Stability of viscoplastic flow

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1 Introduction

Air and water, the most common fluids on the Earth, are Newtonian fluids. It means that the viscous behaviors of these fluids can be described by Stokes' law of viscosity. In other words, the viscosity, which is defined as the ratio of shear stress versus shear rate, is constant. However, there are many liquids that do not obey Stokes' law of viscosity. For example, many geological and industry materials, such as mud, ice, lava, painting oil, toothpaste, drilling mud, chocolate and so forth, are not Newtonian fluids. Any fluid that does not follow the constant-viscosity law is called non-Newtonian fluid. Non-Newtonian fluids often exhibit some very interesting behaviors.

There are many types of non-Newtonian fluids: shearing thinning fluid, viscoplastic fluid and viscoelastic fluid... In this report, we will focus on the viscoplastic fluid. Viscoplastic fluid is also called "yield stress" fluid. Such fluid has a property in which the fluid behaves like a solid below some critical stress value (the yield stress), but flows like a viscous liquid when the yield stress is exceeded. It is often associated with highly aggregated suspensions. Flow of the muddy rivers is a typical example. Among many viscoplastic fluids, there is a special class called Bingham plastics. For Bingham plastic fluid, the shear stress beyond the yield stress is linearly proportional to the shear rate. If the yield stress approaches zero, the Bingham plastic fluid can be approximately treated as Newtonian fluid. Mathematically, this model can be represented as ([11]):

$$\tau_{ij} = \left(\nu + \frac{\tau_Y}{\dot{\gamma}}\right)\dot{\gamma}_{ij} \quad for \quad \tau > \tau_Y, \tag{1}$$

and

$$\dot{\gamma}_{ij} = 0 \quad for \quad \tau < \tau_Y. \tag{2}$$

where τ_{ij} is the deviatoric stress tensor, ν is the viscosity, the rate-of-strain tensor is:

$$\dot{\gamma}_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}. \tag{3}$$

where v_j represents the velocity field, and

$$\tau = \sqrt{\frac{1}{2}\tau_{ij}\tau_{ij}} \quad and \quad \dot{\gamma} = \sqrt{\frac{1}{2}\dot{\gamma}_{ij}\dot{\gamma}_{ij}}. \tag{4}$$

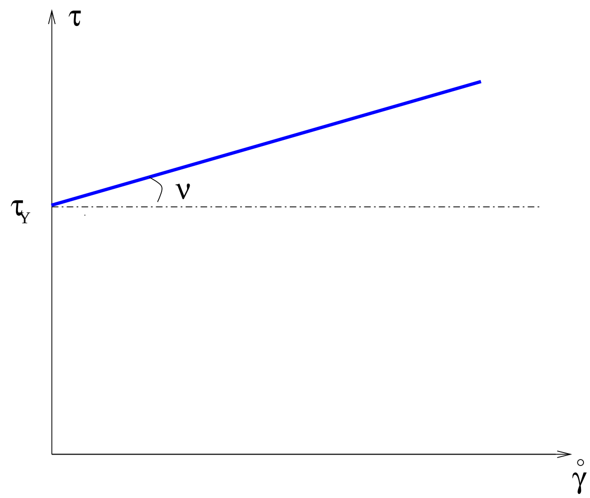

Figure 1: The Rheology of the Bingham model.

are the second invariants of τ_{ij} and $\dot{\gamma}_{ij}$. The rheology of the Bingham fluid is shown in Figure 1.

For the flow of the Bingham fluid, the stress varies in space and time. There can be regions in the fluid where the yield stress is exceeded, and other regions in which it is not. The boundaries between the two regions are the yield surfaces. Tracking the yield surfaces as the flow evolves is one of the most complicated problems associated with the Bingham model. The stability of the viscoplastic flows depends on what happens to these non-material surfaces when a sudden perturbation is introduced? Do the yield surface remain intact and merely displaced, or do they disappear and the plug is "broken" by the perturbation? In the study of the channel flow of the Bingham fluid, Friggard ([4]) address that an infinitesimal perturbation to the flow should displace the yield surfaces but otherwise leave them intact, since the unyielded region is "an elastic solid that would not break up". However, the identification of the yield surface leads to some confusion in the free surface flow of the Bingham fluid down an inclined plane and through narrow conduit ([2]). In this kind of problems, asymptotic expansion can be used to reduce the governing equations due to the small aspect ratio of the fluid. The leading order asymptotic solution contains apparent yield surface, however, the theory subsequently predicts that fluid flow is extensional even in the supposedly non-yielding regions. The resolution of this paradox is to re-interpret the apparent yield surfaces as "fake" ones and apparent "plug flow" as a weakly yielding flow, or a "pseudo-plug" ([1]; [2]).

