Chapter 7. The Eady problem: Effect on the mean and a heuristic explanation of the instability.

7.1 Introduction

In this chapter we take up several fundamental questions and use the Eady model to illustrate those issues. First of all, we will use the mean fluxes calculated in the last chapter to discuss how the eddy field will affect the mean flow. Then we will develop two heuristic arguments to help clarify the physical nature of the baroclinic instability.

Starting from the equations for the mean flow developed in chapters 3 and 4 we obtained the following for the change in the zonal mean flow (*not the basic flow which remains unchanged*). From (3.2.10 a,b),

$$\frac{\partial \overline{u}}{\partial t} = -(\overline{v'u'})_y + f_o \overline{v}_a$$

$$\frac{\partial \overline{b}}{\partial t} = -(\overline{v'b'})_y - \overline{w} N^2 \qquad \text{(7.1.1 a,b,c)}$$

$$\overline{v}_{a_y} + \overline{w}_z = 0.$$

It is clear that the perturbation fluxes of momentum and buoyancy will be of order of the square of the disturbance amplitude. What is not so clear is that both the x-averaged meridional, ageostrophic velocity and the vertical velocity (x-averaged) is also of order amplitude squared leading to changes of the mean zonal velocity and geostrophically balanced mean buoyancy. In fact, as we discussed in chapter 3, it is far more efficient to determine the changes in the mean flow by considering the potential vorticity equation for the mean. Since from (2.4.8)

$$\bar{q} = -\bar{u}_y + \frac{f_o}{\rho_s N^2}(\rho_s \bar{b})_z \tag{7.1.2}$$

we obtain for the mean field(3.2.21) or,

$$\frac{\partial \bar{q}}{\partial t} = -(\overline{v'q'})_y \tag{7.1.3}$$

The change in the mean flow has been denoted in chapter 3 as $\bar{\psi}'$ which is an awkward symbol. Let's call it instead $\Psi(y,z,t)$ so that

$$\bar{u} = -\Psi_y,\ \bar{b} = f_o \Psi_z \tag{7.1.4}$$.

From (7.1.3) it follows that Ψ will be of order amplitude squared and hence so will be the corrections to the mean velocity and buoyancy.

For the Eady model in which q_{o_y} =0, we see from (4.2.16) that the eddy flux of potential vorticity is zero. Hence the correction to the mean potential vorticity must be zero if it is initially zero. Hence the governing equation for Ψ is,

$$\Psi_{yy} + \frac{f_o^2}{N^2}\Psi_{zz} = 0 \tag{7.1.5}$$

We have used the fact that in the Eady model the background density is taken as nearly constant and that the buoyancy frequency, N is constant.

The boundary conditions on z =0 and z =D are that $\bar{w} = 0$ or,

$$f_o \frac{\partial \Psi_z}{\partial t} = -(\overline{v'b'})_y, \qquad z = 0, D \tag{7.1.6}$$

while on y =0,L the condition that both the geostrophic and ageostrophic velocity vanish yields, from (7.1.1 a) that $\partial \bar{u}/\partial t = 0$, or

$$\Psi_y = 0,\ \ y = 0, L \tag{7.1.7}$$

We know from our results of chapter 6 that the buoyancy flux appearing in (7.1.6) is independent of z and hence the same on each boundary. In terms of our results of chapter 6,

$$\overline{v'b'} = \frac{f_o}{4} kc_i |a_o|^2 U_{oz} \sin^2(ly) e^{2kc_i t} \tag{7.1.8}$$

where a_O is an arbitrary amplitude related to the initial amplitude of the wave perturbation.

Thus, (7.1.6) becomes,

$$f_o \Psi_{zt} = -\frac{f_o}{4} kc_i l |a_o|^2 U_{oz} \sin 2ly\, e^{2kc_i t}, \quad z = 0, D \tag{7.1.9}$$

Integrating in t and using the condition that the correction to the mean flow is zero at the start of the perturbation’s growth,

$$\Psi_z = -\frac{l}{8} \sin 2ly |a_o|^2 \left(e^{2kc_i t} - 1\right) \quad z = 0, D \tag{7.1.10}$$

Before proceeding to the solution of (7.1.5), (7.1.7) and (7.1.10) there are several important points to notice. 1) In this problem the entire alteration of the mean flow is forced by buoyancy fluxes at the boundaries. These are the only inhomogeneous forcing terms. This again emphasizes the dynamical character of the boundary conditions. 2) The mean flow correction will vanish if the wave is stable since if c_i =0 the right hand side of (7.1.10) will vanish. 3) Even for unstable waves, the correction to the mean flow will vanish if the wave perturbation were independent of y. (*l=0)*. If the channel were infinitely broad , the linear solution would then be an exact solution of the quasi-geostrophic nonlinear stability equations (check that the Jacobian term would be identically zero in that case). In quasi-geostrophic theory it is the cross stream structure that is vital in determining the nonlinear behavior of the perturbations.

7.2 Solutions for the change in the mean state

There are several ways to solve the set of equations (7.1.5),(7.1.7) and (7.1.10). Since the derivative in z is given on z =0 and D, and since the coefficients of (7.1.5) are constant and the operator is a second derivative in z, it is appropriate to search for solutions in the form

$$\Psi = \sum_{n=1} \Psi_n(y,t) \cos n\pi z / D \qquad (7.2.1)$$

We must be very careful in deriving the equation for Ψ_n since the boundary conditions on z=0,D are not homogeneous for the derivative. Multiply the governing equation (7.1.5) by each cosine function and integrate by parts♦ noting that

$$\int_0^D \Psi_{zz} \cos(n\pi z/D) dz$$

$$= \cos(n\pi z/D)\Psi_z\Big|_0^D + \frac{n\pi}{D}\Psi \sin(n\pi z/D)\Big|_0^D - \left(\frac{n\pi}{D}\right)^2 \int_0^D \Psi \cos(\frac{n\pi}{D}z)dz \qquad (7.2.2)$$

$$= \left[(-1)^n \Psi_z(D) - \Psi_z(0)\right] - \left(\frac{n\pi}{D}\right)^2 \frac{D}{2}\Psi_n$$

It is because we know the boundary condition on Ψ_z at z=0 and D that allows us to use the cosine series. If we knew the function itself, it is clear from (7.2.2) we would have used a sine series instead. The last step in (7.2.2) follows from the standard orthogonality properties of the cosine functions.

