Chapter 6. The Eady problem

6.1 Introduction

It is useful to consider a specific example to clarify and illustrate the general ideas of the first 5 chapters. We will begin by considering the classic model of Eric Eady (1949, Tellus , vol. 1, 33-52) of baroclinic instability. Decades of observations have confirmed that the major source of synoptic scale activity in the atmosphere is due to this form of instability in which the available potential energy, manifested in the sloping density surfaces in the region of the westerly winds (from the west) is released by cyclone scale motions with wavelengths of the order of 500 – 1000 km. The sloping isentropic surfaces, by the thermal wind relation , must be accompanied by the vertical shear of the zonal flow. It is important to emphasize again that the energy source that goes along with that vertical shear is the potential energy of the sloping buoyancy surfaces and not a release of the kinetic energy of the shear. The vertical transport of momentum by a Reynolds stress of the form $\overline{u'w'}$ is negligible for quasi-geostrophic motions because of the smallness of w'. Similar instabilities are observed in the ocean, both in the major currents like the Gulf Stream and in the mid ocean . The oceanic case is often more complex because the flow is rarely purely zonal. Nevertheless, the basic mechanism and the predictions about spatial scale, growth rate and structure are well anticipated by Eady's model.

Historically, the formulation of the model by Eady occurred sometime prior to its publication (1949) and was independent of the model of Charney (1947, to be discussed later). As in Charney's case, the formulation as described in the original paper is hard to follow because both the development of the quasi-geostrophic approximation and the formulation of the baroclinic instability problem had to be done simultaneously. Without the qg approximation the instability problem is very nearly intractable. With the passage of time, and the formulation of quasi-geostrophic scaling the Eady problem looks very simple in retrospect and it is important to realize just how great a creative and difficult accomplishment it was. Eady had a very deep physical insight in to the basic nature of the problem which allowed him to remove from his model all elements that are extraneous to the basic mechanism for instability.

6.2 The Eady model

The basic model consists of an incompressible fluid, so that in our stability equation for normal modes (5.14 a), the density ρ_s can be taken as constant. Further, he assumed that the buoyancy frequency, *N*, is constant and that the horizontal boundary at z=0 was flat (*h* = 0). Most radically, he ignored the effect of sphericity and took β=0. None of these simplifications can be justified by an appeal to a realistic model for the atmosphere, and aside for the simplification about constant density it is equally poor as a simulation of the ocean. The current is contained in a zonal channel of width L. The genius of Eady's formulation was to understand that the model was just rich enough to expose the fundamental nature of the instability.

Perhaps the most artificial (from the atmospheric viewpoint) aspect of the model is the use of a rigid upper boundary at *z=D*. He had in mind a model of the earth's troposphere and later considered a more complex model in which the upper lid is replaced by an infinite region of much larger static stability (i.e. larger *N*) with only quantitative changes in the results. In addition the basic velocity profile is a linear function of z and independent of y.

$$U_o(z) = U_z z \qquad (6.2.1)$$

where U_z is *a constant*. Accompanying this vertical shear there is a buoyancy gradient (due either to a temperature gradient or density gradient).

$$b_o = -f_o U_{oz} y \qquad (6.2.2)$$

leading to a slope of the isopycnals

$$\left(\frac{\partial z}{\partial y}\right)_{b_o} = -\left(\frac{b_{oy}}{b_{s_z}}\right) = \frac{f_o U_z}{N^2} \tag{6.2.3}$$

It is illuminating to compare this slope with the ratio of the characteristic geometrical aspect ratio D/L where D is the vertical dimension of the region and L is the horizontal dimension.

$$\frac{\left(\frac{\partial z}{\partial y}\right)_{b_o}}{D/L} = \frac{f_o(DU_{o_z})L}{N^2 D^2} = \frac{\left(DU_{o_z}\right)}{f_o L}\frac{f_o^{\,2} L^2}{N^2 D^2} = \varepsilon F \tag{6.2.4}$$

where ε is the Rossby number based on the characteristic basic zonal flow velocity DU_{o_z} associated with the shear and F is the square of the ratio of the length scale L to the deformation radius i.e.

$$F = \frac{f_o^{\,2} L^2}{N^2 D^2} = \left(\frac{L}{L_d}\right)^2, \; L_d = ND/f_o \tag{6.2.5}$$

For quasi-geostrophic scaling the parameter F is order one so that scaled with the geometry of the region the slope of the buoyancy surfaces is small, the order of the Rossby number.

Note that with the assumptions already made the potential vorticity gradient of the basic state ,

$$q_{o_y} = \beta - U_{o_{yy}} - \frac{f_o^{\,2}}{N^2} U_{o_{zz}} = 0 \tag{6.2.6}$$

It is the absence of the potential vorticity gradient within the fluid that leads to the essential simplification of Eady's model. According to (5.2.7) this implies that without the presence of an upper boundary (or the equivalent continuity condition at z=D in the presence of a sudden increase in *N)* the flow would be stable. Instability in the model is only possible because of the non trivial condition at z=D, again an illustration of the *dynamical* character of the boundary conditions in this problem.