Then, when the "plug flow" can be treated as "pseudo-plug flow"? How the stability criterion changes as we interpret the "plug flow" differently? What is the meaning of the yield surface and yield stress for the Bingham plastic flow? In this paper, we will try to answer these questions by studying various kinds of Bingham flow phenomena: free surface flow, channel flow and thermal convection. Based on the understanding of the yield surface and yield stress in the Bingham fluid, we then get an improved vertical average mode for the free surface flow of the Bingham thin film.

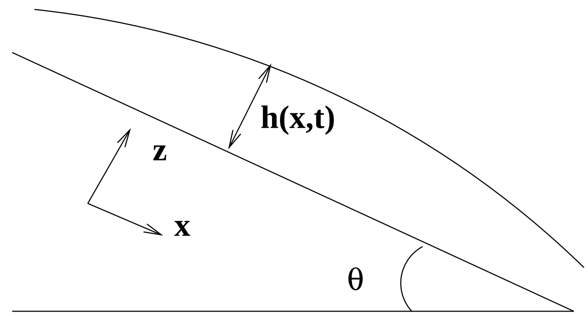

Figure 2: Fluid flowing down the inclined plane.

2 Free surface flow

Mud, the common geological fluid, has Bingham rheology. Every year, accompanying the heavy and persistent rainfalls in mountainous areas, mudflow can be induced by mixture of the water and mud flowing down the hill. It can move stones, boulders and even trees. It threatens the lives of the people who inhabit in the mountain area and is a main source of the natural hazards. Mudflows caused by Hurricane Mitch in 1998 have incurred devastating floods in Central America. In Honduras alone more than 6000 people perished. Half of the nations infrastructures were damaged([9]).

River with a large amount of clay suspension can also be characterized as Bingham fluid. The mud concentration at low water in the Yellow river of China is known to reach 50% by volume ([10]). In Jiang-xia Ravine China, the mudflow surges down during the wet season in groups of successive bores. The maximum wave height reaches $4m$ and the maximum wave velocity $13ms^{-1}$. The wavelength varied between 20 and 100m, while the period of each wave ranges from 5 to 60s. The bore fronts splattered with so much force that even large stones were thrown into the air. The flow in the rear of the waves was much shallower, slower and essentially laminar, and frequently stagnant before next surge. ([10])

Better understanding of the free surface flow of the Bingham fluid and accurate derivation for the stability criterion can help us to monitor the mudflows down the hill and mud surges in the rivers. In this section, we first give the governing equations for two-dimensional free surface flow, and then we introduce the pseudo-plug theory for the Bingham flow under the lubrication approximation. After that, we briefly describe the vertical average mode and its limitations. We also compare the pseudo-plug theory for the Bingham model with the bi-viscous model. Finally, we derive an improved vertical average model.

2.1 Governing equations

Consider a two-dimensional laminar flow of a thin layer of mud flowing down a plane with inclination θ. We define an (x, z) coordinate system with the x-axis along and the z-axis normal to the plane. We denote the longitudinal and transverse velocity components by $u(x, z, t)$ and $w(x, z, t)$ respectively, the pressure by $p(x, z, t)$ and the depth normal to the bed by $h(x, t)$ (Shown in figure 2).

The momentum equations along x and z directions are:

$$\rho(u_t + uu_x + wu_z) = -p_x + \partial_x \tau_{xx} + \partial_z \tau_{zx} + \rho g \sin(\theta), \tag{5}$$
$$\rho(w_t + uw_x + ww_z) = -p_z + \partial_z \tau_{xz} + \partial_z \tau_{zz} - \rho g \cos(\theta), \tag{6}$$

with the continuity equation:

$$u_x + w_z = 0. \tag{7}$$

At the bottom $z = 0$, the velocity vanishes (non-slip boundary condition):

$$u = w = 0. \tag{8}$$

On the free surface $z = h$, the kinematic boundary condition requires:

$$h_t + u(x, h, t)h_x = w(x, h, t); \tag{9}$$

and the stress free boundary condition states:

$$\begin{pmatrix} \tau_{xx} - p & \tau_{xz} \\ \tau_{xz} & \tau_{zz} - p \end{pmatrix} \begin{pmatrix} h_x \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10}$$

