

Chapter 8: The initial value problem and non modal instability

8.1 Introduction

One of the obvious features of the Eady normal mode problem is that for each wavenumber k , there are only two modes in z . For example, for k in the range of unstable modes the two modes are complex conjugates of each other, one for the growing solution and one for the decaying solution. These are clearly not sufficient to represent arbitrary initial conditions whose z structure need not be a linear combination of those two modes. More to the point, the modes have zero potential vorticity and there is no reason why an arbitrary initial condition needs to have zero potential vorticity. Our governing equation (6.2.7) is equivalent to the statement that,

$$\left[\frac{\partial}{\partial t} + U_o \frac{\partial}{\partial x} \right] q = 0 \quad (8.1.1)$$

Now this equation only implies that an initial pv anomaly is advected with the basic flow at that level, not that q need be zero. The normal modes correspond to that choice but as described in chapter 6 there are singular solutions with delta function singularities where $U_o(z) - c = 0$ that we have ignored. The question is how do such solutions complete the discrete, nonsingular normal modes and how could we use them for an initial value problem? Furthermore, can we be sure those singular modes are stable? We shall see that the last question has a somewhat ambiguous answer.

For the Eady problem it is useful to introduce the following nondimensionalization:

$$\begin{aligned}
z &= Dz', \\
x &= Lx' \\
t &= \left[L / (DU_{0z}) t' \right]
\end{aligned}
\tag{8.1.2}$$

Then, in terms of the non dimensional variables the Eady problem is,

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + z \frac{\partial}{\partial x} \right) \left[\phi_{xx} + \frac{1}{S} \phi_{zz} \right] &= 0, \\
\left(\frac{\partial}{\partial t} + z \frac{\partial}{\partial x} \right) \phi_z - \phi_x &= 0, \quad z = 0, 1
\end{aligned}
\tag{8.1.3 a,b,c}$$

where we have dropped the primes on the dimensionless variables and where the parameter

$$S = \frac{N^2 D^2}{f_o^2 L^2}.$$

For simplicity we will look at the problem in the infinite y-domain and

consider solutions independent of y. At time $t=0$, initial data must be specified and we will shortly see what data is needed to completely determine the problem. If we were to look for solutions which were wavelike in x and exponential in t we would recover the problem of chapter 6 which gave us the Eady normal modes. Now we are looking for the full solution.

8.2 The Laplace transform approach

A direct way to approach the problem is to introduce the Laplace transform

$$\tilde{\phi}(x, p) = \int_0^{\infty} e^{-pt} \phi(x, t) dt \tag{8.2.1}$$

Applying the Laplace transform to (8.1.3) yields,

$$\left(p + z \frac{\partial}{\partial x}\right) \left[\tilde{\phi}_{xx} + \frac{1}{S} \tilde{\phi}_{zz} \right] = q_o(x, z, 0) \quad (8.2.2 \text{ a,b,c})$$

$$\left(p + z \frac{\partial}{\partial x}\right) \tilde{\phi}_z - \tilde{\phi}_x = \phi_z(x, z, 0) \equiv \vartheta_o(x, z), \quad z = 0, 1$$

so that the required initial data at $t=0$ are the perturbation potential vorticity within the fluid and the buoyancy perturbation on the boundary at the initial instant. To proceed, we could do a Fourier analysis in x in the most general case but for our purposes it is sufficient to examine the problem for each x wavenumber separately. Writing,

$$[\tilde{\phi}, q_o, \vartheta_o] = [F, Q_o, \Theta_o] e^{ikx} \quad (8.2.3)$$

where the real part is implied, we obtain as our final set of equations,

$$(p + izk) [F_{zz} - \kappa^2 F] = S Q_o, \quad (8.2.4)$$

$$(p + izk) F_z - ikF = \Theta_o, \quad z = 0, 1$$

where $\kappa = kS^{1/2}$.

Since Q_o and Θ_o are arbitrary, it turns out that without loss of generality for the point we wish to make we may take $\Theta_o = 0$ on both boundaries. Dividing through the first equation by its leading factor,

$$F_{zz} - \kappa^2 F = \frac{S Q_o}{ik(z - ip/k)} \quad (8.2.5)$$

There is a singularity on the right hand side which corresponds to all points in z where $z = -ip/k$ or, equivalently, where $p = -izk$. Those of you adept at Laplace transforms will recognize that the singularity corresponds to frequencies such that $\omega = kz$, or where the perturbation has a phase speed equal to the basic flow at that level. These singularities are the ones ignored when we search for regular normal modes and the singularity on the

right hand side of (8.2.5) will restore those singular modes to the problem and the Laplace transform will show us how to compose that infinite number of continuous modes (since z is a continuum between $z=0$ and $z=1$).