The normal mode equation (5.2.7) then becomes,

$$(U_o - c)\left[\frac{f_o^{\,2}}{N^2}\Phi_{zz} + \Phi_{yy} - k^2\Phi\right] = 0, \tag{6.2.7}$$

with boundary conditions,

$$(U_{oz}z - c)\Phi_z - U_{oz}\Phi = 0, \quad z = 0, D$$

$$\Phi = 0, \qquad\qquad y = 0, L \tag{6.2.8 a,b}$$

A solution can be found which automatically satisfies the boundary conditions on y=0, L and is suitable to the governing equation, in the form,

$$\Phi = A(z)\sin ly, \quad l = m\pi/L \tag{6.2.9}$$

leading to the following ordinary differential equation for $A(z)$

$$(z - c/U_{oz})\left[A_{zz} - \kappa^2 A\right] = 0,$$

$$(z - c/U_{oz})A_z - A = 0, \qquad z = 0, D \tag{6.2.10 a,b,c}$$

$$\kappa^2 = \frac{N^2}{f_o^{\,2}}(k^2 + l^2)$$

If we assume that $z - c/U_{o_z}$ never vanishes in the interval *(0,D)* i.e. if the phase speed never equals the basic flow velocity, we can divide (6.2.9) by the first factor and find the non-singular solution,[♦]

$$A = a\cosh\kappa(z - D/2) + b\sinh\kappa(z - D/2) \tag{6.2.11}$$

We apply the boundary conditions (6.2.10 a,b) to obtain, from z=0,

$$a\left[\kappa c\sinh(\kappa D/2) - U_{o_z}\cosh(\kappa D/2)\right] + b\left[U_{o_z}\sinh(\kappa D/2) - c\kappa\cosh(\kappa D/2)\right] = 0 \tag{6.2.12}$$

while from z=D,

$$\begin{aligned} &a\left[\kappa(DU_{o_z} - c)\sinh(\kappa D/2) - U_{o_z}\cosh(\kappa D/2)\right] \\ &+b\left[-U_{o_z}\sinh(\kappa D/2) + (DU_{o_z} - c)\kappa\cosh(\kappa D/2)\right] = 0 \end{aligned} \tag{6.2.12}$$

The set (6.2.11) and (6.2.12) are two homogeneous, linear algebraic equations for *a* and *b*. Non trivial solutions for those constants require that the determinant of the coefficients vanish. As anticipated we have an eigenvalue problem that will yield a condition *c=c(k)*. After a considerable amount of algebra, the solution for *c* resulting from solving the above equations is,

$$c = DU_{o_z}/2 \pm \frac{U_{o_z}}{\kappa}\left\{\left[\frac{\kappa D}{2} - \tanh(\kappa D/2)\right]\left[\frac{\kappa D}{2} - \coth(\kappa D/2)\right]\right\}^{1/2} \tag{6.2.13}$$

[♦] The general solution of the algebraic equation x f(x) =0 is f(x)=Bδ(x) where B is an arbitrary constant. The nonsingular solution takes B=0 as we do for the solution (6.2.11).

As a function of κD the radicand of (6.2.13) can be either positive or negative. In the former case the two roots for c are both real and the disturbance oscillates but does not grow. In the latter case c has a non-zero imaginary part. Note that in that case there is a growing mode and a decaying mode with numerically equal Imag(c). We remarked earlier that that had to be the case since a solution and its complex conjugate were always possible. The figure below shows the nature of the two factors in (6.2.13).

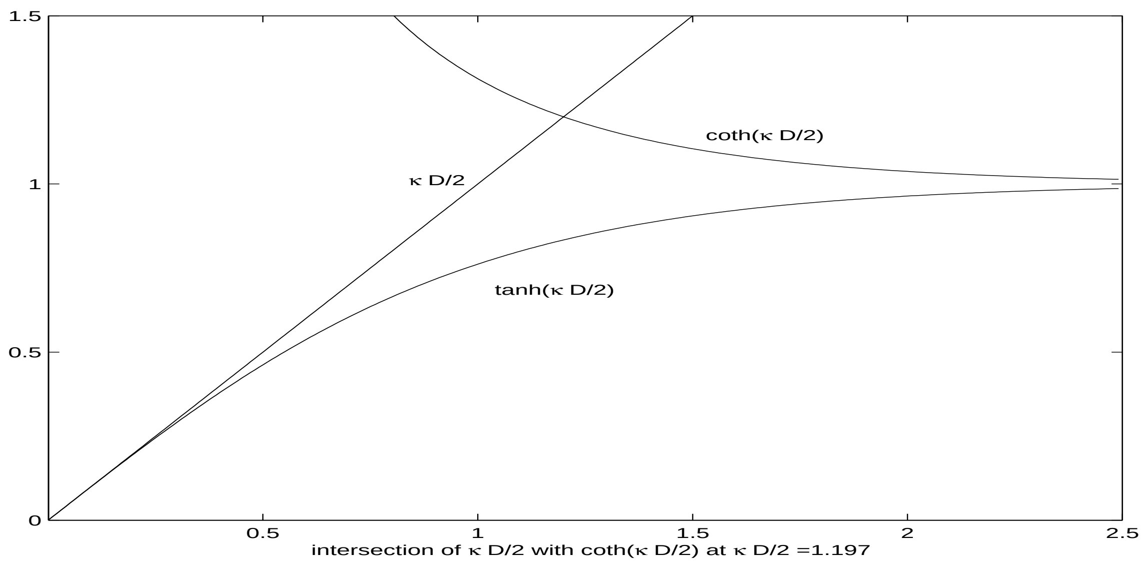

Figure 6.2.1 The terms in the dispersion relation (6.2.13).