The constitutive law of the Bingham fluid is:

$$\begin{cases} \tau_{ij} = (\nu + \frac{\tau_Y}{\dot{\gamma}})\dot{\gamma}_{ij} & if \quad \tau > \tau_Y, \\ \dot{\gamma}_{ij} = 0 & if \quad \tau < \tau_Y, \end{cases} \tag{11}$$

where τ_{ij} is the stress tensor, $\dot{\gamma}_{ij}$ is the rate-of-strain tensor and $\dot{\gamma}$ is the second-invariant of the rate-of-strain tensor.

For the slow flow of the thin mud layer, the height of the fluid layer H is much less than the length L and the vertical velocity V is much less than the horizontal velocity U. We can define the aspect ratio: $\epsilon = \frac{H}{L}$ ($\epsilon \ll 1$) and non-dimensionalize the equations in the following way: $p = \rho g H cos(\theta) p'$, $z = Hz'$, $x = Lx'$, $u = Uu'$, $w = \frac{H}{L}Uw'$, and $t = \frac{L}{U}t'$. Drop the primes and the scaled momentum equations are:

$$\epsilon R_e(u_t + uu_x + wu_z) = -p_x + \epsilon^2 \partial_x \tau_{xx} + \partial_z \tau_{zx} + S, \tag{12}$$
$$\epsilon^3 R_e(w_t + uw_x + ww_z) = -p_z + \epsilon^2 \partial_z \tau_{xz} + \epsilon^2 \partial_z \tau_{zz} - 1. \tag{13}$$

where S is the dimensionless slope defined as: $S = \frac{L}{H} \tan\theta$. R_e is the Reynolds number defined as: $\epsilon R_e = \frac{UH}{\nu}$.
Under this scaling, the stress tensor can be expressed as:

$$\tau = \nu \frac{U}{H} \begin{pmatrix} 2\epsilon u_x & u_z + \epsilon^2 w_x \\ u_z + \epsilon^2 w_x & 2\epsilon w_z \end{pmatrix}. \tag{14}$$

We can scale the stress as $\nu \frac{U}{H}$ and define the Bingham number as $B = \frac{\tau_Y H}{\nu U}$, which is the dimensionless yield stress. Then the largest element in the stress tensor will be:

$$\tau_{zx} = \tau_{xz} = \left(1 + \frac{B}{|u_z|}\right) u_z. \tag{15}$$

2.2 Lubrication Approximation

There are several ways to analyze the stability of the flow. Lubrication approximation is the simplest way. Under the standard lubrication approximation, we let ϵ be zero. The momentum equation can be simplified as:

$$\begin{aligned} 0 &= -p_x + \partial_z \tau_{xz} + S, \\ 0 &= -p_z - 1. \end{aligned} \tag{16}$$

The stress free boundary condition at the top surface requires: $p = 0$ and $\tau_{xz} = 0$ (at $z = h$). Solution of the momentum equations shows that the vertical distribution of shear stress is:

$$\tau_{xz} = (h_x - S)(z - h). \tag{17}$$

The yield surface Y is defined at the place where the shear stress τ_{xz} is equal to the Bingham number:

$$B = \tau_{xz} = (h_x - S)(Y - h). \tag{18}$$

Thus,

$$Y = h + \frac{B}{|h_x - S|}. \tag{19}$$

Below the yield surface ($0 \leq z \leq Y$), the material is yielded. Above the yield surface ($h \geq z > Y$), the material remains to be unyielded. We usually call the region above the yield surface as the plug region.

Integrating the momentum equation (16) along the x-direction gives the vertical distribution of the horizontal velocity u:

$$\left\{ \begin{array}{lllll} u &=& (S - h_x)\frac{z}{2}(2Y - z) & for & z < Y, \\ u &=& (S - h_x)\frac{Y^2}{2} & for & z > Y. \end{array} \right. \tag{20}$$

Below yield surface ($0 \leq z \leq Y$), the velocity u has parabolic dependence on height z. Above the yield surface ($h \geq z > Y$), the velocity u is independent of height z and called plug velocity.

