).

Indeed, for large F a very simple approximation to the growth rate yields,

$$\lambda / F^{1/2} = h_1 h_2 \tilde{l}_o V \left\{ \frac{\left[1 - \tilde{l}_o{}^2\right]}{\left[1 + \tilde{l}_o{}^2\right]} \right\}^{1/2} \tag{13.2.23}$$

where $\tilde{l}_o = l_o / F^{1/2}$, $\qquad V = k\overline{A}$.

In this form the growth rate is reminiscent of the growth rate for the Eady problem for an f-plane flow of magnitude V. It is not exactly the same except in the case when the two layer thicknesses are equal. The third term on the right hand side of (13.2.12b) will obviously not lead to a contribution to a resonant term while it does enter the Eady problem. Nevertheless, the result (13.2.23) shares some similar properties. The short wave cut-off is of the order of the deformation radius and the growth rate is of the order of $VF^{1/2}$ or, in dimensional units, V_{dim}/L_d. As a function of y wavenumber the maximum growth rate occurs for $\tilde{l}_o = 0.64359$, very close to the maximum value for the Eady problem. Corresponding to that meridional wavenumber $k_o = -0.8284k$ while $k_1 = 0.1716\ \ k$ so that the barotropic mode is very nearly x-independent (remember the y wavenumber is $O(F^{1/2})$.

It is important to keep in mind the remarkable feature of the result, namely that the flow in the basic wave is unstable to the perturbations represented by the two remaining members of the triad no matter how large the parameter β is, or equivalently, no matter how weak the vertical and horizontal shear is in the wave. The further nonlinear evolution of the wave amplitudes can be calculated by direct integration of the amplitude equations (13.2.14)

Figure 13.2.3 shows the amplitude evolution for the most unstable triad for the same parameter values as in figure 13.2.2.

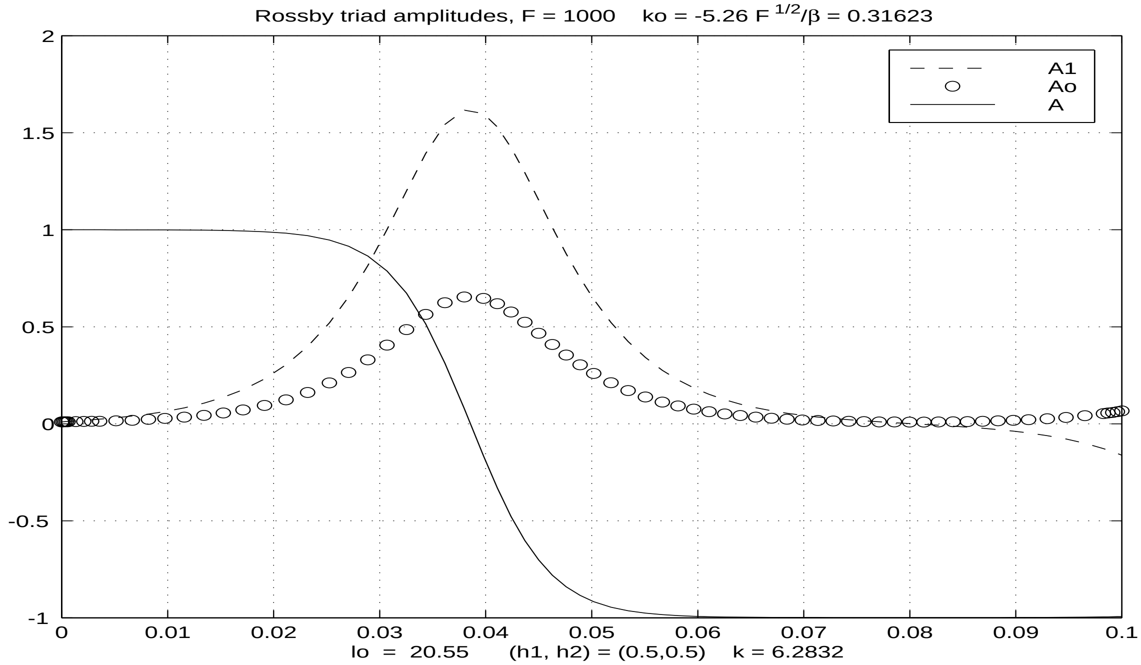

Figure 13.2.3 Amplitude evolution of the wave triad.

We see that initially both of the perturbation waves grow exponentially but then saturate. The amplitude of the original basic wave is nearly constant during the period of initial exponential growth in accordance with (13.2.19a) but when the parasitic waves reach a large amplitude the amplitude of the basic wave diminishes as energy is passed from the large scale wave to the perturbations (whose horizontal wavenumber puts their scale at the deformation radius). Since the amplitude equations are reversible with time in the absence of dissipation, the solutions are periodic in time and, in principle, will execute an endless repetitive nonlinear oscillation as shown in Figure 13.2.4.