This yields an equation for each Fourier amplitude,

$$\Psi_{n_{yy}} - \frac{f_o^{\,2}}{N^2}\left(\frac{n\pi}{D}\right)^2 \Psi_n = \frac{2}{D}\left[1-(-1)^n\right]\Psi_z(0)\frac{f_o^{\,2}}{N^2}$$

(7.2.3)

$$= -\frac{2}{D}\left[1-(-1)^n\right]\frac{l}{8}\frac{f_o^{\,2}}{N^2}\sin 2ly\,|a_o|^2\left(e^{2kc_i t}-1\right)$$

using the fact that the derivative is the same at z=0 and z=D. The solution which satisfies (7.1.7) is,

$$\Psi = \Psi_o \sum_{n=1} \frac{\left[1-(-1)^n\right]}{\left(4l'^2+[n\pi]^2 F\right)}\left[e^{2kc_i t}-1\right]\cos(n\pi z')\left[\sin 2l'y' - \frac{2l'}{n\pi F^{1/2}}\frac{\sinh n\pi F^{1/2}(z'-1/2)}{\cosh(n\pi F^{1/2}/2)}\right]$$

$$\Psi_o \equiv \frac{lf_o^{\,2}L^2}{8DN^2}|a_o|^2 f_o, \quad y' = y/L, \quad z' = z/D,$$

$$F = \frac{f_o^{\,2}L^2}{N^2D^2}$$

(7.2.4)

We have introduce nondimensional y and z variables for ease and to expose, once again, the parameter *F*, the square of the horizontal length scale to the deformation radius squared. Note that for instability we showed in the last chapter that *F* must be greater than 1.722 and that the larger *F* is the greater the range of unstable wavenumbers and the greater the maximum growth rate. Now that Ψ is known we can calculate directly the correction to the mean zonal velocity and the buoyancy.

$$\bar{u} = -\Psi_o 2l \sum_{n=1} \frac{\left[1-(-1)^n\right]}{\left(4l'^2+[n\pi]^2 F\right)}\left[e^{2kc_i t}-1\right]\cos(n\pi z')\left[\cos 2l'y' - \frac{\cosh n\pi F^{1/2}(z'-1/2)}{\cosh(n\pi F^{1/2}/2)}\right]$$

(7.2.5)

♦ Note that if you (incorrectly) just substituted the series into the differential equation and differentiated term by term, the inhomogeneous boundary conditions would not enter your problem for Ψ_n and you would obtain the silly answer $\Psi=0$.

This correction function is contoured in Figure 7.2.1 while Figure 7.2.2 shows the profile of the zonal flow correction near the upper surface (z/D =0.9).

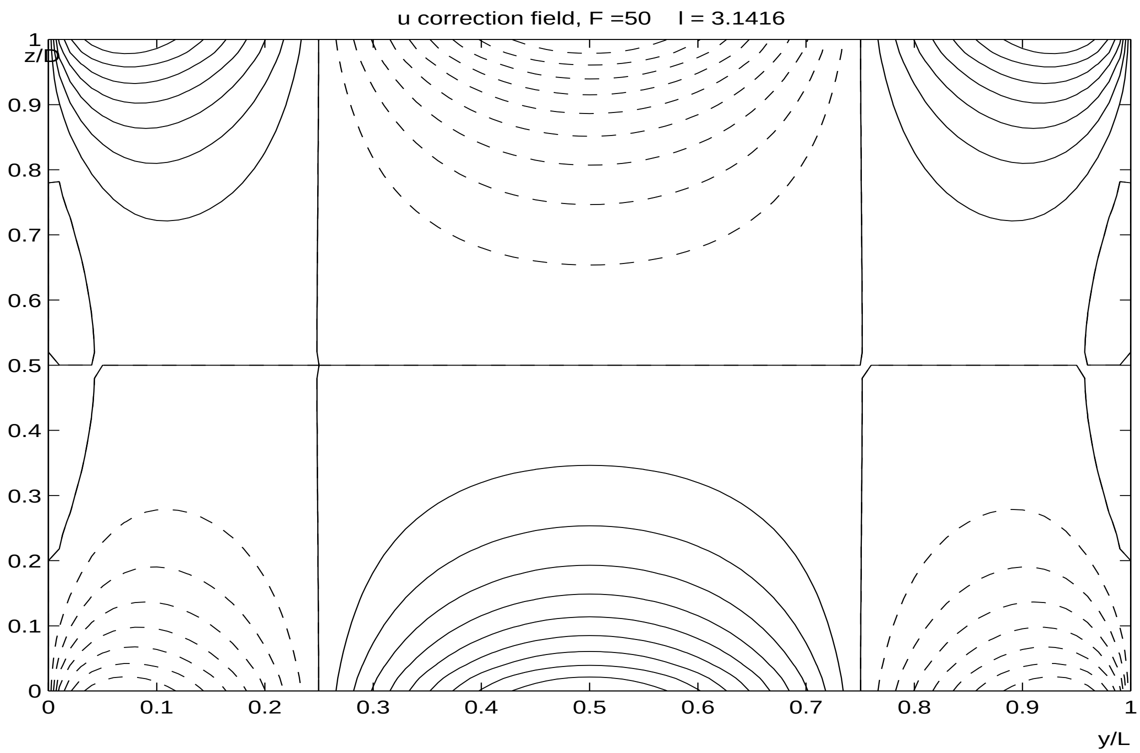

Figure 7.2.1 The correction to the mean zonal flow in due to the perturbations. Positive values contoured in solid, negative values are shown with dashed contours.