Integrating the kinematic boundary condition (9)across the layer, we arrive:

$$h_t + \frac{\partial}{\partial x} \int_0^h u dz = 0. \tag{21}$$

Substituting the horizontal velocity $u(z)$ (Equation 20)into this equation, we find that the height of the free surface satisfies:

$$h_t + \frac{1}{6} \frac{\partial}{\partial x} \left[Y^2 (3h - Y)(S - h_x) \right] = 0. \tag{22}$$

Linear stability analysis with an infinitesimal perturbation shows that the profile is linearly unconditionally stable. However, instabilities has been observed for the free surface flow of Bingham fluid down the inclined plane both in nature ([8]) and in the laboratory . Then, the lubrication approximation cannot be used to describe the stability of the Bingham fluid flowing down the inclined plane. Since the Reynolds number for the free surface flow of Bingham fluid can be very large, then $\epsilon Re \sim O(1)$, the inertia term must be included for correctly describing the stability of the free surface flow for the Bingham fluid.

However, the lubrication approximation reveals some interesting features about the Bingham fluid. During the initiation period, the viscoplastic flow usually contains perturbation, which can be the noise in the laboratory or natural environment. Then the velocity at the plug region can treated as a sum of the plug velocity $u_0(x,t)$ and the $O(\epsilon)$ perturbation velocity $u_1(x,z,t)$:

$$u = u_0(x,t) + \epsilon u_1(x,z,t). \tag{23}$$

Notice that the velocity in the plug region will not be independent of height any more. We can treat the plug as a "fake" one, and call this region as " pseudo-plug" region. The stress tensor in the pseudo-plug region can be written as:

$$\{\tau_{ij}\} = \epsilon\left(1 + \frac{B}{\dot{\gamma}}\right)\begin{pmatrix} 2u_{0x} & u_{1z} \\ u_{1z} & -2u_{0x} \end{pmatrix}. \tag{24}$$

Substituting the strain rate: $\dot{\gamma} = \epsilon\sqrt{4u_{0x}^2 + u_{1z}^2}$ into the above expression, we then have:

$$\tau_{ij} = \frac{B}{\sqrt{4u_{0x}^2 + u_{1z}^2}}\begin{pmatrix} 2u_{0x} & u_{1z} \\ u_{1z} & -2u_{0x} \end{pmatrix}. \tag{25}$$

Therefore, as the flow reaches the steady state, the variation of the zero order plug velocity along the x direction goes to zero: $u_{0x} \to 0$, the stress tensor through the pseudo-plug region can be written as:

$$\tau_{ij} = B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} sgn(u_{1z}) + O(\epsilon). \tag{26}$$

As the flow settles down to a steady state, the stress in the pseudo-plug region will be $O(\epsilon)$ above the yield stress B. It implies that the flow in the pseudo-plug region will relax to a slightly yielded state if we consider the noise in the initialization period.

In figure 3, we compared the vertical stress distribution obtained in the uniform true equilibrium state (17) with that obtained in the pseudo-plug state. From this figure we can see that the uniform true equilibrium state will produce a true plug, but the limiting lubrication solution will always produce a slightly yielded pseudo-plug.

For the experiment carried out in the laboratory and geological Bingham fluid flowing down the inclined plane, the uniform true equilibrium state is very hard to achieve. In most cases, the flow in the plug region will always relax to a slightly yielded pseudo-plug state after the flow is initiated.

2.3 Vertical averaged model

For describing stability of the Bingham fluid flowing down the inclined plane, the more accurate boundary layer approximation including the inertia term can be used. In boundary layer approximation, the momentum equations are:

$$\begin{aligned} R(u_t + uu_x + wu_z) &= 1 - p_x + \frac{\partial}{\partial z}\tau_{xz}, \\ 0 &= -1 - p_z. \end{aligned} \tag{27}$$