To solve (8.2.5) for general Q_0 we introduce the Greens function, $G(z, z')$

$$G_{zz} - \kappa^2 G = \delta(z - z'), \quad (8.2.6 \text{ a,b,c})$$

$$(p + izk)G_z - ikG = 0, \quad z = 0, 1$$

where the right hand side of (8.2.6a) is the dirac delta function centered at z' . By multiplying (8.2.6) by F and (8.2.5) by G and using the boundary conditions we easily obtain,

$$F(z) = \int_0^1 \frac{SQ_0(z')G(z, z')}{(p + iz'k)} dz' \quad (8.2.7)$$

The standard way to find the Greens function for (8.2.6) is to find a solution that satisfies the boundary condition at $z=0$, call it aU_1 and another solution which satisfies the boundary condition at $z=1$, call it bU_2 . The first solution will be valid in the interval $z \leq z'$ while the second will be valid in $z \geq z'$ where z' is a point in $(0, 1)$. It is easy to show that since for all points in z not equal to z' , the right hand side of (8.2.6a) is zero,

$$G = aU_1 = a \left\{ \cosh \kappa z + \frac{ik}{p\kappa} \sinh \kappa z \right\}, \quad z \leq z', \quad (8.2.8a, b)$$

$$G = bU_2 = b \left\{ \cosh \kappa(z-1) + \frac{ik}{(p+ik)\kappa} \sinh \kappa(z-1) \right\}, \quad z \geq z'$$

To determine the constants a and b we use two matching conditions, easily derived from the equation (8.2.6) by integrating in a small neighborhood including the point $z=z'$.

Namely, that G is continuous across the point z' while there is a jump in the first derivative such that

$$\begin{aligned} aU_1(z') - bU_2(z') &= 0 \\ aU_1'(z') - bU_2'(z') &= -1 \end{aligned} \quad (8.2.9 \text{ a,b})$$

The solution for a and b yields the final expression for G ,

$$G(z, z') = \frac{U_1(z_{<})U_2(z_{>})}{W(U_1, U_2)} \quad (8.2.10)$$

where $z_{<}$ is the lesser of z and z' while $z_{>}$ is the greater of the two of them. The denominator of (8.2.10) is the Wronskian of the two independent solutions of the homogeneous problem U_1 and U_2 , i.e.

$$W(U_1, U_2) = U_1 U_2' - U_1' U_2 \quad (8.2.11)$$

It follows from the homogeneous part of (8.2.6) that the Wronskian in this case is independent of z , i.e. it can be evaluated at any convenient point. The prescient student may make the connection with the independence of the buoyancy flux with z in the normal mode problem and the constancy of the Wronskian. If W is evaluated at $z=0$, it is easy to show that,

$$\begin{aligned} W &= -\frac{\kappa \sinh \kappa}{p(p + ik)}(p - p_1)(p - p_2), \\ p_1 &= -ik/2 + \frac{k}{2} \left[\left(\frac{\kappa}{2} - \tanh \frac{\kappa}{2} \right) \left(\coth \frac{\kappa}{2} - \frac{\kappa}{2} \right) \right]^{1/2}, \\ p_2 &= -ik/2 - \frac{k}{2} \left[\left(\frac{\kappa}{2} - \tanh \frac{\kappa}{2} \right) \left(\coth \frac{\kappa}{2} - \frac{\kappa}{2} \right) \right]^{1/2} \end{aligned} \quad (8.2.12)$$

For the range of wave number for which the Eady mode is unstable, the two roots of the Wronskian will correspond to poles in the Laplace transform plan. The first, at p_1 will give a time dependence of the unstable Eady mode while the second, p_2 , will give the time dependence of the stable Eady mode as we shall shortly see. Putting our results for the Greens function together,

$$G = \frac{-p(p+ik)}{\kappa \sinh \kappa(p-p_1)(p-p_2)} [U_1(z_<)U_2(z_>)] \quad (8.2.13)$$