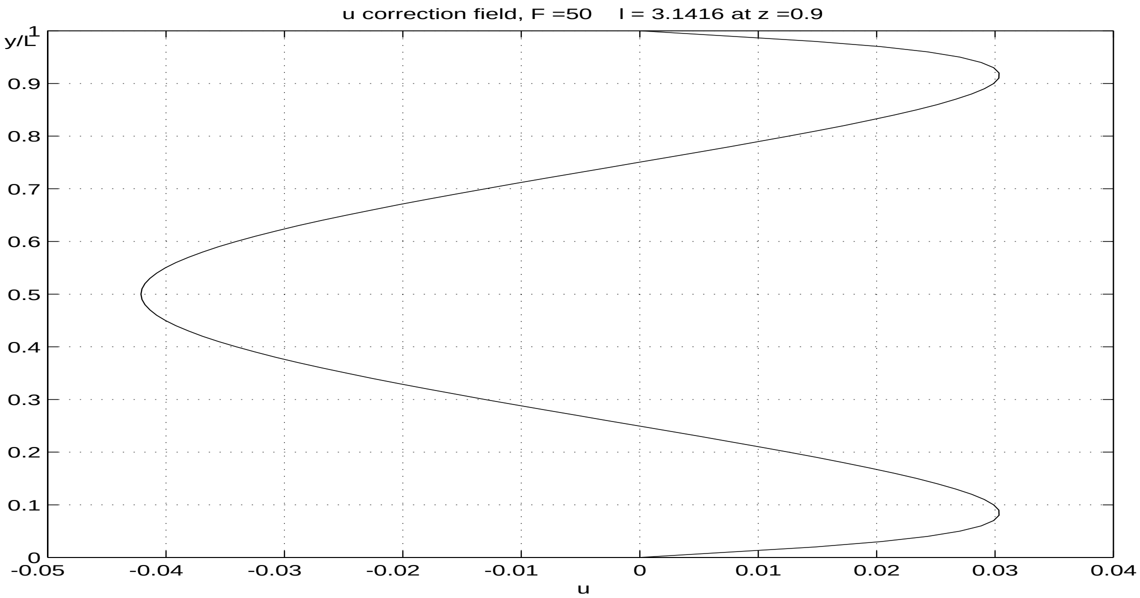

Figure 7.2.2 The correction profile of the zonal velocity at z/D =0.9

There are several interesting things to note. First of all, as we see from the contour plot the correction to the mean zonal flow is purely baroclinic. At each y the vertical averaged $\overline{u}$ is unchanged. This follows from the absence of any Reynolds stress so that the change in the mean flow, from (7.1.1. a) must be due solely to the Coriolis force of the ageostrophic meridional flow. To conserve mass flux across each latitude circle the vertical average of that ageostrophic meridional velocity must be zero, Hence the purely baroclinic character of $\overline{u}$. Second, we see from both the contour plot and the profile that in the center of the channel the zonal velocity is reduced at upper levels and increased at lower levels (again this follows from the equal and opposite meridional velocities). Thus, in the center of the channel, where the perturbation eigenfunction has its maximum value, the vertical shear of the mean flow is reduced by the effect of the perturbation. This we might intuitively expect. Note however, that the zonal flow correction actually has the opposite sign in two narrow regions near the boundaries at y = 0 and L. This is due to structure of the basic unstable mode which gives rise to a buoyancy flux proportional to $\sin 2ly$ and a zonal flow correction proportional to $\cos 2ly$ whose structure is evident in the figure. The homogeneous solution adds only a narrow boundary layer of width $F^{-1/2}$

(deformation radius in dimensional units) to bring the correction to the zonal velocity to zero on the boundary. This homogeneous solution of the potential vorticity equation, giving rise to this term, is forced by the boundary condition of vanishing $\overline{u}$ at the boundaries. The larger is F the smaller these regions will be. So, fundamentally the action of the perturbations is to reduce the vertical shear.

This becomes clear if we look at the profiles of the buoyancy correction. The buoyancy correction has a vertical structure that is, by the thermal wind, that of the vertical derivative of $\overline{u}$ and hence is the same above and below the mid-level depth. The profile of the correction is shown in Figure 7.2.3.