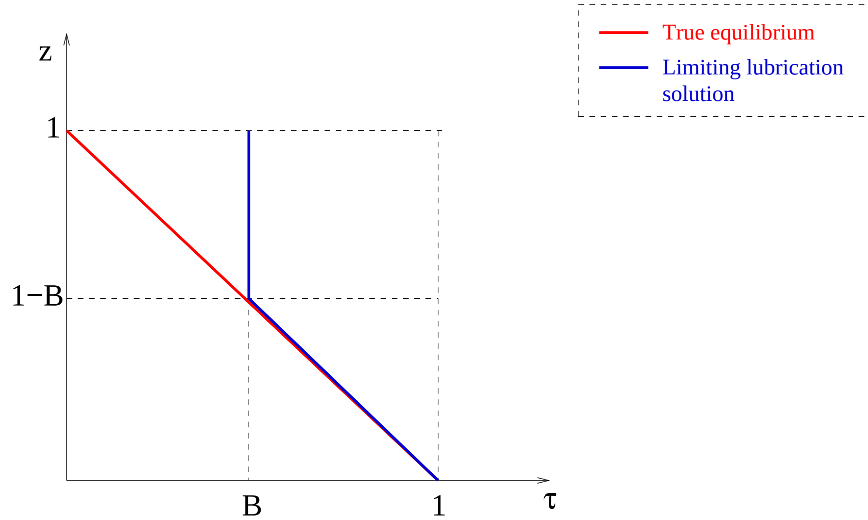

Figure 3: Compare the vertical stress distribution between the true equilibrium state and the pseudo plug state.

where $R = \epsilon R_e$. Following the study of the Newtonian fluid flowing down the inclined plane ([15]), further simplification of this mode can be obtained for Bingham fluid by vertically integrating the momentum equation assuming the following velocity profile ([8]):

$$\begin{cases} u = U_P(x,t)\frac{z}{Y}(2-\frac{z}{Y}) & for \quad z < Y, \\ u = U_P(x,t) & for \quad z > Y. \end{cases} \tag{28}$$

where $U_P(x,t)$ is the plug velocity. This velocity profile is obtained from the lubrication approximation. Integrating the momentum equation along the x-direction for the plug region $Y < z < h$ gives:

$$R(U_Pt + U_PU_{Px}) = 1 - h_x - \frac{B}{h-Y}. \tag{29}$$

And integrating the x-direction momentum equation for the yielded region $0 < z < Y$ gives:

$$R\left(\frac{2}{3}YU_{Pt} - \frac{1}{3}U_PY_t + \frac{2}{5}YU_PU_{Px} - \frac{2}{15}U_P^2Y_x\right) = (1-h_x)Y - \frac{2U_P}{Y}sgn(U_P). \tag{30}$$

Finally, integration of the kinematic boundary condition across the whole layer yields:

$$h_t + \frac{\partial}{\partial x}\left[\frac{U_P(x,t)}{3}(3h-Y)\right] = 0. \tag{31}$$

The vertical average model (VAM) is composed of the above three equations (29, 30, 31). Linear stability analysis of this model has been conducted by Liu and Mei ([8]). However, even for the Newtonian fluid flowing down the inclined plane, VAM does not give the correct Reynolds number ([3]; [15]).
For Newtonian fluid, VAM is usually used to study the turbulent flow. Since the flow of the

Bingham fluid down to the inclined plane is usually laminar, it is not necessary to apply the vertical average model.
Furthermore, VAM is built on the assumption that the vertical profile of the velocity u is the same with or without considering the inertia term in the momentum equation. Due to the existence of the yield stress and yield surface in the Bingham fluid, there is no justification for assuming this velocity profile. A better understanding of the Bingham fluid is necessary for obtaining the stability criteria.

2.4 Long wave expansion

For Newtonian fluid flowing down the inclined plane, it is well known that the long wave expansion for the boundary layer model (27) yields the correct Reynolds number ([3],[15]). To see how good the vertical averaged model for Bingham fluid is, let us check the critical Reynolds number for the Bingham fluid using a long wave expansion for the boundary layer model (27). First, we will carry out the stability analysis under the pseudo-plug assumption. After that, we will do the stability analysis under the true plug assumption.
Under the boundary layer approximation (See equation 27), we can integrate the momentum equation along the z-direction and considering $p = 0$ on the free surface $z = h$. Then, we get the vertical distribution of the pressure: $p = h - z$. Substituting it to the momentum equation along the x-direction, we get:

$$R(u_t + uu_x + wu_z) \quad = \quad 1 - h_x + \frac{\partial}{\partial z}\tau_{xz}. \tag{32}$$

For simplification, we set the slope to be unity: $S = 1$. In equilibrium, the horizontal velocity U is:

$$\begin{cases} U = \frac{1}{2}z(2Y - z) \quad & for \quad z < Y, \\ U = \frac{1}{2}Y^2 \quad & for \quad z > Y. \end{cases}$$

