Chapter 4: Linear Stability Theory

1. Introduction

The governing equation for the perturbation geostrophic stream function (let's call it ϕ instead of ψ' to avoid the use of primes everywhere) is ,

$$\left(\frac{\partial}{\partial t}+U_o\frac{\partial}{\partial x}\right)\underbrace{\left(\nabla^2\phi+\frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\rho_s\frac{f_o^{\,2}}{N^2}\frac{\partial\phi}{\partial z}\right)\right)}_{q'}+\frac{\partial\phi}{\partial x}\frac{\partial q_o}{\partial y}+J(\phi,q')=\delta\Sigma_i \qquad (4.1.1)$$

The nonlinear term $J(\phi,q')=\vec{u}'\cdot\nabla q'$ represents the self-advection of perturbation potential vorticity by the perturbation velocity. If we could solve this nonlinear problem we would but it is just too hard although later we shall describe some approaches to the nonlinear problem that yield some useful insights. It would be interesting to test the stability of U_O to perturbations of arbitrary size and to be able to follow the evolution of disturbances as their amplitudes increase. To some degree this can be done numerically by a direct numerical integration of (4.1.1) but even so problems remain. The numerical methods are often limited by resolution and it is especially difficult to use them to find the thresholds for instability. It is also difficult to ask questions about the inviscid character of the instability in numerical models that possess some intrinsic numerical dissipation. For these and other technical reasons we are force to linearize the problem by neglecting the Jacobian term in (4.1.1). Since the term is O(amp)^2 this means considering disturbances which, at least initially are small. There are some virtues in this approach (Always a good idea to find virtue in necessity).

1) If small amplitude perturbations can be shown to grow spontaneously this provides a natural explanation for the existence of the observed finite amplitude fluctuations since small background fluctuations are always present to act as seeds for the instability.

2) Dealing with small perturbations reduces the question to the stability properties of the basic flow and not the amplitude level of the perturbations. If the perturbation is large, as large as the basic flow, it is a bit difficult to understand what we might mean by instability if we are perturbing the flow with another flow of the same energy level.
3) Small amplitude perturbations must get their energy directly from the basic flow while larger amplitude perturbations could get energized by interacting with the ultimate energy source via other perturbations by interacting with one another.

Let a be a measure of the amplitude of the perturbations with respect to the basic flow then to O(a^2) the stability equation (4.1.1) simplifies to,

$$\left(\frac{\partial}{\partial t}+U_o\frac{\partial}{\partial x}\right)\left(\nabla^2\phi+\frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{f_o^2}{N^2}\frac{\partial\phi}{\partial z}\right)\right)+\frac{\partial\phi}{\partial x}\frac{\partial q_o}{\partial y}=\delta\Sigma_i \tag{4.1.2}$$

The associated linear boundary condition on z =0 (the lower boundary) is from (3.1.9),

$$\left(\frac{\partial}{\partial t}+U_o\frac{\partial}{\partial x}\right)\phi_z-U_{oz}\phi_x+\phi_x h_x\frac{N^2}{f_o}=(-\delta_E\nabla^2\phi+\delta H)\frac{N^2}{f_o} \tag{4.1.3}$$

depending on the nature of the upper boundary at $z=z_T$ condition of the above type will enter as discussed in the previous chapter. The boundary condition on the lateral boundaries, say at *y1* and *y2* are simply that *v'* vanish or,

$$\phi_x=0,\ \ y=y1,y2 \tag{4.1.4}$$

If the disturbance can radiate energy to large meridional distances this boundary condition would have to be modified to consider the radiation properties of the waves.

4.2 Necessary conditions for instability: formulation

Consider the case first where we can ignore $\delta\Sigma_i$ and δH in the above equations. This is really a statement about relative time scales of inertial versus dissipative processes *for the perturbations.* Dissipation and friction may still be essential, as described in Chapter 2, in determining the structure of the basic state. We can now ask under what conditions can the perturbations grow in energy? What conditions does that place on the basic flow? Can we determine some structural requirement of the basic state that is necessary in order that it be unstable?

Let us first form an energy equation by multiplying (4.1.2) by the density ρ_S times the perturbation streamfunction ϕ and rearranging terms. For example,

$$\rho_s \phi \frac{\partial}{\partial t}\left[\phi_{xx} + \phi_{yy} + \frac{1}{\rho_s}\left(\rho_s \frac{f_o^2}{N^2} \phi_z \right)_z \right] =$$

$$= -\frac{\partial}{\partial t}\left[\frac{\rho_s}{2}\left\{ \phi_x{}^2 + \phi_y{}^2 + \frac{f_o^2}{N^2}\phi_z{}^2 \right\} \right] \tag{4.2.1}$$

$$+\left\{ \frac{\partial}{\partial x}(\rho_s \phi \phi_{xt}) + \frac{\partial}{\partial y}(\rho_s \phi \phi_{yt}) + \frac{\partial}{\partial z}\left(\rho_s \frac{f_o^2}{N^2} \phi \phi_{zt} \right) \right\}$$