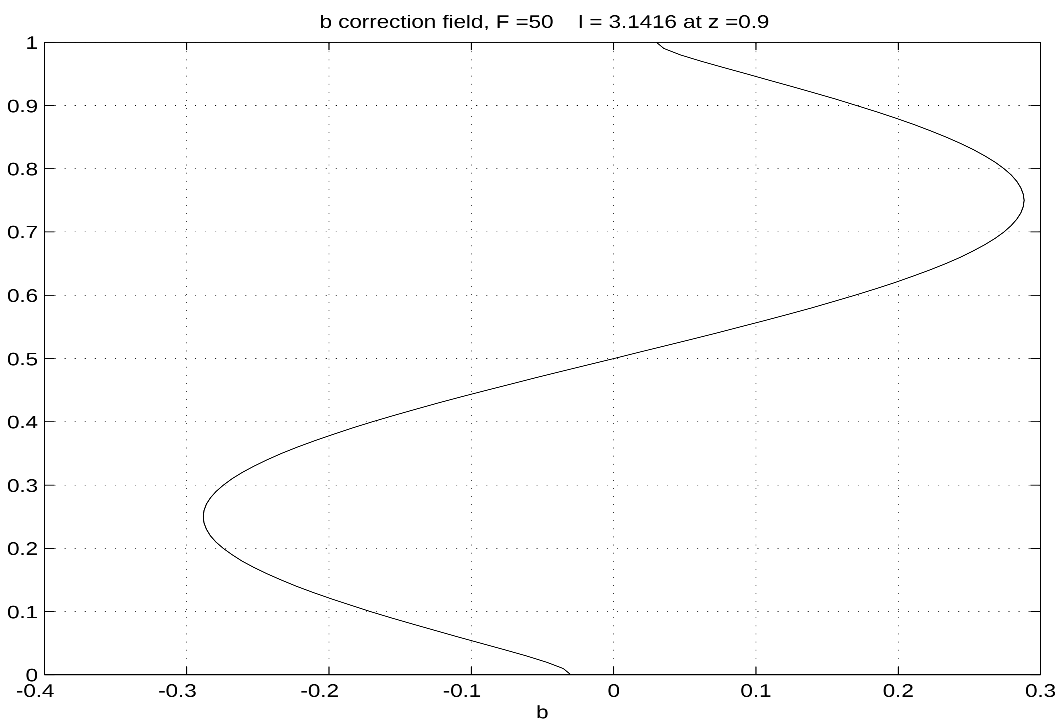

Figure 7.2.3 The correction to the x-averaged buoyancy.

If we think of the buoyancy as proportional to the temperature perturbation we see that the net effect of the eddy buoyancy flux is the warm northern latitudes and cool southern latitudes as the perturbations flux heat across the current to the north. If we add this alteration to the temperature profile that exists in the basic state we see clearly in

Figure 7.2.4 that the effect of the perturbation heat flux is to weaken the unstable temperature gradient in the middle of the domain.

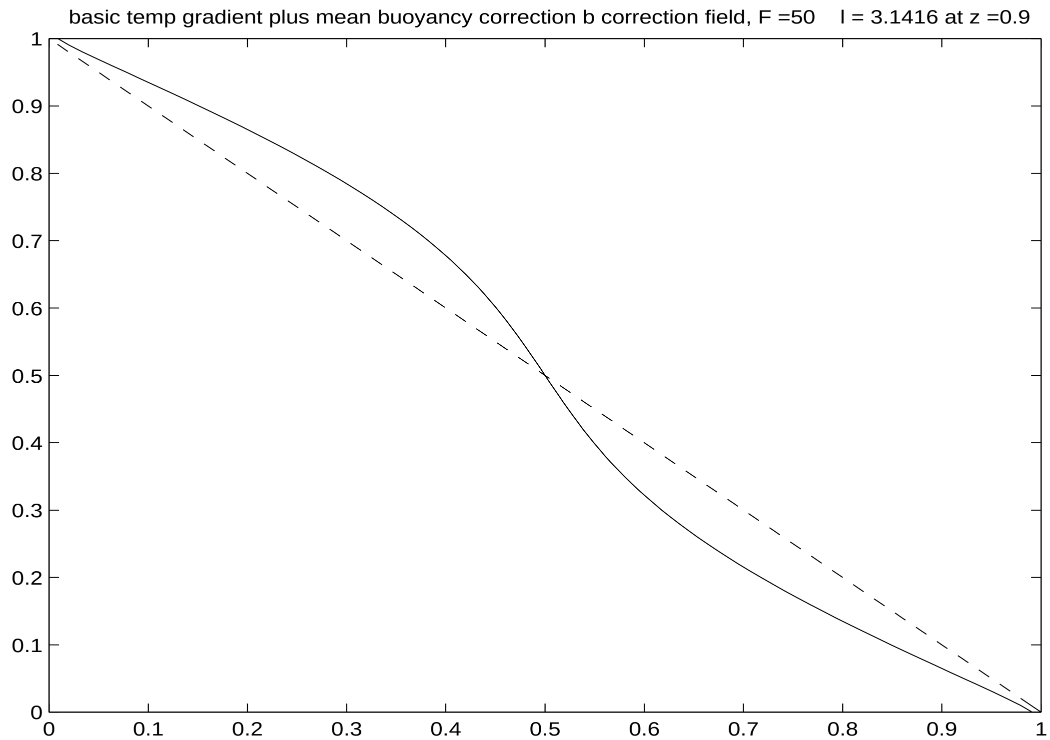

Figure 7.2.4 The basic state temperature (buoyancy) profile is shown as a dashed curve. When the effect of the perturbations is added the temperature profile shown by the solid curve results.

The similarity of the instability to classical convection is suggested by Figure 7.2.4. Instead of the unstable temperature gradient being in the z-direction it is here a horizontal temperature gradient that drives the instability and the eddy flux, just as in the classic convection problem tends to smooth out the unstable temperature profile. That is one reason why baroclinic instability is often referred to as *horizontal convection*. We will return to that idea further on.