Thus, the solution for $F(z)$ is given as,

$$F(z,p) = - \int_0^1 \frac{SQ_o(z')p(p+ik)U_1(z_<)U_2(z_>)}{\kappa \sinh \kappa(p-p_1)(p-p_2)(p+iz'k)} dz' \quad (8.2.14)$$

This gives us the solution in the Laplace transform space. To go back to the time domain and the perturbation stream function $\phi(x,z,t)$ we have to take the inverse Laplace transform of F . Note that F has simple *poles* in the p -plane. There is a pole at each of the Eady normal mode roots, p_1 and p_2 there is also a pole at $p = -izk$ corresponding to a frequency of oscillation $\omega = kU_o(z) = kz$ in our dimensionless units. The third pole corresponds to a continuum of roots since z is distributed continuously between $z=0$ and $z=1$.

Figure 8.2.1 shows the position of the poles and the path of the contour integral for the Laplace inversion.

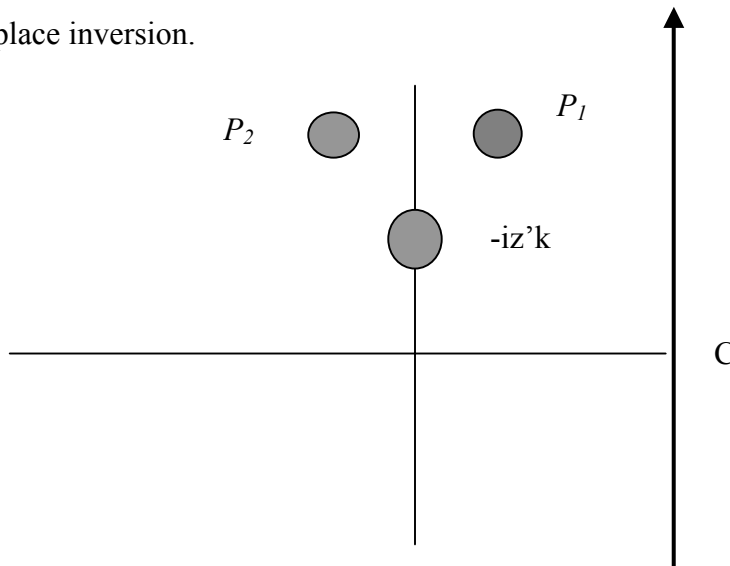


Figure 8.2.1 The Laplace inversion contour , C, to the right of the three poles of F .

To find the function ϕ we have to invert the transform by using,

$$\phi = \frac{1}{2\pi i} \int_C F(z, p) e^{pt} e^{ikx} dp \quad (8.2.15)$$

where the contour C runs from $-i\infty$ to $i\infty$ on a line to the right of all the singularities of F . For $t > 0$ one closes the contour in the right half plane picking up the contributions from the three poles. Note that the third pole, $p = -izk$ comes from the equation

$$(p + izk)q = 0 \text{ so that either } q=0 \text{ or } q \text{ is a delta function,}$$

$$q = B\delta(z - p/ik)$$

In the original system in the time domain, (8.1.1) the general solution of which is

$$q = Q_0(x - zt, z) \quad (8.2.16)$$

a special case of which is the delta function $\delta(x - zt, z)$. These solutions correspond to a delta function in the potential vorticity with amplitude B, each one limited to a layer a $z=c$ where c lies in the range of U here between 0 and 1. What the application of the Laplace transform and the Greens function will do for us is to show us automatically the projection of the initial conditions on the three parts of the solution, the two Eady modes and the continuous spectrum corresponding to the delta function potential vorticity modes which together allow us to represent an arbitrary initial condition. Carrying out the Laplace integral finally yields,

$$\begin{aligned}
\phi = & -\int_0^1 dz' \frac{SQ_0(z')}{\kappa \sinh \kappa} \left\{ \frac{p_1(p_1 + ik)U_1(z_{<}, p_1)U_2(z_{>}, p_1)}{(p_1 - p_2)(p_1 + iz'k)} \right\} e^{p_1 t} e^{ikx} \\
& -\int_0^1 dz' \frac{SQ_0(z')}{\kappa \sinh \kappa} \left\{ \frac{p_2(p_2 + ik)U_1(z_{<}, p_2)U_2(z_{>}, p_2)}{(p_2 - p_1)(p_2 + iz'k)} \right\} e^{p_2 t} e^{ikx} \\
& -\int_0^1 dz' \frac{SQ_0(z')}{\kappa \sinh \kappa} \left\{ \frac{z'(z' + ik)U_1(z_{<}, -ikz')U_2(z_{>}, -ikz')}{(p_1 + iz'k)(p_2 + iz'k)} \right\} e^{-iz'kt} e^{ikx}
\end{aligned} \tag{8.2.17}$$