We recognize the first term on the right hand side as (minus) the rate of change of the total energy (kinetic plus potential) in the perturbation field,

$$E(\phi) = \frac{\rho_s}{2}\left[\phi_x{}^2 + \phi_y{}^2 + \frac{f_o^2}{N^2}\phi_z{}^2 \right] \tag{4.2.2}$$

Carrying out the same process with the remaining terms from (4.1.2) yields an equation for the rate of change of perturbation energy in linear theory. After a considerable amount of algebra (which you should do on your own) we obtain,

$$\left\{ \frac{\partial}{\partial t} + U_o \frac{\partial}{\partial x} \right\} E(\phi) + \nabla \cdot \vec{J} = \rho_s \phi_x \phi_y U_{oy} + \rho_s \phi_x \phi_z \frac{f_o^2}{N^2} U_{oz} \tag{4.2.3}$$

The *flux vector* $\vec{J}$ is a three dimensional vector and the divergence operator in (4.2.3) is the usual three dimensional divergence operator, while:

$$
\vec{J} = \hat{i}\left\{ -\rho_s\phi[\phi_{xt} + U_o\phi_{xx}] + \rho_s\phi\phi_y U_{o_y} + \rho_s\phi\phi_z \frac{f_o{}^2}{N^2} U_{o_z} - \rho_s\phi^2 q_{o_y}/2 \right\}
$$

$$
+\hat{j}\left\{-\rho_s\phi\left[\phi_{yt} + U_o\phi_{xy}\right]\right\} + \hat{k}\left\{ -\rho_s\phi\frac{f_o{}^2}{N^2}[\phi_{zt} + U_o\phi_{zx}] \right\} \qquad (4.2.4)
$$

It would probably be helpful to the student at this point to go back to the discussion of energy propagation in Rossby waves and compare the energy flux vector in that problem, in which U_O is a constant and q_{Oy} is simply β to see the connection and generalization to (4.2.4).

Let's integrate (4.2.3) over the domain of the fluid. At the boundaries at *y1* and *y2* the geostrophic meridional velocity ϕ_x must vanish. Using the linearized form of the zonal x-averaged zonal flow equation we can show that,

$$
\overline{\phi\left[\frac{\partial}{\partial t} + U_o\frac{\partial}{\partial x}\right]\phi_y} = \overline{f_o\phi v_a} \qquad (4.2.5)
$$

and so this must vanish at the meridional boundaries where the ageostrophic meridional velocity must also vanish. Similarly, using the boundary condition at $z = z_T$ it is easy to show that,

$$
\overline{\phi\left[\frac{\partial}{\partial t} + U_o\frac{\partial}{\partial x}\right]\phi_z} = 0, \; z = 0, z_T \qquad (4.2.6)
$$

Thus the integral of the divergence of $\vec{J}$ is zero and the energy equation for the total perturbation energy is simply,

$$\frac{\partial}{\partial t}\langle E(\phi)\rangle = \iint \overline{\rho_s \phi_x \phi_y} U_{o_y}\, dydz + \iint \rho_s \overline{\phi_x \phi_z} \frac{f_o{}^2}{N^2} U_{o_z}\, dydz$$

$$= -\iint \overline{\rho_s u' v'} U_{o_y}\, dydz + \iint \rho_s \overline{v' b'} \frac{f_o}{N^2} U_{o_z}\, dydz \tag{4.2.7}$$

To O(a^2) the zonal mean flow , $\overline{u}$ treated in chapter 3 is identical to U_O since the change of the mean from the basic flow depends on the quadratic fluxes of buoyancy and momentum. Therefore, to the order of the linear problem we are discussing they are identical. Comparing (3.2.18) and (4.2.7) we note that the same energy transformation terms occur on the right hand sides of each equation *with opposite sign*. Therefore,

$$\frac{\partial}{\partial t}\left[E(\phi) + E(\overline{\psi})\right] = 0 \tag{4.2.8}$$

The total energy of the mean plus the perturbations is conserved. The terms on the right hand sides of (3.2.18) and (4.2.7) are true transformation terms. Energy drained from the mean flow shows up as perturbation energy and vice-versa. As discussed before, in order for the perturbations to grow the perturbations must either flux momentum from region of high zonal momentum to regions of lower zonal momentum (and thus smooth out the velocity profile in latitude) and/or the perturbations must flux buoyancy from regions of high buoyancy (e.g. temperature) to regions of low zonal mean buoyancy. The eddies in the atmosphere are observed to do the latter. That is they typically flux heat northward in the overall equator to pole decrease of temperature. However, it is observed that on average they also flux zonal momentum *up the gradient of zonal momentum* returning some of the energy to the mean flow (and sharpening the profile of the jet stream). Of course the sum of the two terms has the sign such that total energy flows from the zonal mean to the perturbations.

It is particularly revealing to rewrite the energy transformations in terms of the potential vorticity flux of the perturbations.