Once $\overline{u}$ is known we can calculate the x-averaged ageostrophic meridional velocity, $\overline{v}_a$ directly from (7.1.1.a) since the Reynolds stresses are zero. It is proportional to the time derivative of $\overline{u}$ and so will have its structure. However, since it is proportional to $\partial\overline{u}/\partial t$ its time factor is different, i.e.

$$\overline{v}_a = -2kc_i t\Psi_o(2l/f_o)\sum_{n=1}\frac{\left[1-(-1)^n\right]}{\left(4l'^2+[n\pi]^2F\right)}\left[e^{2kc_i t}\right]\cos(n\pi z')\left[\cos 2l'y' - \frac{\cosh n\pi F^{1/2}(z'-1/2)}{\cosh(n\pi F^{1/2}/2)}\right] \tag{7.2.6}$$

Note that while the correction to the x-averaged zonal flow and the buoyancy vanish at t=0, the ageostrophic meridional velocity (also of order amplitude squared as forecast) is immediately different from zero as long as the flow is unstable, i.e. as long as there is a non-zero growth rate. This is due to the fact that the perturbations, from the first instant, while growing, transport buoyancy northward and force a Eulerian meridional circulation in the y-z plane. Indeed, using (7.1.1 c) we can introduce a stream function for the x-averaged meridional circulation,

$$\overline{v}_a = -\frac{\partial\chi}{\partial z}, \qquad \overline{w} = \frac{\partial\chi}{\partial y} \tag{7.2.7 a.b}$$

and from (7.2.6) we can immediately integrate to find χ,

$$\chi = 2kc_i t\Psi_o(2l/f_o)D\sum_{n=1}\frac{\left[1-(-1)^n\right]}{\left(4l'^2+[n\pi]^2F\right)n\pi}\left[e^{2kc_i t}\right]\sin(n\pi z')\left[\cos 2l'y' - \frac{\cosh n\pi F^{1/2}(z'-1/2)}{\cosh(n\pi F^{1/2}/2)}\right] \tag{7.2.8}$$

The streamfunction is plotted in Figure 7.2.5

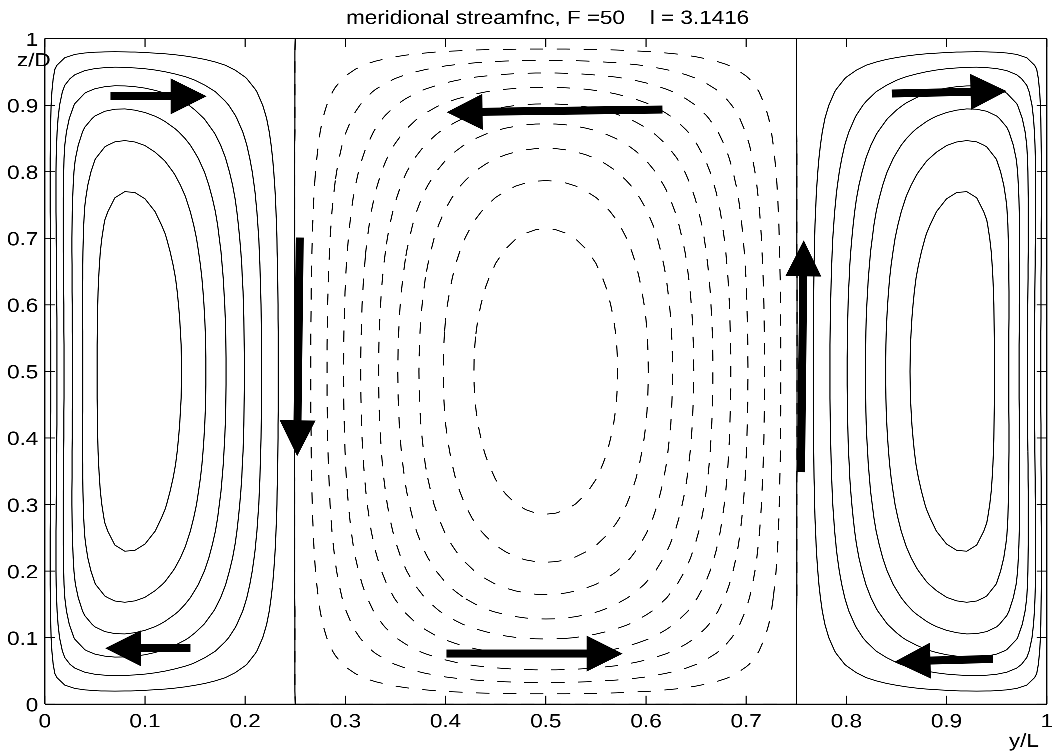

Figure 7.2.5 The meridional streamfunction for the perturbation induced x-averaged flow in the meridional plane.

Arrows have been added to the figure to show the sense of the circulation which you should be able to deduce from the sense of the change in the mean zonal velocity. It is important to realize that since the overall flow is time dependent the above streamfunction gives the pattern of the Eulerian overturning circulation. Fluid particle trajectories will not follow those isolines. Indeed, by using the theory of the Transformed Eulerian Mean equations discussed in 12.802 it can be shown that the particle trajectories are quite different. (A particularly illuminating discussion of this is found in Shepherd 1983, GAFD, vol 27. pp 35-72) in which it is shown that the actual particle trajectories rise in the south and flow northward in an upper Lagrangian branch of the circulation and sink at high latitudes before returning southward, a circulation much more in tune with our intuition of a "convectively" driven instability. Unfortunately, a full discussion of that feature is, for reasons of time, beyond the scope of this discussion.

Overall, we note that the net effect of the instabilities is to drive the mean flow towards stability by reducing the storehouse of energy. Alternatively, recognizing that the necessary conditions for instability in this problem involve the meridional buoyancy gradients on the boundary, the effect of the instability is to reduce those gradients, again driving the mean flow towards stability. Please note that throughout all this change the *basic state* remains the same. These changes form part of the perturbation field and are O(amplitude)^2.