We note that for all κ, $\kappa D/2 \geq \tanh(\kappa D/2)$ while $\kappa D/2 < \coth(\kappa D/2)$ for a critical value of κ $D/2$ =1.197. Thus for instability to occur,

$$\kappa D < 2.394 \tag{6.2.14}$$

or, using our definition of κ

$$k^2 + l^2 < 5.731/L_d^2, \qquad L_d = ND/f_o \tag{6.2.15}$$

We will later discuss the alternative, singular solution where B is not zero.

or,

$$k < \left(\frac{5.731}{L_d^{\,2}} - l^2 \right)^{1/2} \tag{6.2.16}$$

corresponding to a wavelength in the zonal direction,

$$\lambda = 2\pi \left(\frac{5.731}{L_d^{\,2}} - l^2 \right)^{-1/2} \tag{6.2.17}$$

In a channel of width L the minimum value of l is π/L. Thus for instability to occur at all it is necessary that (for real λ)

$$L > \frac{\pi}{2.3994} L_d = 1.309 L_d \tag{6.2.18}$$

Therefore, the current must be at least as broad as the deformation radius to allow instability. Narrow currents will be stable, *even if they satisfy the necessary conditions for instability.* For a very broad channel, i.e. much broader than the deformation radius, the l term in (6.2.17) can be neglected and this yields a critical wavelength,

$$\lambda_{crit} = \frac{2\pi}{2.3994} L_d = 2.62 L_d \tag{6.2.19}$$

so that the distance from peak to trough in the wave (the quarter wavelength) is about $0.65L_d$. Thus we would expect to see unstable disturbances with scales of the order of the deformation radius. This fact is the fundamental reason why both the atmosphere and the ocean are filled with perturbations with horizontal spatial scales of the order of the deformation radius.

For wavenumbers which lie in the unstable region the real part of the phase speed is simply the flow at the mid-level, $DU_{o_z}/2$. The growth rate is the imaginary part of *c* multiplied by the *x wavenumber*, *k,* or

$$kc_i = \frac{U_{o_z}k}{\kappa}\left\{\left[\frac{\kappa D}{2} - \tanh(\kappa D/2)\right]\left[\coth(\kappa D/2) - \frac{\kappa D}{2}\right]\right\}^{1/2} \tag{6.2.14}$$

Note that the phase speed is a function only of the total wavenumber $K = \left[k^2 + l^2\right]^{1/2}$ so that, by (6.2.14) the growth rate is maximized by making the ratio of the x-wavenumber to the total wavenumber as large as possible, i.e. by reducing the y wavenumber as much as possible for a given *K*. Figure 6.2.2a shows the dispersion relation for very broad channel (i.e. $L=100L_d$.)