Since,

$$\rho_s \overline{v'q'} = -\rho_s(\overline{u'v'})_y + \left(\rho_s \overline{v'b'}\frac{f_o}{N^2}\right)_z$$
$$= \rho_s(\overline{\phi_x \phi_y})_y + \left(\rho_s \overline{\phi_x \phi_z}\frac{f_o{}^2}{N^2}\right)_z \tag{4.2.9}$$

an integration by parts of (4.2.7) yields,

$$\frac{\partial}{\partial t}\langle E(\phi)\rangle = \iint \overline{\rho_s \phi_x \phi_y} U_{o_y}\, dydz + \iint \rho_s \overline{\phi_x \phi_z} \tfrac{f_o{}^2}{N^2} U_{o_z}\, dydz$$

$$= -\iint \overline{\rho_s(\phi_x\, \phi_y)_y} U_o dydz - \iint (\overline{\rho_s \phi_x \phi_z} \tfrac{f_o{}^2}{N^2})_z U_o dydz \tag{4.2.10}$$

$$+ \int dy \rho_s \overline{\phi_x \phi_z} \tfrac{f_o{}^2}{N^2} U_o \Big]_{z=z_T} - \int dy \rho_s \overline{\phi_x \phi_z} \tfrac{f_o{}^2}{N^2} U_o \Big]_{z=0}$$

or

$$\frac{\partial}{\partial t}\langle E(\phi)\rangle = -\iint dydz\, U_o \overline{v'q'} + \int dy\, \rho_s \overline{v'b'}\frac{f_o}{N^2} U_o \Bigg]_{z=z_T} - \int dy\, \rho_s \overline{v'b'}\frac{f_o}{N^2} U_o \Bigg]_{z=0} \tag{4.2.11}$$

The energy transformations leading to energy growth for the perturbations are clearly intimately tied to the potential vorticity transport and its correlation with the basic zonal velocity and with the buoyancy flux on the horizontal boundaries and its correlation with the basic zonal flow on the boundaries. Note again, the equivalence between buoyancy fluxes on the boundaries and pv fluxes in the interior. We already now from chapter 3 that on average the pv flux must be down the gradient of the basic potential vorticity gradient, i.e. that $\left\langle \overline{v'q'} q_{o_y} \right\rangle < 0$ for instability

In the absence of dissipation and friction there is a simple connection between the particle displacements and the pv flux. Let's define the Lagrangian particle displacement in the y direction as η, then in linear theory

$$v' = \left(\frac{\partial}{\partial t} + U_o \frac{\partial}{\partial x}\right)\eta \tag{4.2.12}$$

At the same time the linearized potential vorticity equation in the absence of dissipation is,

$$\left(\frac{\partial}{\partial t} + U_o \frac{\partial}{\partial x}\right)q' + v' q_{o_y} = 0 \tag{4.2.13}$$

which with (4.2.12) implies that

$$\left(\frac{\partial}{\partial t} + U_o \frac{\partial}{\partial x}\right)\left[q' + \eta q_{o_y}\right] = 0 \tag{4.2.14}$$

If q' =0 when η =0 it ,i.e. if the only perturbation in potential vorticity comes about by the particles carrying their old potential vorticity to a new location, then

$$q' = -\eta \frac{\partial q_o}{\partial y} \tag{4.2.15}$$

or using (4.2.12) with (4.2.15) to evaluate the pv flux,

$$\overline{v'q'} = -\frac{\partial}{\partial t}\frac{\overline{\eta^2}}{2}\frac{\partial q_o}{\partial y} \tag{4.2.16}$$

$$b' = f_o \phi_z,$$

Similarly at z = 0, where

$$\left(\frac{\partial}{\partial t} + U_o \frac{\partial}{\partial x}\right) b' - f_o U_{o_z} v' + v' h_y N^2 = 0$$

(4.2.17)

The use again of 4.2.12) implies that

$$b' = \eta \left[f_o U_{o_z} - N^2 h_y \right] \tag{4.2.18}$$

Bringing these results together allows us to rewrite (4.2.11) in the suggestive form,

$$\frac{\partial}{\partial t}\langle E(\phi)\rangle = \iint dydz\, \rho_s U_o q_{o_y} \overline{(\eta^2)_t}$$

$$+ \int dy\, \rho_s \overline{(\eta^2)_t} \left[\frac{f_o^2}{N^2} U_{o_z} - f_o h_y \right] U_o \Bigg]_{z=0}^{z=z_T}$$

(4.2.19)

Here we have introduced the useful notation,

$$G(z)\big]_{z=0}^{z=z_t} = G(z_t) - G(0) \tag{4.2.20}$$

Note that (4.2.19) can be written as a conservation statement,

$$\frac{\partial}{\partial t}\left[\langle E(\phi)\rangle - \iint dydz\, \rho_s U_o q_{o_y} \overline{(\eta^2)} - \int dy\, \rho_s \overline{(\eta^2)} \left[\frac{f_o^2}{N^2} U_{o_z} - f_o h_y \right] U_o \Bigg]_{z=0}^{z=z_T} \right] = 0$$

(4.2.21)

Of course, at the upper boundary we have to interpret properly the value of h (it will usually be zero there, and, if $z_T \rightarrow$ infinity the term itself will be zero if the perturbations decay with z.