7.3 A heuristic model

We have the full results of the stability calculation but can we try to understand in a heuristic way some of the key features of the problem? In particular, the critical wavenumber for the instability and the reason for the phase shift with height of the disturbance (although we are aware the latter is necessary if we want to release energy). Note that the condition (6.2.14) that the wavelength be of the same order as the deformation radius, or larger, can be restated by saying that the two boundaries of the fluid must be close enough so that each is aware of the other, The exponential character of the solutions of (6.2.10) implies that if D is too large the disturbance will be negligibly small at the other boundary, as if the other boundary were removed infinitely far away. The necessary conditions for instability tell us that for the Eady problem both boundaries must enter. We can try to sharpen these ideas a little by considering just that situation. Suppose we redo the Eady model but instead of having two boundaries we start with a single boundary at z=0 and insist that the solution by finite as z → infinity. We can find solutions of (6.2.10) for ϕ in the form,

$$\phi = Ae^{-kNz/f_o}\cos(k[x-ct]) \tag{7.3.1}$$

where for simplicity we are considering perturbations independent of y in an region infinitely wide. The single remaining boundary condition (6.2.8a) (the other condition has been replaced by finiteness at infinity that is already satisfied by (7.3.1)). Application of that boundary condition yields,

$$c = \frac{f_o U_{o_z}}{kN} \tag{7.3.2}$$

so that the wave always moves to the right with respect to the lower boundary.

The buoyancy and meridional velocity perturbations associated with (7.3.1) are,

$$b = -kNAe^{-kN/f_o}\cos(k[x - ct])$$
$$v = -Ake^{-kN/f_o}\sin(k[x - ct]) \qquad (7.3.3\text{ a,b})$$

If we look instead at a model where there is an upper boundary at z=D and no lower boundary, we would look for solutions instead, of the form, (the subscript u for the upper wave),

$$\phi_u = A_u e^{kN(z-D)/f_o}\cos(k[x - c\,t] + \gamma)$$
$$b_u = kNA_u e^{kN(z-D)/f_o}\cos(k[x - ct]) \qquad (7.3.4\text{ a,b,c})$$
$$v_u = -kA_u e^{kN(z-D)/f_o}\sin(k[x - ct])$$

Note that the dependence on z is chosen so that the solution decays exponentially downward, The constant γ is an arbitrary phase chosen with respect to the lower wave. In particular, note the difference in the phase relation between the pressure field ϕ and the buoyancy field for the upper wave when compared to the lower wave.

Now applying the boundary condition at z=D yields,

$$c = DU_{o_z} - \frac{f_o U_{o_z}}{kN} \qquad (7.3.5)$$

so that the wave propagates *against* the mean flow (i.e.. the left with respect to the current).

In the figures the solid curves show the stream functions, the dashed curves are the buoyancy perturbations and the dotted curves are the meridional velocities. In each case the functions are evaluated on their respective boundary.

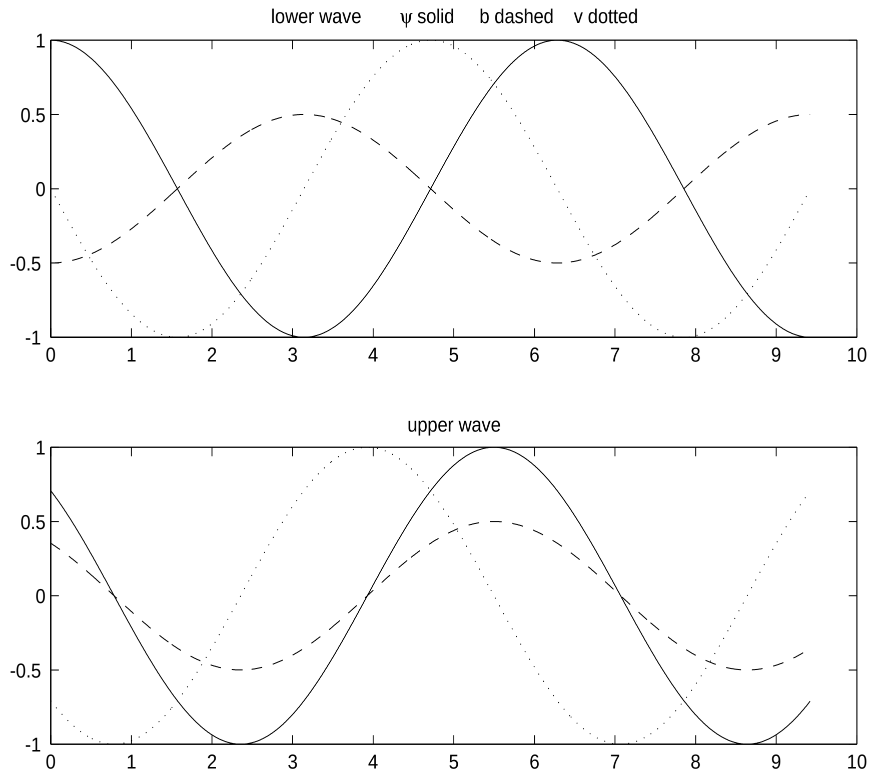

Figure 7.3.1 a,b,

On the boundaries the buoyancy perturbation is advected directly with the meridional velocity. Note that *v* is 90^0 out of phase with *b* so that there is no net flux of buoyancy (the wave is stable). Examining figure 7.3.1a, for example, we see that the meridional velocity will shift the phase of the buoyancy wave to the right since *v* is phase shifted to the right of *b*. The opposite is true at the upper boundary so that the wave is speed shifts against the current to the left. In each case this is consistent with the phase speed relations (7.3.2) and (7.3.5).

Now let's suppose we try to just superpose these two waves to build a single mode in a domain with two lateral boundaries. The first condition that must be fulfilled is that the waves have a common phase speed. Equating the results of (7.3.2) and (7.3.5) yields,

$$k = 2f_o / ND = 2/L_d \,,$$

$$c = DU_{o_z}/2 \qquad (7.3.6)$$

The phase speed is exactly the phase speed of the marginally neutral Eady wave and the critical wave number is very close $2/L_d$ rather than $2.399/L_d$. We can't expect to obtain the exact criterion since the sum of the two solutions fails to exactly match the boundary conditions at each boundary—but it's close!