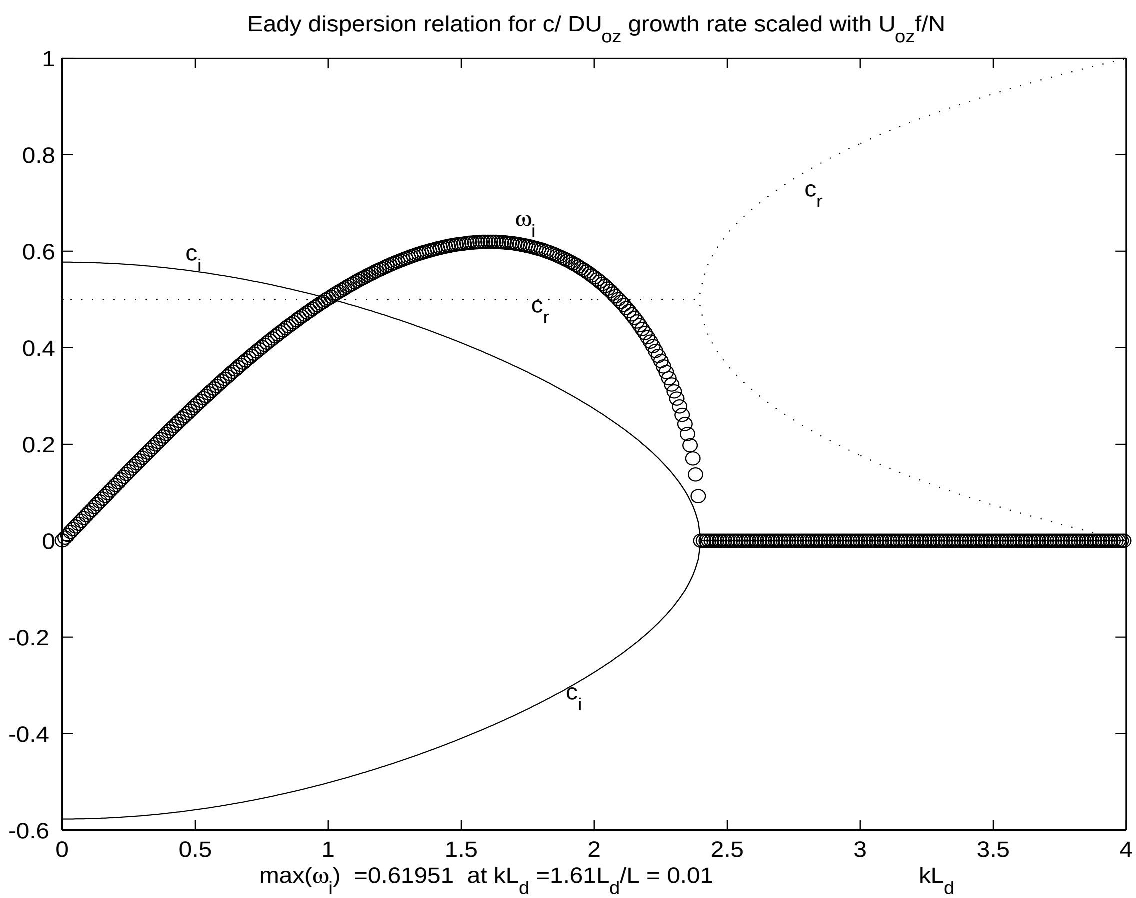

Figure 6.2.2 The growth rate and complex phase speed for the Eady model as a function of the x wavenumber (scaled with the deformation radius) for the case where $L=100L_d$.

For a narrower channel, for example $L=2L_d$ we the dispersion relation is shown below,

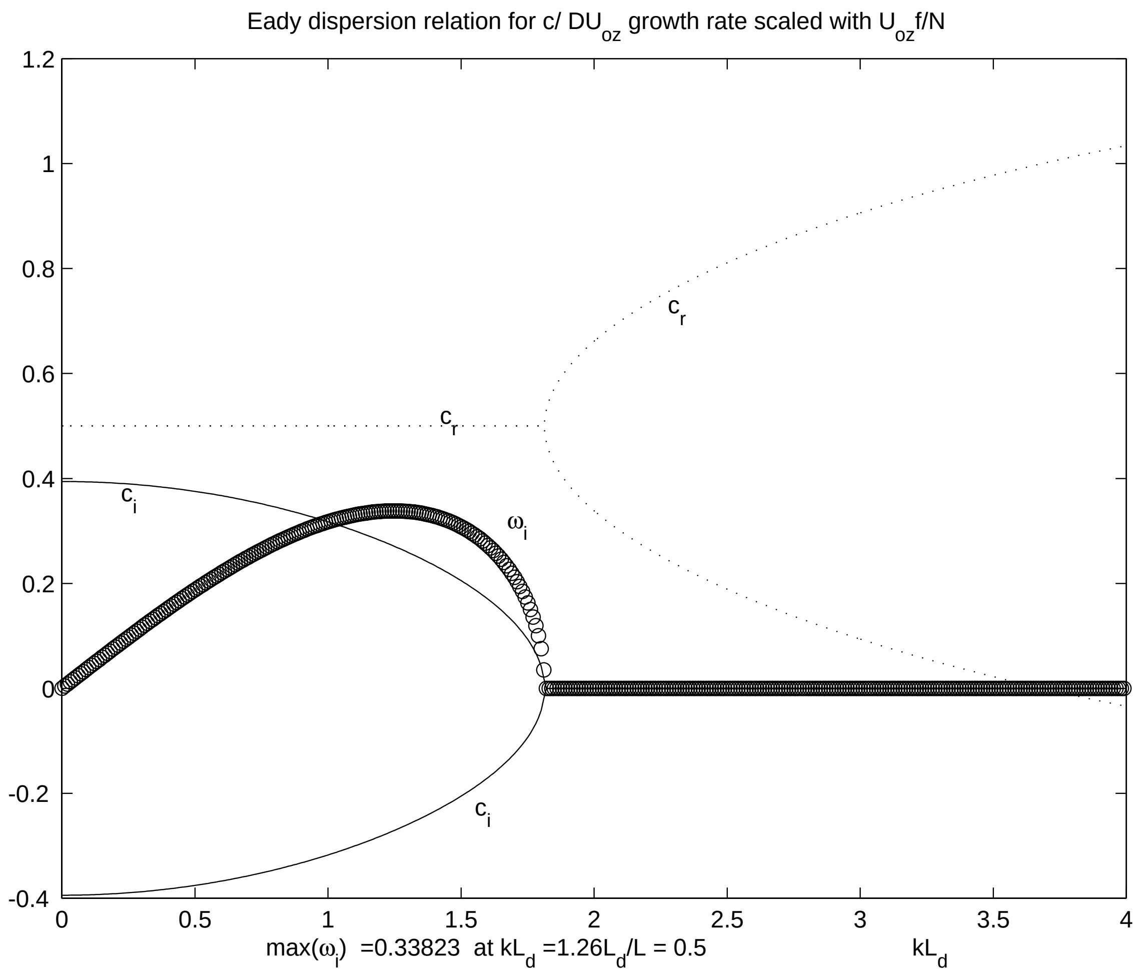

Figure 6.2.3 In comparison to the broad channel case this example shows the dispersion relation when $L=2L_d$. Note the reduced growth rate and the shift of the x wavenumber for the most unstable wave to longer wavelengths (smaller k).