It is useful to represent the potential vorticity flux itself in this form, Integrating (4.2.9) over the meridional plane and using the fact that *v'* vanishes at the lateral boundaries,

$$\iint \overline{\rho_s v'q'dydz} = \int dy \frac{f_o}{N^2}\rho_s \overline{v'b'}\Big\}_{z=0}^{z=z_t} \tag{4.2.22}$$

Using (4.2.16) and 4.2.18) we obtain,

$$\frac{\partial}{\partial t}\left[\iint dydz\rho_s \frac{q_{o_y}}{2}\overline{\eta^2} + \int dy\overline{\eta^2}\left(\frac{f_o{}^2}{N^2}U_{o_z} - f_o h_y\right)/2\right\}_{z=0}^{z=z_t}\Bigg] = 0 \tag{4.2.23}$$

This is clearly a second conservation statement. While the first statement is clearly connected to the conservation of total energy, (4.2.23) can be shown (an exercise for the student) to be a conservation statement for the *total x averaged momentum integrated over the meridional cross section.*

We note in both (4.2.21) and (4.2.23) the occurrence of the term

$$f_o\,U_{o_z} - N^2 h_y$$

which can be rewritten, using the thermal wind relation as (for the atmospheric case)

$$\begin{aligned} f_o U_{oz} - N^2 h_y &= -g\frac{\vartheta_{o_y}}{\vartheta_s} - g\frac{\vartheta_{s_z}}{\vartheta_s}h_y \\ &= N^2\left[\frac{-\vartheta_{o_y}}{\vartheta_{s_z}} - h_y\right] \\ &= N^2\left[\left(\frac{\partial z}{\partial y}\right)_{\vartheta} - h_y\right] \end{aligned} \tag{4.2.24}$$

The term is therefore proportional to the difference between the slope of the isolines of potential temperature at the boundary and the slope of the boundary itself. For the oceanic case a simple rederivation will exchange the isolines of density with those of potential temperature.

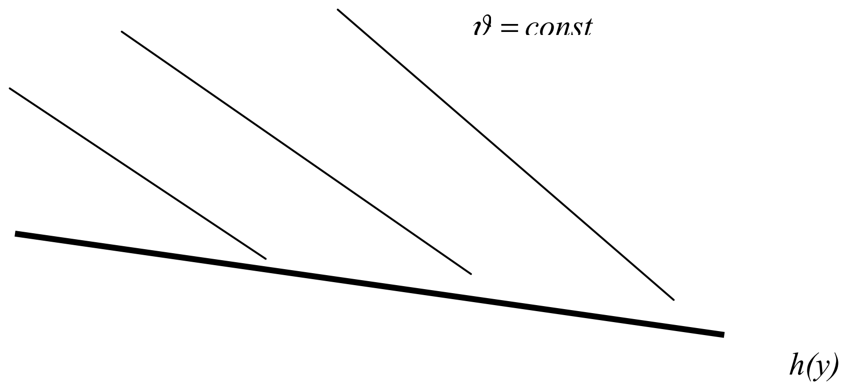

Figure 4.2.1 The difference between the boundary slope and the slope of the buoyancy surfaces determines the boundary factor in (4.2.21) and (4.2.23).

The boundary term will vanish only if the boundary is a surface of constant buoyancy. It will change sign if one slope exceeds the other.

4.3 Necessary conditions for instability. Consequences

Let us examine the consequences of (4.2.23) first. We shall use the fact that if a disturbance is growing the mean dispersion of fluid elements from their initial latitudes mus increase with time, i.e. that for instability,

$$\frac{\partial}{\partial t}\overline{\eta^2} > 0 \qquad (4.3.1)$$

A) Consider the case when the boundaries are coincident with isentropic or isopycnal surfaces so that the boundary terms in (4.2.23) are identically zero. Then (4.2.23) reduce to

$$\frac{\partial}{\partial t}\left[\iint dydz\rho_s \frac{q_{o_y}}{2}\overline{\eta^2}\right] = 0 \tag{4.3.2}$$

or

$$\iint dydz\rho_s \frac{q_{o_y}}{2}\overline{\eta^2} = \iint dydz\rho_s \frac{q_{o_y}}{2}\overline{\eta^2}\Bigg)_{t=0} \tag{4.3.3}$$

However, if q_{o_y} were of a single sign over the whole meridional plane, i.e. if the potential vorticity gradient of the mean flow does not change sign then, (suppose for example it is positive over the whole cross section),

$$q_{o_{y_{\min}}}\left\langle\rho_s\overline{\eta^2}\right\rangle < \iint dydz\rho_s q_{o_y}\left(\overline{\eta^2}\right)_{t=0} \tag{4.3.4}$$

This clearly places an upper bound on the dispersion of fluid particles which goes to zero as the initial disturbance amplitude goes to zero. In this sense the flow must be stable. Thus, when the boundaries at z =0, z_T are isothermal (isopycnal for the ocean) a *necessary* condition for instability is that the meridional gradient of potential vorticity must change sign over the domain.