The first two integrals in (8.2.17) have the time dependence of the Eady normal modes and if k is in the range in which the modes are unstable the exponential factors in (8.2.17) are precisely the same as the complex frequencies of the normal mode problem. In addition, although it requires some algebra, it is not too hard to show that the terms in (8.2.17) of the form $U_1(z_{<}, p_n)U_2(z_{>}, p_n)$ are proportional to the Eady normal modes corresponding to the n^{th} root ($n=1$ or 2). The virtue of solving the initial value problem is that (8.2.17) explicitly shows how the initial condition of the potential vorticity projects onto each normal mode. For details see (Pedlosky, 1964, *Tellus*, XVI, 12-17).

The third integral picks up the remainder of the initial condition and is an integral over the *continuous* spectrum, each component of which is a delta function of potential vorticity being advected by the basic flow, $U(z)$ at the level z . In this case of course, $U(z)=z$. The frequency of oscillation of each component of the continuous spectrum is just ikz and the integral in (8.2.17) shows how all the modes are added together to satisfy the initial data.

For each k such that the real part of p_n is different from zero, standard asymptotic estimates will show that this part of the solution will algebraically decay, at least as fast as $1/t$ for large t . That is, the continuous spectrum is *asymptotically* stable. The interference between neighboring sheets of potential vorticity, each oscillating with a slightly different frequency, will eventually lead to a decay of this part of the perturbation solution leaving the normal modes as the dominant signal for large time. Hence, although the normal modes are not complete they give an adequate *asymptotic* picture of the perturbation behavior. This does not answer the question whether the continuous spectrum might yield significant growth temporarily even if eventually it decays. This

possibility will be examined in the next section. Note that in general, for a completely arbitrary initial condition (8.2.17) forms the solution for a single x wavenumber and the total solution would require a further Fourier integral over k .

8.3 Non normal mode instability.

The issue of the possible temporary growth of the continuous spectrum is a complex one and one to which much work has recently been addressed. We will examine only a simple example here and describe an additional model problem that is illustrative of the basic mathematical character of the problem. For more detail see the work of Farrell and his co-workers, e.g. (Farrell and Ioannou, 1996, J.Atmos. Sci.,53, 2025-2040).

Let us return to the basic Eady model,(8.2.1) and examine the case where the horizontal boundaries are each removed to infinity. This is equivalent to examining an initial wave packet distant from either of the boundaries. As we have described earlier in chapters 5 and 6 without the presence of horizontal boundaries the Eady problem, lacking an internal potential vorticity gradient, would have no unstable normal modes. In such a case any kind of growth would have to come from components of the continuous spectrum. If the lateral boundaries are removed to plus and minus infinity and the shear is linear there is no obvious vertical scale in the problem. The horizontal scale L can be chosen as the wavelength of the perturbation. Absent a geometrical scale, the only reasonable choice of vertical scale is,

$$D = f_o L / N \Rightarrow S = 1 \quad (8.3.1)$$

leading to the governing equation for potential vorticity,

$$\left(\frac{\partial}{\partial t} + z \frac{\partial}{\partial x} \right) q = 0, \quad (8.3.2 \text{ a,b})$$

$$q = \phi_{xx} + \phi_{zz}$$

where for simplicity we have again ignored the y -dependence of the perturbation, i.e. we have chosen the region to be infinite also in the y -direction and let $l = 0$.