It is also important to note that there are two solutions for c , one growing and one decaying (at the same rate). Since the dynamics considered here are inviscid they must also be time reversible. That is, if we have a solution that, for some initial condition A_{int}, diverges from an initial value we must also be able to find initial conditions so that asymptotically the solution can return to that value of A . Just as setting a ball in an unstable solution on a hill we can find the right initial conditions so that it will come to a stop just at the peak of the hill and remain there (unlikely but the physics must contain that possibility).

It is also important to note that for each k there are *only* two solutions with different structures in z. Clearly, these modes are not sufficient to represent arbitrary initial structures in z and this is related to the fact that we have examined only the nonsingular solutions to (6.2.10) i.e. we set B=0 in the general solution described in the footnote on that page. We will return to this point later in discussing the general initial value problem.

6.3 The wave structure:

Let us now examine the wave structure in the x,z plane. Suppressing momentarily the factor of $\sin(\pi y/L)$ we can write the perturbation stream function:

To present the result it is economical to introduce *nondimensional variables.* As we saw from the dispersion relation it is natural to scale the phase speed with the overall change in the basic velocity, the wavenumber with the deformation radius and the z variable with the total depth. This leads to the scaling transformations

$$
\begin{aligned}
&c = DU_{oz}c', \\
&z = Dz' \qquad\qquad\qquad\qquad (6.3.1) \\
&\kappa = \kappa'/D \ \Rightarrow \kappa' = \left[k^2 + l^2\right]^{1/2} ND/f_o
\end{aligned}
$$

In terms of these variables the perturbation function (aside from its y factor) is,

$$\phi = R\left[\cosh\kappa' z' - \frac{1}{\kappa' c'}\sinh\kappa' z'\right]e^{i\theta}, \quad \theta = kx - ct \tag{6.3.2}$$

This form of the solution is equivalent to (6.2.11) but is a bit easier to manipulate. We have also used the boundary condition to relate the relative magnitude of the terms in sinh and cosh. The overall amplitude *R* is, of course, arbitrary in the linear problem.

By explicitly writing *c'* in terms of its real and imaginary parts,

$$\phi = R\left[\cosh\kappa' z' - \frac{c_r'}{\kappa'|c'|^2}\sinh\kappa' z' + i\frac{c_i'}{\kappa'|c'|^2}\sinh\kappa' z'\right]e^{i\theta}$$

$$= R|A(z)|e^{i\alpha(z')}e^{ik(x-c_r t)}e^{kc_i t} \qquad \text{(6.3.3}$$

$$\tan\alpha(z) = \frac{c_i'}{\kappa'|c'|^2}\frac{\sinh\kappa' z'}{\left[\cosh\kappa' z' - \frac{c_r'\sinh\kappa' z'}{\kappa'|c'|^2}\right]} = \frac{c_i'}{\kappa'|c'|^2}\frac{1}{\left\{\coth(\kappa' z') - \frac{c_r'}{\kappa'|c'|^2}\right\}}$$

a,b)

The phase shift α is a function of height and is zero if the wave is neutral, i.e. if the imaginary part of *c* vanishes. We saw in our discussion of the energetics that the phase of the disturbance should "lean" against the shear of the basic current to release the potential energy in the basic flow. The total phase of the perturbation in the x,z plane is $phase = kx + \alpha z'$, therefore,

$$\left(\frac{\partial z'}{\partial x}\right)_{phase} = -\frac{\partial phase}{\partial x}\Big/\frac{\partial phase}{\partial z} = -k\Big/\frac{\partial\alpha}{\partial z'} \tag{6.3.4}$$

To have the phase of the wave sloping upward and westward for a positive shear requires then that $\partial\alpha/\partial z' > 0$ as shown in figure 6.3.1.