Recall that

$$q_{o_y} = \beta - U_{o_{yy}} - \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\rho_s \frac{f_o^{\,2}}{N^2} U_{oz}\right) \tag{4.3.5}$$

In this case (boundaries of constant potential temperature or density in the basic state) a large enough β (i.e. a weak enough basic velocity) will always stabilize the flow. Note that it is the curvature of the basic velocity gradient which enters the determination of q_{oy} (modified by factors involving the stratification in the vertical) rather than an overall shear level itself. There can be a large store of available energy, either in the horizontal shear or vertical shear but it will not be released to a growing perturbations if the potential vorticity distribution does not allow it). This emphasizes the key role potential vorticity dynamics plays in the generation of meso-scale eddy activity in both the atmosphere and the oceans. Note also that the condition (4.3.2) is a *necessary* condition for instability. It is not a *sufficient* condition for instability. That is, if the potential vorticity gradient is of a single sign we have a *sufficient condition for stability* but if it changes sign we are not guaranteed that the flow will be unstable. Indeed, we will later be able to show some counter examples.

Let us use the condition and (4.3.5) to give an estimate of the vertical shear in the mid ocean that would be required to satisfy the condition for instability. The horizontal velocity curvature is on the scale of the oceanic gyre and so is negligible (this is not true for swift currents as the Gulf Stream). If we make the estimate:

$$\frac{\partial}{\partial z}\left[\frac{f_o^2}{N^2}U_{o_z}\right] \approx \frac{f_o^2}{N^2}\Delta U_o / D^2$$

where we have ignored the variation of the background density and taken the buoyancy frequency to be varying less rapidly than the vertical shear (or no larger), then for instability to occur the vertical shear,

$$\Delta U_o \geq \beta\frac{N^2 D^2}{f_o^2} = \beta L_D^2 \tag{4.3.6}$$

Here, D is the vertical scale of the disturbance, (one might imagine it is of the order of the thermocline depth, say about 1 km). The condition in (4.3.6) has been rewritten in terms of the deformation radius based on that depth, $L_D = ND/f_o$. Using mid-latitude values , the deformation radius is about 50 km. Since β is about 2 x 10^{-13} $cm^{-1}sec^{-1}$ this gives a critical value for ΔU_o of 5 cm/sec.

As a historical note. If we ignore the β effect and if the fluid is homogeneous the potential vorticity gradient reduces to the horizontal curvature of the velocity profile yielding the famous Rayleigh theorem, i.e. that instability of classical, homogeneous shear flows requires an *inflection point* in the velocity profile. That condition always seemed physically mysterious but we can see that what it really requires is that the pv gradient not be of a single sign. Otherwise, the pv gradient is qualitatively like the β effect which, as we know, only gives rise to stable oscillations. We shall come back to this fact later in our interpretation of the conditions for instability.

B) Now suppose that the pv gradient is of a single sign throughout the fluid but that the one horizontal boundary is not an isentrope (atmosphere) or an isopycnal (ocean) and the other surface is either removed to infinity or is a surface of constant buoyancy. Then reconsidering the condition (4.2.23) it is clear that somewhere in the fluid one of the following must be satisfied for instability.

$$
\begin{aligned}
&q_{o_y}\left[f_o U_{oz} - N^2 h_y\right]_{z=z_t} < 0, \\
&q_{o_y}\left[f_o U_{oz} - N^2 h_y\right]_{z=0} > 0
\end{aligned}
\qquad (4.3.7\ \text{a,b})
$$

The classical problem of Charney (1947) is one in which the upper boundary is removed to infinity so that (4.3.7b) applies. Further, in Charney's example the potential vorticity gradient is everywhere positive (and constant). Instability is therefore allowed only if the vertical shear at the ground exceeds the slope term or equivalently, only if the slope of the isotherms at the lower boundary exceeds the slope of the bottom boundary. In Charney's problem, the first full discussion of baroclinic instability, the stabilizing effect of β is balanced by the northward temperature gradient at the lower flat surface. Charney succeeded in finding unstable modes (we shall review his calculation later) but we can see from (4.3.7) that a bottom sloping upward to the north could stabilize the entire flow if it is large enough.

C) An extreme case showing the importance of the boundary conditions and that emphasizes the dynamical character of the boundary is one in which the interior potential vorticity gradient is identically zero. This is the problem studied by Eady (1949). In this case a reference to (4.2.23) shows that instability requires that

$$\left[f_o U_{oz} - N^2 h_y\right]_{z=0} \bullet \left[f_o U_{oz} - N^2 h_y\right]_{z=z_t} > 0 \tag{4.3.8}$$

Therefore a domain with a non sloping upper and lower boundary will always satisfy the necessary condition for instability. Note that topography can reverse the sign of one of the brackets and render the flow stable regardless of the amount of vertical shear (horizontal buoyancy gradient) present in the mean flow. The release of that energy must be consistent with pv dynamics and the "wrong" distribution of basic state potential vorticity, or its equivalent, buoyancy gradients on the horizontal boundaries can render the energy in the basic state dynamically unavailable.