Now suppose, we imagine that the upper wave's geostrophic streamfunction is phase shifted to left (against the shear) on the upper boundary with respect to the lower boundary. One obtains a situation shown in Figure (7.3.2).

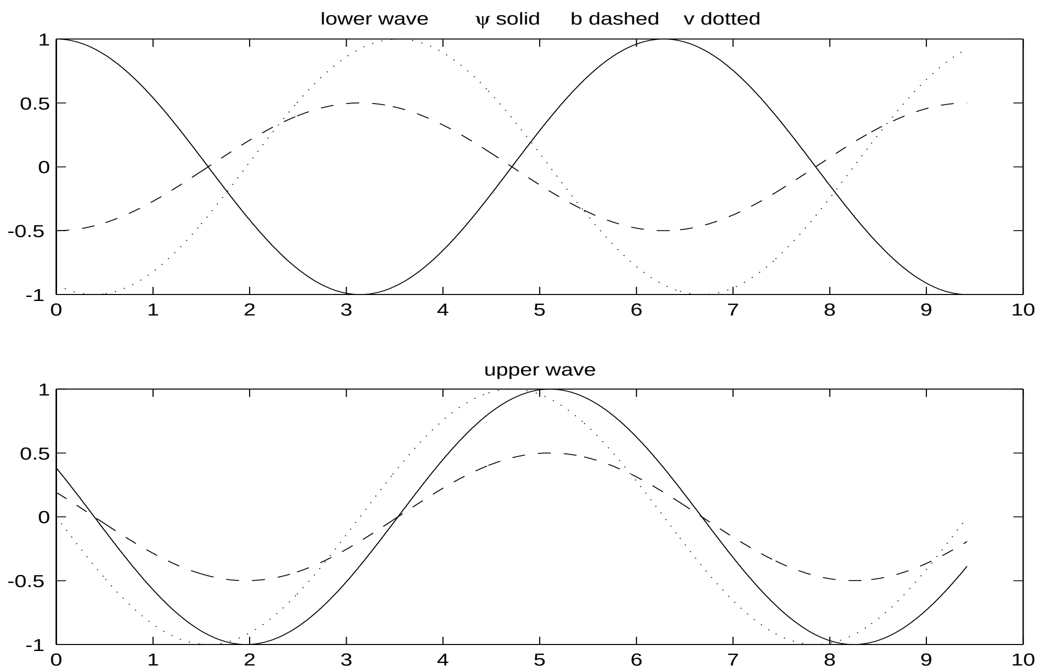

Figure (7.3.2 a,b) The waves on each boundary. The solid curves are the streamfunction. The dashed curves are the buoyancy perturbations and the dotted curves are the meridional velocity due to the wave at the *other* boundary.

In Figure 7.3.2 a,b we have plotted the ϕ (solid curve) on each boundary and its corresponding buoyancy perturbation b (dashed) and the meridional velocity due to the wave on the other boundary assuming that the boundaries are close enough together for the perturbation to "reach" the other boundary. From the solutions (7.3.1) and (7.3.4) this requires ,

$$kND/f_o \leq 1 \qquad (7.3.7)$$

which, roughly speaking is the Eady criterion for instability. Because of the phase shift with height of the pressure field there is now a contribution to the meridional velocity at each boundary by the perturbation at the other boundary that is in phase with the buoyancy perturbation and so will serve to enlarge the amplitude of the perturbation. Algebraically, if we have a buoyancy perturbation at the lower boundary of the form,

$$b_L = \alpha(t)\cos k(x-ct) \tag{7.3.8}$$

the equation, at that boundary for the perturbation is,

$$\frac{db_L}{dt} = -v\frac{\partial b_o}{\partial y} \tag{7.3.9}$$

where v is the total velocity. If we write,

$$\begin{aligned} v &= \alpha(t)\left[A\cos k(x-ct) + B\sin k(x-ct)\right] \\ &= \alpha(t)\sin\left[k(x-ct)+\theta\right] \end{aligned} \tag{7.3.10}$$

it follows that, as anticipated by the figure,

$$\frac{d\alpha}{dt} = \alpha A\left(-\frac{\partial b_o}{\partial y}\right) = \alpha\sin\theta\left(-\frac{\partial b_o}{\partial y}\right) \tag{7.3.11}$$

so that there will be an exponential increase in the amplitude of the buoyancy perturbation caused by the in-phase component of the meridional velocity which is due 1) the phase shift of the upper wave with respect to the lower wave and 2) the sufficient length of the wave so that the velocity perturbation on one boundary can reach the other boundary.

It is important to keep in mind that this argument which looks only kinematic actually depends on two dynamical relations. The conservation of potential vorticity which determines the vertical “reach” of each boundary wave as a function of its wavelength and on the fact that the buoyancy on each boundary is advected by the velocity on the boundary as well as the hydrostatic relation between the geostrophic pressure and the buoyancy. You should check that the kinematic phase relations we have noted here for growth are equivalent to a requirement that there is a net buoyancy flux down the buoyancy gradient in the basic state.

7.4 Eady's description of the instability

There is another heuristic description of the instability which is very suggestive and illuminating and which was suggested by Eady himself and can be found in the discussion section of the Quarterly Journal of the Royal Meteorological Society 1948, 74. It is not free from difficulty and is not completely rigorous but it does, I believe capture the essence of the problem.

As we noted in chapter 6 the vertical shear of the basic state, and the thermal wind equation implies that the buoyancy surfaces in the basic state are sloping at an angle α to the horizontal where,

$$\tan\alpha = -\frac{\rho_y}{\rho_z} = \frac{f_o U_{o_z}}{N^2} \qquad (7.4.1)$$

for the case where the fluid is incompressible and the buoyancy perturbation is due to density anomalies advected by the perturbations. Consider Figure 7.4.1.