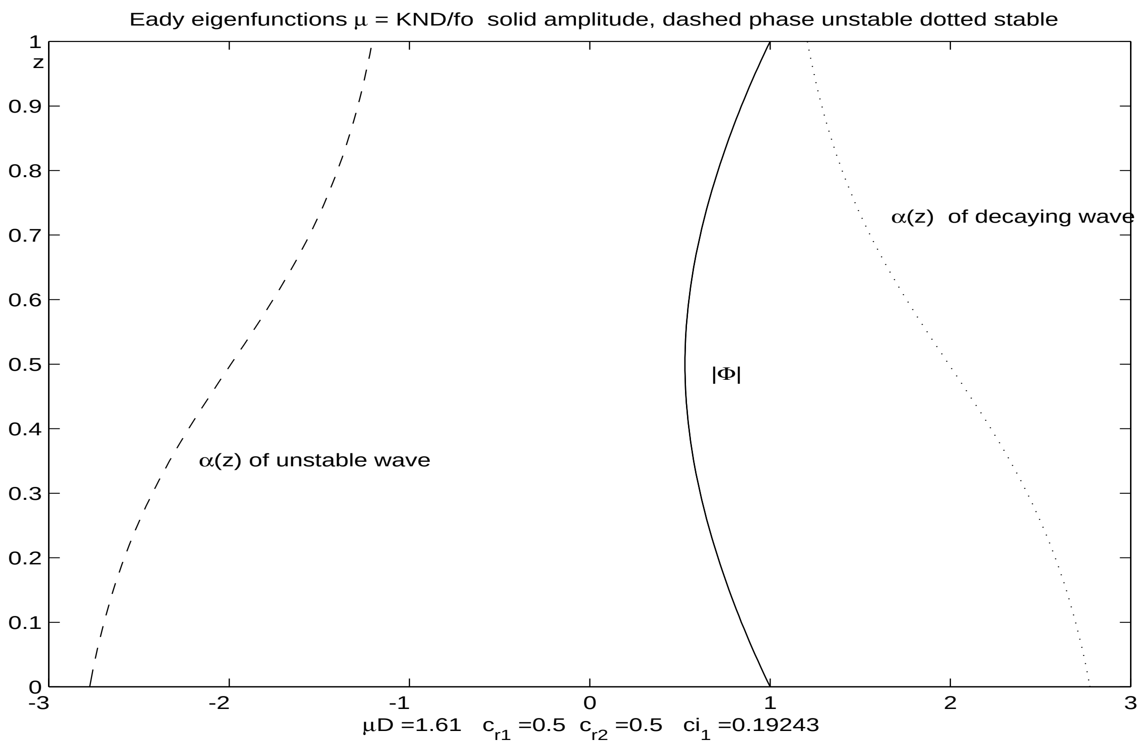

Figure 6.3.1 The structure of the most unstable wave for the case shown in figure 6.2.2.

The solid curve in figure 6.3.1 shows the absolute value of the amplitude as a function of z'. Corresponding to the most *unstable* wave the phase function α is shown on the left side of the figure. Note that α *increases* with height corresponding to a phase of the wave that tilts upward to the left for positive shear in the basic flow. The dotted curve on the right of the figure shows the phase function α for the stable wave possible at the same wavenumber. Note that the α decreases with height and so this wave leans forward *with* the shear of the basic state.

The buoyancy, i.e. either (minus) the density anomaly or the potential temperature, is given by,

$$b' = f_o \phi_z = \frac{f_o}{D}\phi_{z'}$$

$$= \frac{f_o}{D} R\left[\sinh \kappa' z' - \frac{c_r'}{\kappa' |c'|^2}\cosh \kappa' z' + i\frac{c_i'}{\kappa' |c'|^2}\cosh \kappa' z'\right] e^{i\theta}$$

(6.3.5

$$= \frac{f_o}{D}|B(z')| e^{i(\mu(z')+\theta)}$$

$$\tan \mu = \frac{c_i'}{\kappa' |c'|^2}\frac{1}{\tanh \kappa' z - c_r' / \kappa' |c'|^2}$$

a,b)

The figure below shows the amplitude and phase of the buoyancy perturbation.