Return for a moment to the integral over the y-z domain of the x-averaged potential vorticity flux. From (4.2.22) and (4.2.16)

$$\iint \overline{\rho_s v'q' dydz} = \int dy \frac{f_o}{N^2} \rho_s \overline{v'b'} \Big\}_{z=0}^{z=z_t} = -\iint \rho_s \overline{\eta^2}_t q_{o_y} \, dydz \tag{4.3.9}$$

If the potential vorticity gradient of the basic state is positive, i.e. in the beta sense, then for a growing instability there must be a) a potential vorticity flux down the pv gradient, and b) either a compensating northward buoyancy flux at the ground or c) a compensating southward buoyancy flux at the upper surface.

If we consider the equation for the x-averaged zonal flow, (3.2.10) and the relation (4.2.9) between the Reynolds stress, the pv flux and the eddy buoyancy flux,

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} &= -\frac{\partial}{\partial y} \overline{u'v'} + f_o \bar{v}_a \\ &= \overline{v'q'} - \frac{1}{\rho_s} \frac{\partial}{\partial z} \rho_s \frac{f_o}{N^2} \overline{v'b'} + f_o \bar{v}_a \end{aligned} \tag{4.3.10}$$

Now let's integrate this equation over the full depth of the fluid at an arbitrary latitude, y,

$$\frac{\partial}{\partial t}\int_0^{z_t}\rho_s\bar{u}\,dz=\int_0^{z_t}\rho_s\overline{v'q'}dz-\left[\rho_s\frac{f_o}{N^2}\overline{v'b'}\right]_0^{z_t}+f_o\int_0^{z_t}\rho_s\bar{v}_a dz \tag{4.3.11}$$

The last term in (4.3.11) is zero since in quasi-geostrophic theory as much mass moves northward as southward across each vertical surface oriented along a latitude circle, i.e. there is not net storage of mass at any latitude. (The student should review the derivation of quasi-geostrophy to check the validity of this statement and understand its limits of validity).

Using the results of (4.2.15) and (4.2.18) we obtain, at each y, and equation for the vertically integrated zonal mean momentum,

$$\frac{\partial}{\partial t}\int_0^{z_t}\rho_s\bar{u}dz=-\frac{1}{2}\int_0^{z_t}\rho_s\overline{\eta^2}_t\,q_{o_y}-\frac{1}{2}\left[\overline{\eta^2}_t\rho_s\left(\frac{f_o^2}{N^2}U_{o_z}-f_oh_y\right)\right]_0^{z_t} \tag{4.3.12}$$

Clearly, since there are no external forces, the total zonal momentum, if we were to integrate must be conserved. Thus, the integral of (4.3.12) would yield zero for the left hand side and the condition that the y integral of the right hand side also vanishes recovers the necessary condition for instability (4.2.23) which we see is really a momentum conservation statement.

However, it is useful to return to a direct consideration of (4.3.12). Let's suppose that the meridional pv gradient q_{o_y} is always positive. Further let's imagine, for the atmospheric case that the upper boundary moves to infinity and so the upper term at z_t vanishes, then for instability to occur we know that the lower boundary term must have the opposite sign to the term involving q_{o_y}. Consider a positive zonal current over a flat bottom in which the vertical shear at the ground is very strong at the jet axis where the meridional temperature gradient is largest and weak on the flanks of the jet. Such instabilities due to the meridional buoyancy gradients in the flow are called baroclinic instabilities. Then in such cases the right hand side of (4.3.12) will be positive near the jet axis and negative on its flanks. What that predicts is that the effect of the perturbations will be to *accelerate* the vertical average of the jet at its core where it is already large and *decelerate* the jet at its flanks where it is weak! Such counter intuitive behavior is, in fact, observed in the atmosphere and is a basic consequence of baroclinic instability. Of course nothing is for free and the mean field suffers a loss of energy to the growing perturbations and this is manifested by a decay of the vertical shear of the mean current and, by thermal wind, and accompanying loss of potential energy of the mean as the supporting meridional temperature gradient declines in balance with the reduced vertical shear of the mean current. This important feature of the atmospheric general circulation is not mirrored in the oceanic case because of the change in the position of the active boundary and the student is encouraged to think out the difference in the resulting behavior and to ask what would happen to westward mean flows.