And the vertical velocity W is zero. We can also normalize the height of the free surface at the equilibrium to be unity $h = 1$.
Applying the following infinitesimally small perturbations to the equilibrium state:

$$u = U(z) + \hat{u}(x, z, t), \qquad w = \hat{w}(x, z, t), \qquad h = 1 + \hat{h}(x, z, t). \tag{33}$$

Dropping the hats, the momentum equation can be rewritten as:

$$R(u_t + Uu_x + wU_z) = 1 - h_x + \frac{\partial}{\partial z}\tau_{xz}. \tag{34}$$

Since the velocity perturbation satisfies the following continuity equation: $u_x + w_z = 0$, we can define a stream function Ψ and the perturbations can be rewritten as:

$$u = \Psi_z, \qquad w = -\Psi_x. \tag{35}$$

First, we assume the periodic perturbations: $\Psi, h \propto exp(\lambda t + ikx)$ carry out the stability analysis for the pseudo-plug assumption. We divide the layer into two parts: the upper part is the pseudo-plug region and the low part is the yielded region. If we use the superscript

t and b to express the function of the top and bottom part respectively, the momentum equations in these two regions can be written as:

$$\Psi^t_{zz} = 0, \qquad for \quad z > Y, \tag{36}$$

and,

$$\Psi^b_{zzz} = ik + R(\lambda\Psi_z + ikU\Psi_z - ik\Psi U_z) \qquad for \quad z < Y. \tag{37}$$

From the expression of the stream function in the region above the yield surface, we can see that the motion is allowed in this region under the pseudo-plug assumption. The linearized kinematic boundary condition at the free surface $h_t + Uh_x = w$ is:

$$\lambda = -\frac{ik}{2}Y^2 - ik\Psi^t, \qquad at \qquad z = 1. \tag{38}$$

Non-slip boundary condition at the bottom $z = 0$ requires:

$$\Psi^b_z(z = 0) = \Psi^b(z = 0) = 0. \tag{39}$$

Under the pseudo-plug assumption, the zero order stress in the pseudo-plug region is equal to the yield stress B. And the stress free boundary condition at the top surface is satisfied by the first order correction (See the discussion in the section of the lubrication approximation).

At the interface of the two parts (the perturbed yield surface), the continuation of the stress implies:

$$U^t_z + \Psi^t_{zz} = U^b_z + \Psi^b_{zz}, \tag{40}$$

Since $U^t_z = 0$ and $\Psi^t_{zz} = 0$ at the interface, we then have:

$$YU^b_{zz}(z = Y) + \Psi^b_{zz}(z = Y) = 0. \tag{41}$$

Recall the perturbation of the yield surface is: $Y' = 1 - ikB - R(\lambda + iU^t)\Psi_z B$, the interface condition (41) can be written as:

$$(1 - ikB - R(\lambda + ikU^t)\Psi_z B)U^b_{zz}(z = Y) + \Psi^b_{zz}(z = Y) = 0. \tag{42}$$

Similarly, the continuation of the perturbed velocity u across the interface implies:

$$\Psi^b_z(z = Y) = \Psi^t_z(z = Y), \tag{43}$$

and the continuation of the stream function across the interface needs,

$$\Psi^b(z = Y) = \Psi^t(z = Y). \tag{44}$$

We can then do long wave expansion by taking $k << 1$:

$$\lambda = k\lambda_1 + k^2\lambda_2 + ... \tag{45}$$
$$\Psi = k\Psi_1 + k^2\Psi_2 + ... \tag{46}$$

where λ_1, λ_2, Ψ_1, and Ψ_2 denote the expansion coefficients. To the leading order, the momentum equations can be written as:

$$\Psi^t_{0zz} = 0, \tag{47}$$
$$\Psi^b_{0zzz} = 0, \tag{48}$$

with the interface conditions:

$$U^b_{zz}(z=Y)+\Psi^b_{0zz}(z=Y)=0, \tag{49}$$
$$\Psi^b_{0z}(z=Y)=\Psi^t_{0z}(z=Y), \tag{50}$$
$$\Psi^b_{0}(z=Y)=\Psi^t_{0}(z=Y), \tag{51}$$