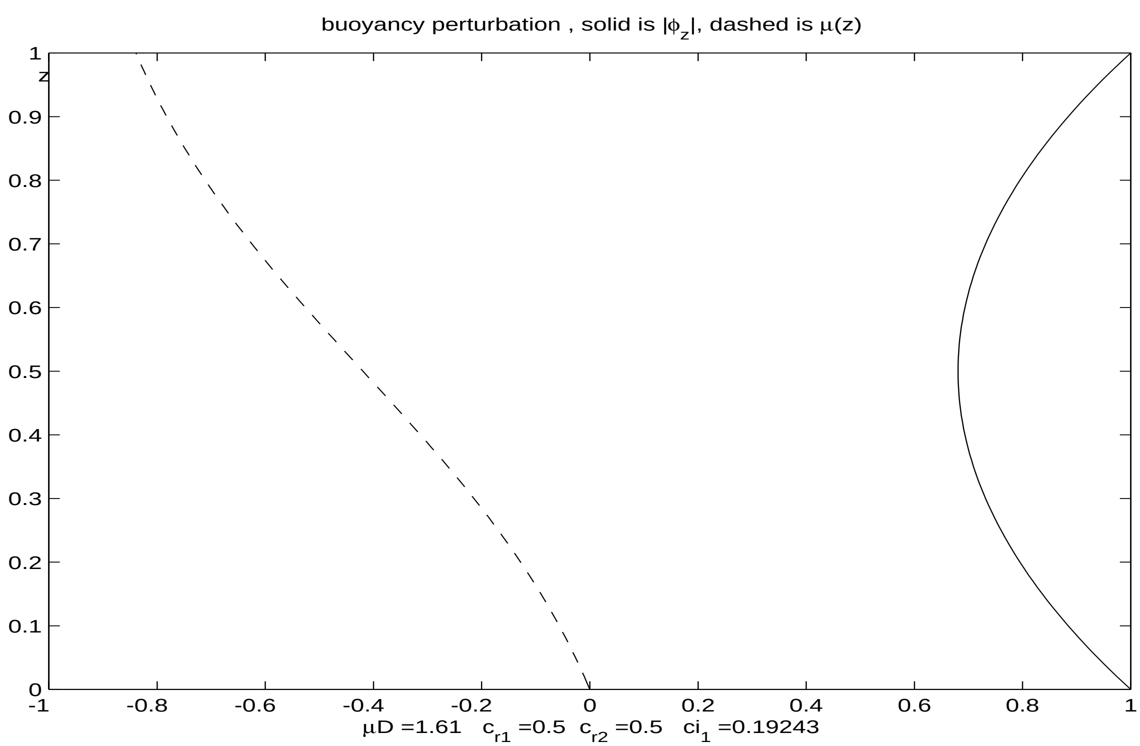

Figure 6.3.2 The amplitude and phase of the buoyancy perturbation for the unstable wave of figure 6.3.1.

Note that the function μ decreases with z' so that the phase of the *buoyancy* perturbation of the *unstable* wave leans with the shear instead of against it as does the pressure perturbation. This is very important to keep in mind if you were to examine observations and were trying to use the phase of a density record to deduce the nature of the perturbations. Figure 6.3.3 shows the isolines of *b* and ϕ in the x,z plane,

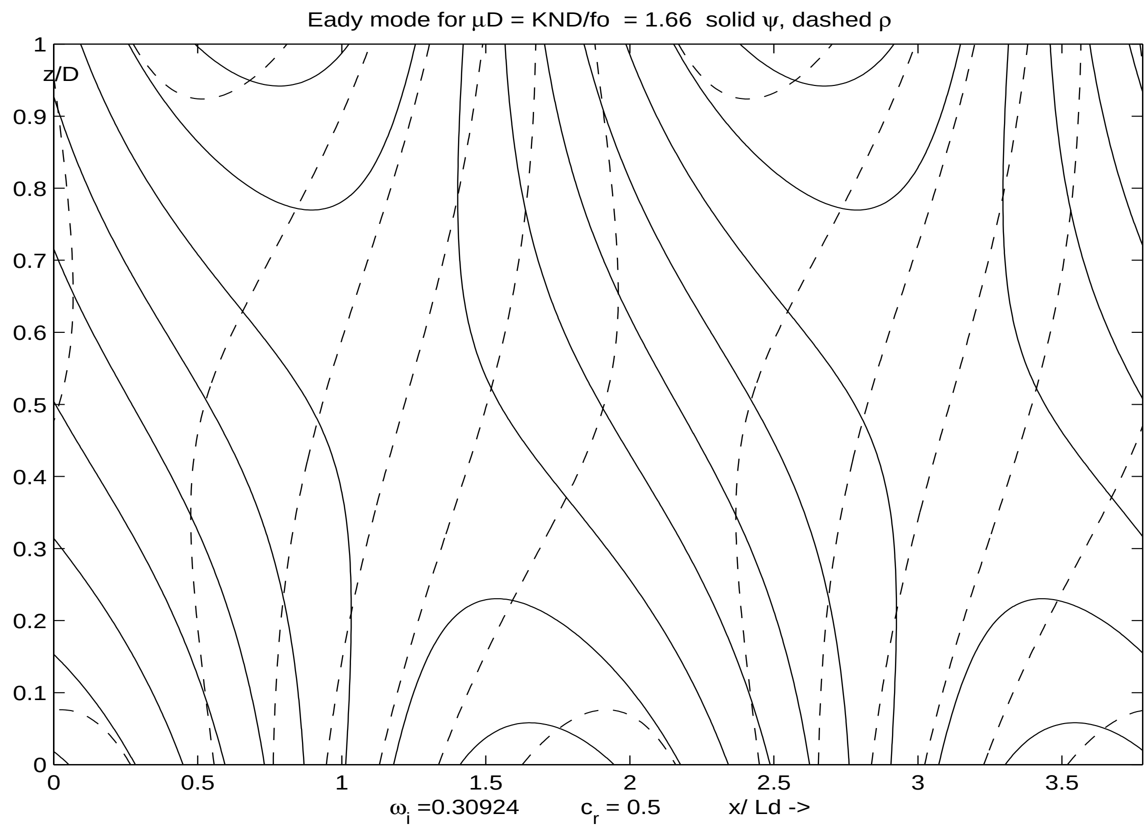

Figure 6.3.3. The solid lines are the lines of constant geostrophic streamfunction (the pressure) for the perturbation. The dashed lines show the isolines of *b,* the buoyancy.