The solution for (8.3.2a) is simply,

$$q = Q(x - zt, z) \quad (8.3.3)$$

where $Q(x, z)$ is an arbitrary function. Suppose that at $t=0$ the potential vorticity is given by

$q_o(x, z)$. Then, the solution of (8.3.2 a) which satisfies that condition will be, according to (8.3.3)

$$q = q_o(x - zt, z) \quad (8.3.4)$$

Consider the special case where the initial potential vorticity is a plane wave,

$$q_o = \tilde{q}_o e^{i(kx + mz)} \quad (8.3.5)$$

Note that this non zero potential vorticity can not be represented by either of the Eady normal modes which have, of course, zero potential vorticity. According to (8.3.4) the potential vorticity at subsequent times will be,

$$q = \tilde{q}_o e^{i(k[x - zt] + mz)} = \tilde{q}_o e^{i(kx + z[m - kt])} \quad (8.3.6)$$

The solution remains a plane wave and the phase of the wave is,

$$\Theta = kx + z(m - kt) \quad (8.3.7)$$

Before discussing (8.3.7) further it might be helpful to rewrite the phase in dimensional units. Recall that we used L to scale x , D to scale z and $L/U_z D$ to scale time.

Temporarily denoting dimensional variables with asterisks,

$$\Theta = k_* x_* + z_*(m_* - k_* U_z t_*) \quad (8.3.8)$$

At any time t the wave vector is given by the spatial gradient of the phase. In nondimensional units, from (8.3.7)

$$\vec{K} = k \hat{x} + (m - kt) \hat{z} \quad (8.3.9)$$

so that the vertical component of the wave vector $m - kt$ changes with time. In dimensional units the vertical component of the wave vector is

$$m_* - k_* U_z t_* \quad (8.3.10)$$

At $t=0$ the vertical component is m (nondimensional) and the horizontal component is k . Because of the shear of the basic current which advects the potential vorticity, the wave crests tilt over and the vertical component of the wave vector decreases with time. The x component, as we see remains fixed.

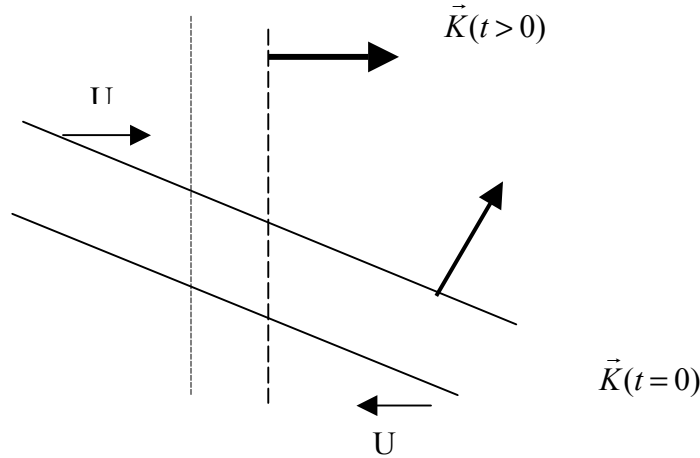


Figure 8.3.1 The phase lines of the potential vorticity perturbation being tilted by the shear (see [8.3.10]) of the basic flow reducing the vertical component of the wave vector. Note that the horizontal distance between phase lines remains fixed during the tilting.

Now that the perturbation vorticity is determined (and note its amplitude remains fixed with time) we can use (8.3.2) to determine the perturbation streamfunction.

$$\phi = -\text{Re} \frac{\tilde{q}_0}{k^2 + [m - kt]^2} e^{i(kx + z(m - kt))} \quad (8.3.11)$$

where we have reminded ourselves that it is the real part of the right hand side of (8.3.11) that we want. That is,

$$\phi = -\frac{1}{2} \left\{ \frac{\tilde{q}_o e^{i[kx+z(m-kt)]} + \tilde{q}_o^* e^{-i[kx+z(m-kt)]}}{k^2 + (m-kt)^2} \right\} \quad (8.3.12)$$

The energy of the perturbation is given by

$$E = \frac{1}{2} (\phi_x^2 + \phi_z^2) \quad (8.3.13)$$

(recall that the second term represents the potential energy; the scale factor f_o^2/N^2 is absent due to our non-dimensionalization). Using (8.3.12) and averaging over a wavelength in x, yields

$$\bar{E}(t) = \frac{1}{4} \frac{|\tilde{q}_o|^2}{[k^2 + (m-kt)^2]} \quad (8.3.14)$$

In particular note that

$$\frac{\bar{E}(t)}{\bar{E}(0)} = \frac{k^2 + m^2}{k^2 + (m-kt)^2} \quad (8.3.15)$$

The maximum value of the ratio occurs at $t = m/k$ and is

$$\left[\frac{\bar{E}(t)}{\bar{E}(0)} \right]_{\max} = \frac{k^2 + m^2}{k^2} = 1 + \left[\frac{m}{k} \right]^2 \quad (8.3.16)$$

To get a substantial magnification the initial perturbation must, 1) be leaning against the shear so that m/k is positive (otherwise the ratio in (8.3.15) always gets

smaller with time, and 2) the vertical wavenumber must be much larger than the horizontal wavenumber. Figure 8.3.2 shows the nature of the amplification.