From the leading order calculation, we obtain: $\lambda_1 = -iY$. To the second order, the momentum equations are:

$$\Psi^t_{1zz} = 0, \tag{52}$$
$$\Psi^b_{1zzz} = i + R(\lambda_1\Psi^b_{0z} + iU^b\Psi^b_{0z} i\Psi^b_0 U^b_z). \tag{53}$$

and the interface conditions are:

$$(-iB - R(\lambda_1 + iU^b)\Psi^b_{0z}B)U^b_{zz}(z=Y) + \Psi^b_{0zz}(z=Y) = 0, \tag{54}$$
$$\Psi^b_{1z}(z=Y)=\Psi^t_{1z}(z=Y), \tag{55}$$
$$\Psi^b_{1}(z=Y)=\Psi^t_{1}(z=Y). \tag{56}$$

Putting the second order solution into the kinematic boundary condition (38), we then have:

$$\lambda_2 = -\frac{RB^6}{5} + \frac{RB^5}{5} + \frac{RB^4}{3} - \frac{RB^3}{3} + \frac{B^3}{3} - \frac{2RB}{15} + \frac{2R}{15} - \frac{1}{3}. \tag{57}$$

And the critical Reynolds number is then:

$$R = \frac{5(B^2+B+1)}{3B^5 - 5B^3 + 2}. \tag{58}$$

We can compare the critical Reynolds number obtained by the long wave expansion with that gotten from VAM in the long wave length limit (See Figure 4). From this figure, we find that the Reynolds number obtained in VAM is always larger than that in the long wave expansion. When the yield stress equals zero, the Reynolds number gotten from the long wave expansion is $\frac{5}{6}$ of that from VAM, which is consistent with the calculation for the Newtonian fluid. As the yield stress increases (yield surface decreases), the difference increases. If the dimensionless yield stress approaches one, the difference approaches infinity.

Now, let us carry out the stability analysis under the true plug assumption. Since the true plug behaves as the elastic solid, it will not be destroyed by the infinitesimal perturbation. Then the perturbation velocity is zero in the plug region. Under the linear perturbation: $u, w, h, \Psi \propto exp(\lambda t + ikx)$, the stream function in the plug region satisfies:

$$\Psi^t = \Psi^t_z = \Psi^t_{zz} = 0 \qquad for \quad z \geq Y, \tag{59}$$

which means that no motion can be transferred from the yield region to the plug region.

The stream function in the yielded region, the boundary condition, and the interface condition will remain to be the same with that under the pseudo-plug assumption. Taking the long wave expansion, to the leading order we have:

$$\Psi^t_0 = \Psi^b_0 = 0. \tag{60}$$

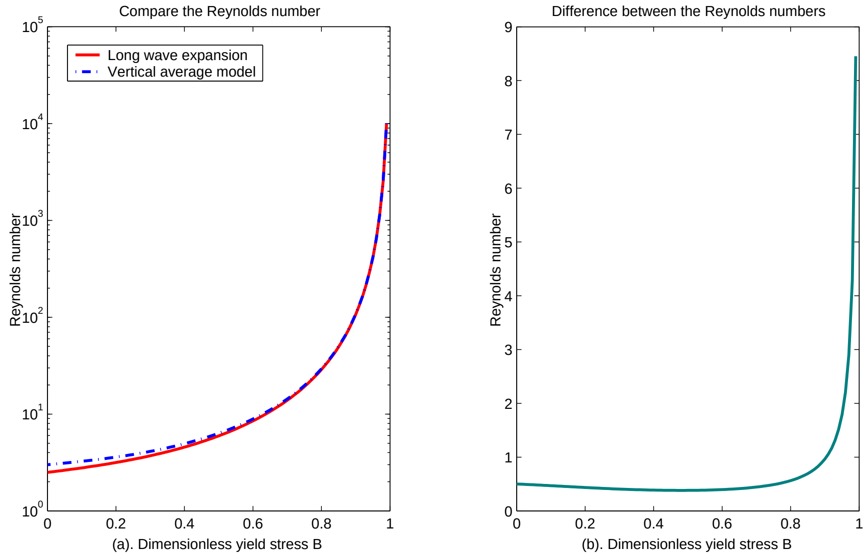

Figure 4: (a) Compare the critical Reynolds number obtained from the long wave expansion with that from the vertical average mode. (b) The difference between the Reynolds number obtained by different methods.

and $\lambda_1 = 0$. To the second order, we have:

$$\Psi_1^t = \Psi_{1z}^t = \Psi_{1zz}^t = 0 \quad \text{for} \quad z \geq Y, \tag{61}$$

$$\Psi_{1zzz}^b = i \quad \text{for} \quad z < Y. \tag{62}$$

There is no solution satisfying these two equations, the boundary conditions, as well as the interface conditions. It means the solution does not exist under the true plug assumption.