6.4 The wave fluxes

The energy equation (4.2.7) shows that any instability must be connected to either fluxes of momentum or buoyancy by the perturbations. Moreover, these must be rectified fluxes in the sense that the wavy perturbations nonlinearly must give rise to fluxes which have an x-independent component in order to release energy from the basic state and send it to the perturbation field. Let's examine the momentum and heat fluxes in the unstable Eady wave.

For the Eady wave,

$$\phi = \operatorname{Re} A(z) e^{i\theta_r} \sin(\pi y/L) e^{kc_i t} = \frac{1}{2}\left(Ae^{i\theta_r} + A^* e^{-i\theta_r}\right) e^{kc_i t} \sin(ly)$$
$$l = \pi/L$$
(6.4.1a)

where

$$\theta_r = kx - c_r t \qquad (6.4.1b)$$

The x-averaged momentum flux,

$$\overline{v'u'} = -\overline{\phi_x \phi_y} = \frac{1}{4}\overline{\left[Ae^{i\theta} + A^* e^{-i\theta}\right] l \cos ly \left[Ae^{i\theta} - A^* e^{-i\theta}\right]} ik \sin ly e^{2kc_i t}$$
$$= \frac{1}{8}\left\{A^2 \overline{e^{2i\theta}} - A^{*2}\overline{e^{-2i\theta}}\right\} lk \sin 2ly \, e^{2kc_i t} = 0 \qquad (6.4.2)$$

The momentum flux in these perturbations has zero x-average since $\overline{e^{2i\theta}} \equiv 0$, and hence there is no release of energy by the Reynolds stresses. This should not be a surprise because, as we saw earlier, in order to release energy by such barotropic processes there must be a phase tilt of the perturbations in the x-y plane and that clearly does not happen for perturbations of the form of (6.4.1). On the other hand there will be a non zero buoyancy flux. There has to be. For the perturbations to grow there must be either a momentum or a buoyancy flux. We know we have a growing perturbation and so the energy, by process of elimination, if nothing else, must come from the buoyancy flux. The x-averaged buoyancy flux is,

$$\overline{b'v'} = f_o \overline{\phi_x \phi_z} = \frac{f_o}{4} \overline{\left[Ae^{i\theta_r} - A^* e^{-i\theta_r}\right]\left[A_z e^{i\theta_r} + A_z{}^* e^{-i\theta_r}\right]} e^{kc_i t}\, ik \sin^2(ly)$$
$$= \frac{f_o}{4}\left(AA_z{}^* - A^* A_z\right) ik \sin^2(ly) e^{2kc_i t} \tag{6.4.3}$$

Note that here there are terms in the product (6.4.3) which are independent of x. The calculation of the buoyancy flux implied by (6.4.3) is made considerably simpler if we recognize that it must be independent of z. This follows directly by a manipulation of (6.2.10a) but more directly from (4.2.9) and (4.2.16). The former shows that when the Reynolds stress is zero that the potential vorticity flux will be equal to the vertical derivative of the buoyancy flux. On the other hand (4.2.16) shows that when, as in the Eady model, that the basic state has zero potential vorticity gradient the buoyancy flux must be independent of z. We can therefore evaluate it at any convenient location in the z interval of our problem. If we return to the form of the solution given by (6.2.11), i.e.

$$A = a \cosh \kappa (z - D/2) + b \sinh \kappa (z - D/2) \tag{6.4.4}$$

It follows that at z=D/2,

$$AA^*{}_z - A^* A_z = \kappa\left(ab^* - a^* b\right) \tag{6.4.5}$$

Use of the boundary condition (6.2.12) allows us to write the above,

$$b/a = \frac{\kappa c \sinh(\kappa D/2) - U_{o_z} \cosh(\kappa D/2)}{\kappa c \cosh(\kappa D/2) - U_{o_z} \sinh(\kappa D/2)} \tag{6.4.6}$$

and after some algebra,

$$ab^* - a^* b = -2i\kappa c_i \frac{|d|^2}{\left|\kappa c \cosh \kappa D/2 - U_{o_z} \sinh(\kappa D/2)\right|^2}, \qquad (6.4.7)$$

so that,

$$\overline{v'b'} = \frac{f_o}{4} k c_i |a_o|^2 U_{o_z} \sin^2(ly) e^{2kc_i t}$$

$$(6.4.8)$$

$$|a_o|^2 = |d|^2 \frac{\kappa}{\left|\kappa c \cosh(\kappa D/2) - U_{o_z} \sinh(\kappa D/2)\right|^2}$$

The important point that follows from (6.4.8) is that if the shear is positive, (temperature decreasing northward) the heat flux in the wave will be northward, i.e.*down the temperature gradient of the basic state.* As we discussed in chapter 4 this is the condition for the release of potential energy by the perturbations. We shall see later that the effect this has on the mean temperature gradient (more generally the buoyancy gradient) is to smooth out the mean gradient. The potential energy lost is exactly what feeds, by (4.2.9) the energy of the perturbations.

In the next chapter we will discuss in a preliminary way the alteration of the mean flow by the perturbations that we can calculate from linear theory. Since the flux is increasing exponentially with time there will come a time when the linearization hypothesis must fail and we will have to reconsider the further evolution of both the perturbation and the mean flow to take that into account.