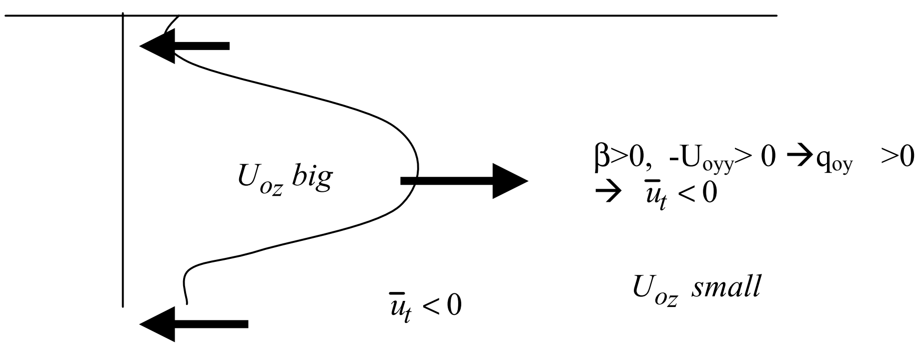

Figure 4.3.1 The role of unstable perturbations for a zonal flow whose pv gradient is positive but with a large meridional buoyancy gradient at the jet center and weak gradients at the jet edges. The perturbations accelerate the vertically averaged zonal flow in the core and decelerate it at the edges.

Let us now examine the second condition for instability, (4.2.21), the one expressing the energy balance for the perturbations. If we suppose that the first condition, (4.2.23) is satisfied (otherwise there is no reason to examine an additional condition for instability for the flow would be stable) we can subtract an arbitrary multiple of (4.2.23) from (4.2.23). That multiplicative factor we will call c it is an arbitrary constant and has the dimensions of a speed although at this point we do not identify it with any particular speed.

Then (4.2.21) can be written,

$$\frac{\partial}{\partial t}\left[\langle E(\phi)\rangle - \iint dydz\,\rho_s(U_o-c)q_{o_y}\overline{(\eta^2)} - \int dy\,\rho_s\overline{(\eta^2)}\left[\frac{f_o^2}{N^2}U_{o_z}-f_oh_y\right](U_o-c)\Bigg]_{z=0}^{z=z_T}\right]=0$$

(4.3.13)

The terms multiplied by c sum to zero by (4.2.21).

Clearly for instability $\frac{\partial}{\partial t}\langle E(\phi)\rangle > 0$. Hence

$$\frac{\partial}{\partial t}\left[\iint dydz\,\rho_s(U_o-c)q_{o_y}\overline{(\eta^2)} - \int dy\,\rho_s\overline{(\eta^2)}\left[\frac{f_o^2}{N^2}U_{o_z}-f_oh_y\right](U_o-c)\Bigg]_{z=0}^{z=z_T}\right]>0 \qquad (4.3.14)$$

Let's examine a few examples to see how (4.3.14) might be used. Suppose the upper boundary is moved to infinity and does not enter the condition and let's suppose the lower boundary is a surface of constant mean buoyancy so that it does not enter the condition either, Then a necessary condition for instability is simply,

$$\left[\iint dydz\,\rho_s(U_o-c)q_{o_y}\overline{(\eta^2{}_t)}\right]>0 \qquad (4.3.15)$$

If we can find *any* value of c that makes the integrand always negative the flow must be stable even if q_{o_y} changes sign in the domain of the flow. A simple but not quite trivial example occurs when the scale is small enough to ignore the β effect and U_o is a function of y alone. Suppose that

$$U_{oyy}(y_c)=0$$

so that q_{o_y} vanishes at that point. The condition for instability (4.3.15) becomes

$$(U_o - U_o(y_c))\left[-U_{o_{yy}}\right] > 0 \qquad (4.3.16)$$

Consider the following velocity field $U_o = U_{oo}(1 + e^{a(y-L)} - e^{-ay})$. The pv gradient is just (minus) the second derivative of the velocity and always vanishes at y =L/2 where the basic velocity is always U_{oo}. The flow profile, the velocity curvature and the condition (4.3.16) is shown below.