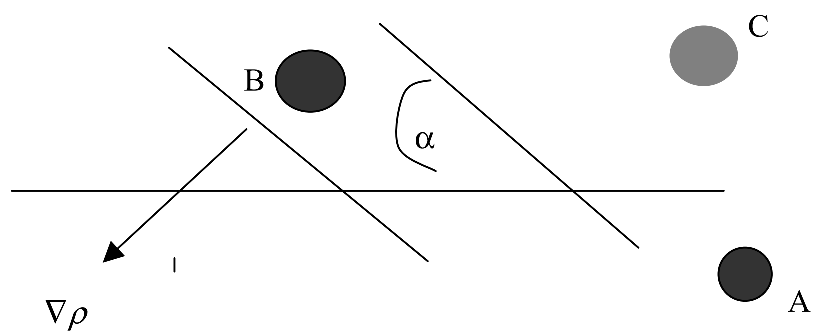

Figure 7.4.1 The wedge opened by the thermal wind between the isolines of the basic state buoyancy and the horizontal surface.

Consider moving the fluid element A from its position to some other position, say either B or C. Let the small distance it's moved by represented by the virtual displacement vector $\delta\vec{r} = \hat{j}\,\eta + \hat{k}\zeta$. Then assuming that the density is conserved by the perturbation the density anomaly at any point, say B, will be the difference between the ambient density at B and the density of the fluid element which arrives from A carrying that value of the density.

Thus, at any point near A,

$$\rho(\vec{r}+\delta\vec{r}) \approx \rho_A + \frac{\partial\rho}{\partial y}\eta + \frac{\partial\rho}{\partial z}\zeta,$$

$$\rho_A - \rho_B \approx -\eta\rho_y - \zeta\rho_z \tag{7.4.2}$$

Now let's consider these small displacements such that $\zeta/\eta = \tan\varphi$,

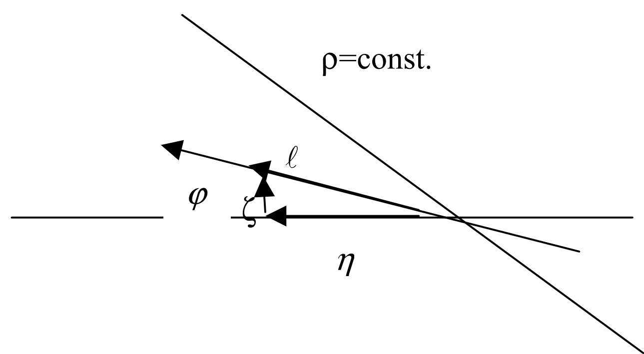

Figure 7.4.2 the trajectory of a displacement of a fluid element within the wedge opened up by the vertical shear.

If we calculate the gravitational buoyancy force along the path of the displacement,

$$\frac{(\rho_A - \rho_B)}{\rho_s} g \sin\varphi = -\left[\frac{\eta\rho_y + \zeta\rho_z}{\rho_s}\right] g \sin\varphi$$

$$= -g\frac{\rho_z}{\rho_s}\left[1 + \frac{\eta}{\zeta}\frac{\rho_y}{\rho_z}\right]\zeta \sin\varphi = N^2\left[1 - \frac{z_y)_\rho}{\zeta/\eta}\right]\zeta \sin\varphi \qquad (7.4.3)$$

$$= N^2\left[1 - \frac{\tan\alpha}{\tan\varphi}\right]\ell \sin^2\varphi$$

where $\ell = \zeta/\sin\varphi$ is the distance along the direction of displacement, The work done on the particle is by the buoyancy force is just the force times the displacement distance, or,

$$Work\,done = N^2\ell^2\left[1 - \frac{\tan\alpha}{\tan\varphi}\right]\sin^2\varphi \qquad (7.4.4)$$

If the shear were zero, so that α were zero, there would be a net restoring force in the direction of the motion and the net work done would be simply $N^2\ell^2$. It would be positive and this would lead to an oscillation whose maximum frequency would be *N*. On the other hand, in the presence of the shear the angle between the density surface and the horizontal opens up. Thus, from (7.4.4) if $\tan\varphi \leq \tan\alpha$ the work done on the fluid element is *negative*, that is the gravitational force will accelerate the fluid element away from its initial position rather than tend to restore it to its starting point. What we see then, because of the slope of the density surfaces is lighter fluid *rising* into heavier fluid. This is possible only if there is a vertical shear of the current. Note that at displacement outside the wedge of instability opened up by the sloping density surfaces, e.g. a displacement from A to C would not release energy. It is a simple matter to show that the displacement angle which maximizes the energy release is one for which $\tan 2\varphi = \tan\alpha$ or for small angles $\varphi = \alpha/2$.

The condition for instability is thus,

$$\frac{\zeta}{\eta} = \frac{w}{v} < \frac{f_o U_{o_z}}{N^2} \qquad (7.4.5)$$

From the vorticity equation we can estimate the size of *w* with respect to *v* by,

$$f_o w_z \approx O(U_o v_{xx}) = O(vU_o / L^2) \qquad (7.4.6)$$

thus, the condition (7.4.5) is

$$\frac{w}{v} = O\left(\frac{U_o}{f_o L}\frac{D}{L}\right) < \frac{f_o U_{o_z}}{N^2} = O\left(\frac{f_o U_o}{N^2 D}\right) \qquad (7.4.7)$$

or,

$$L > \frac{ND}{f_o} = L_d \qquad (7.4.8)$$

This is qualitatively the condition for instability we derived directly from the Eady problem and its interpretation here is that the width L must be large enough to allow fluid trajectories to lie within the wedge of instability. If L is too small the trajectories must be tilted upward too much to stay within the wedge and the energy release mechanism will be lost.