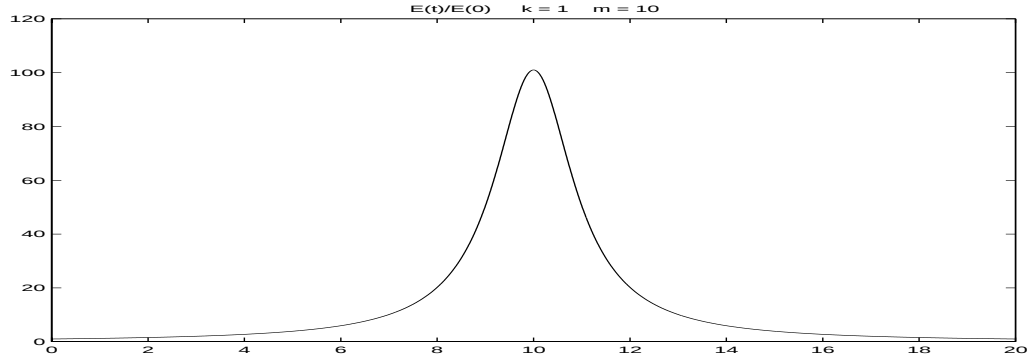


Figure 8.3.2 The temporary magnification of the non-normal mode perturbation.

Note that eventually the perturbation energy decreases like t^{-2} in accordance with the asymptotic result quoted earlier. Nevertheless for finite time a well chosen initial condition can experience a large amplification if its structure is just right. To understand the source of this perturbation energy, we need only refer to (4.2.3), the energy equation for the perturbations. As long as $\overline{\phi_x \phi_z} > 0$ we will be releasing energy through a down gradient buoyancy. Indeed, using (4.2.3) leads directly to (8.3.15). At the initial time, km is positive and there is a down-gradient buoyancy flux. As the current tilts the wave crests vertically the energy increases, after passing through vertical the sign of km changes and becomes negative and the perturbations put energy back into the mean flow and eventually decay. The distinction between the temporary growth of these non normal mode disturbances and the normal modes is that the latter are able to maintain the necessary phase shift with height of the disturbance for all t and are not tilted by the current.

Another way to understand the temporary growth is to note that there is a simple relation between the *enstrophy* of the disturbance and its energy. The potential enstrophy is

$$\overline{q^2}/2 = \frac{1}{2} \overline{(\phi_{xx} + \phi_{zz})^2} \quad (8.3.17).$$

For a plane wave this leads to the simple relation,

$$\overline{q^2} = 2\overline{E}(k^2 + m^2) \quad (8.3.18)$$

or

$$\overline{E}(t) = \frac{1}{2} \frac{\overline{q^2}}{(k^2 + m^2)} \quad (8.3.19)$$

In the Eady problem the potential vorticity present initially (and this must be in the non normal mode part of the solution) is conserved in time. Therefore, if, as in the case where km is >0 , the vertical wavenumber decreases with time the energy must, at least temporarily, increase. As the wave vector tilts over so that $km < 0$, m again increases and the perturbation energy decreases asymptotically.

Especially in meteorology, such temporary growth, not described by the normal modes can be very important in describing particular, extreme cyclogenesis. There is less emphasis on such transient extreme events in oceanic dynamics but the potential is there for particular transient episodes of growth. Associated with this process is, nowadays, a rather complex analysis to determine, given a particular meteorological model, the initial conditions that would maximize the growth of the non normal modes. This is beyond the scope of this course but it leads to a formulation which has much in common with the oceanographic inverse problem. That is, given a particular amplification what is the initial data that is consistent with that growth under the constraint of a particular model dynamics. Nevertheless, in all cases the basic physics of the growth is similar to the simple model we have described here.