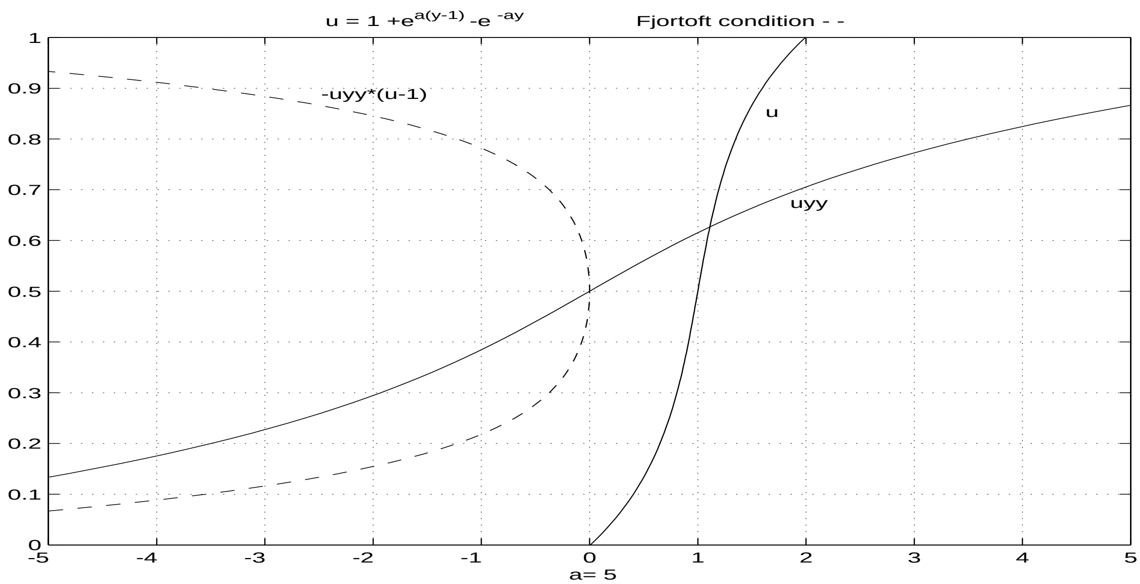

Figure 4.3.2 The velocity profile, its curvature and the condition (4.3.16) are plotted as a function of y/L.

Note that the potential vorticity gradient vanishes as y =L/2 so that the profile satisfies the first necessary condition for instability. Examining the figure though it is clear that the condition (4.3.16) is never satisfied. The dashed line shows the left hand side of the condition and it is always negative. Thus although the potential vorticity gradient changes sign and although there is a substantial amount of shear in the flow, it must be stable according to the energy condition we have derived. This condition is often called the Fjörtoft condition after the Norwegian meteorologist who derived it in this simple context (in quite another way).

Suppose instead that the velocity field, again a function only of y, is

$$U_o = U_{oo}(1-\cos^2 \pi y/2L)$$

$$U_{o_{yy}} = U_{oo}(\pi^2/2L^2)\cos(\pi y/L)$$

Here the pv gradient $-U_{o_{yy}}$ vanishes at y=L/2 where the velocity is $U_{oo}/2$. The figure below shows the profiles of the velocity, the curvature of the profile and the Fjörtoft condition. In this case both the Rayleigh condition (the first condition) and the second condition are satisfied. Indeed, the flow is unstable as shown by direct calculation.

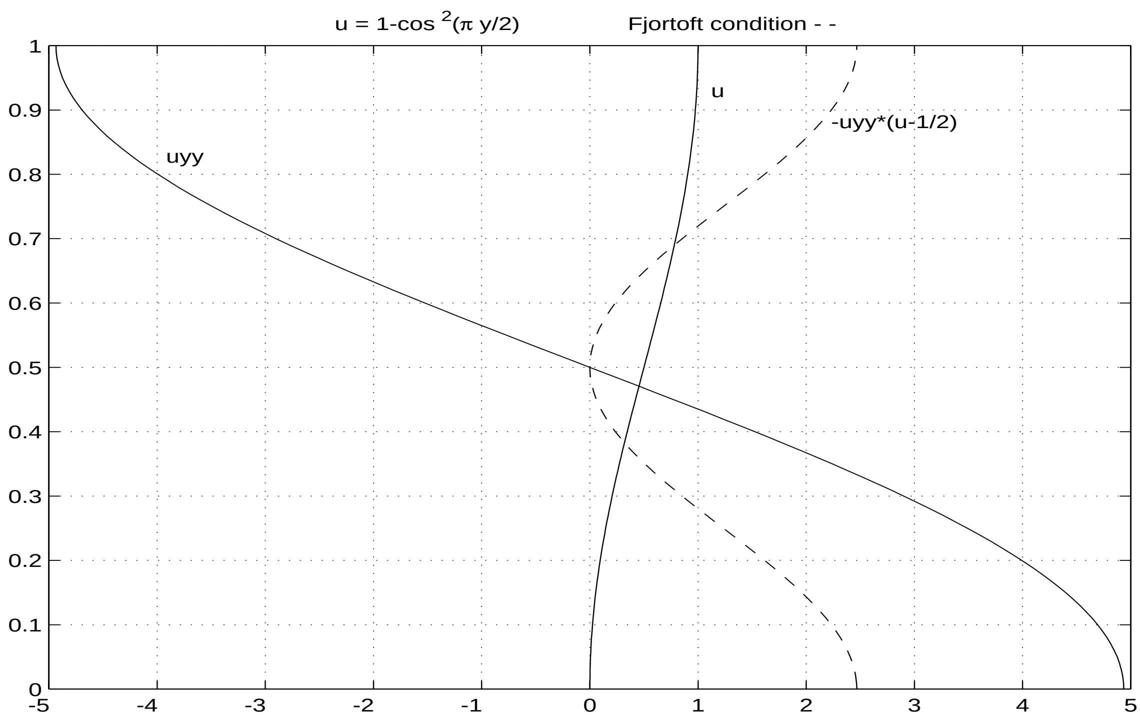

Figure 4.3.3 The velocity profile, its second derivative for the cosine profile discussed above. Here the second necessary condition as well as the first condition is satisfied.