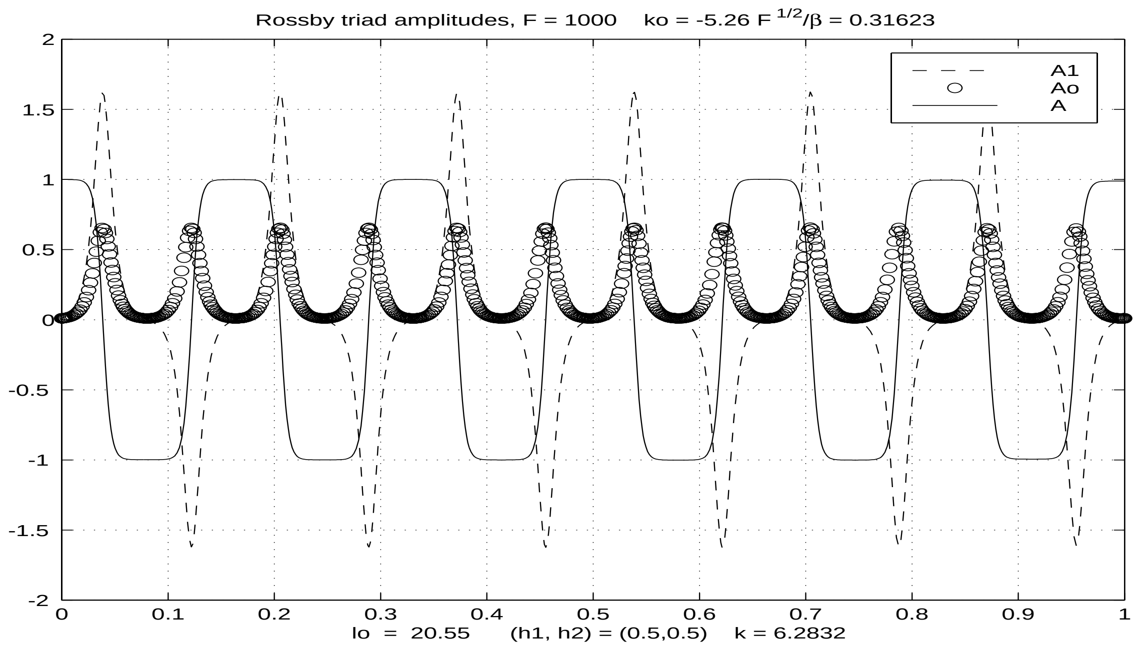

Figure 13.2.4 The periodic, nonlinear oscillation which continues the solution of Figure 13.2.3.

In fact, once the amplitudes of the other two waves become large the triad will be able to share energy with other triads that can be linked to any one of the members of the original triad and a turbulent cascade to fill out the wavenumber spectrum begins.

13.3 General considerations of Enstrophy.

In addition to the energy constraint (13.2.17) for the triad it is possible to also show that the enstrophy of the triad is conserved, that is,

$$\left(K^2+F\right)^2|A|^2+\left(K_o{}^2+F\right)^2|A_o|^2+K_1{}^4|A_1|^2=V=const. \qquad (13.3.1)$$

which is the triad version of the conservation of mean squared perturbation potential vorticity or *enstrophy*. If we define the energy of each component of the triad as,

$$E = (K^2 + F)|A|^2,$$

$$E_o = (K_o^2 + F)|A_o|^2, \qquad \text{(13.3.2 a,b,.c)}$$

$$E_1 = (K^2)|A_1|^2$$

then the combination of t energy and enstrophy conservation yields,

$$(K^2 + F)\dot{E} + (K_o^2 + F)\dot{E}_o + (K_1^2)\dot{E}_1 = 0,$$

$$\dot{E} + \dot{E}_o + \dot{E}_1 = 0 \qquad \text{(13.3.3 a,b)}$$

where an over-dot stands for a time derivative. If we define the total wavenumber as,

$$\kappa_j^2 = K_j^2 + pF \qquad \text{(13.3.4)}$$

where j is an index pertaining to any of the three waves and p =0 for a barotropic wave and p = 1 for the baroclinic wave in the two-layer model then (13.3.3 a,b) is just

$$(\kappa^2)\dot{E} + (\kappa_o^2)\dot{E}_o + (\kappa_1^2)\dot{E}_1 = 0,$$

$$\dot{E} + \dot{E}_o + \dot{E}_1 = 0 \qquad \text{(13.3.5 a,b)}$$

From these two equations it is easy to show that,

$$\frac{\dot{E}}{(\kappa_1^2 - \kappa_o^2)} = \frac{\dot{E}_o}{(\kappa^2 - \kappa_1^2)} = \frac{\dot{E}_1}{(\kappa_o^2 - \kappa^2)} \qquad \text{(13.3.6)}$$

Since, in our case $\kappa_o{}^2 > \kappa^2 > \kappa_1{}^2$ it follows that energy must go to scales larger and smaller than the basic wave in order to conserve energy and enstrophy in the triad and this is consistent with the results of our instability calculation.