2.5 Compare pseudo-plug theory for the Bingham fluid with bi-viscous model

Under the pseudo-plug assumption, the motion can exist in the plug region for the Bingham fluid. Then the relationship between the rate of strain tensor and the stress is like the modified constitutive model:

$$\tau_{ij} = \left(\nu + \frac{\tau_Y}{\sqrt{\Delta^2 + \dot{\gamma}^2}} \right) \dot{\gamma_{ij}} \tag{63}$$

When the parameter Δ becomes small, the model becomes more and more Bingham-like, with flow regions that resemble the fully plastic regions of the Bingham fluid, and other regions in which the flow is just slightly yielding and reminiscent of the pseudo-plug. Here, we will consider the simplest regularized model: the bi-viscous model.

The constitutive relationship for the bi-viscous fluid is illustrated in figure 5, with a lower viscosity ν for the high shear rate and much higher viscosity ν for the lower shear rate. We can use α to express the viscosity ratio: $\alpha = \frac{\nu}{\nu}$. If α goes to zero, the bi-viscous

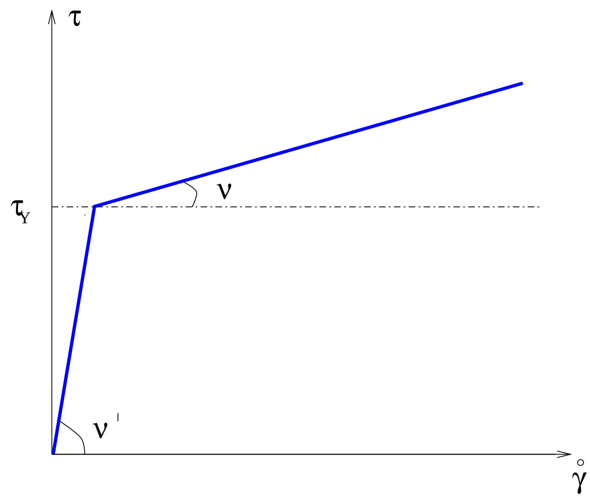

Figure 5: The Rheology of the bi-viscous fluid.

fluid will approaches the Bingham fluid. If α goes to one, the bi-viscous fluid is reduced to the Newtonian fluid.

$$\tau_{ij} = \nu\dot{\gamma}_{ij} \qquad for \qquad \tau \leq \tau_Y, \tag{64}$$

$$\tau_{ij} = \nu\dot{\gamma}_{ij} + (1-\alpha)\frac{\tau_Y}{\dot{\gamma}}\dot{\gamma}_{ij} \qquad for \qquad \tau > \tau_Y. \tag{65}$$

We use the same scaling as free surface flow of the Bingham fluid. The boundary layer approximation requires:

$$R(u_t + uu_x + wu_z) = 1 - p_x + \frac{\partial}{\partial z}\tau_{xz}, \tag{66}$$

$$0 = -1 - p_z. \tag{67}$$

For the bottom layer, the shear stress is $\tau_{xz} = u_z + B(1-\alpha)$. For the top layer, the shear stress is $\tau_{xz} = \frac{1}{\alpha}u_z$ Substitute the expression for the shear stress into the momentum equation, we can then get the equilibrium velocity profile:

$$U^t = -\frac{\alpha z^2}{2} + \alpha z + \frac{Y^2}{2} - \frac{\alpha Y^2}{2} \quad for \quad z > Y, \tag{68}$$

$$U^b = -\frac{z^2}{2} + (-\alpha Y + Y + \alpha)z \quad for \quad z < Y. \tag{69}$$

with $Y = 1 + \frac{B}{|h_x - S|}$. Here the superscript t and b denotes the function for the top and bottom layer respectively.

As before, we perturb the equilibrium by infinitesimal amount: $u = U(z) + u'(x,z,t)$ and $w = w'(x,z,t)$. We also use a stream function Ψ to express the perturbation velocity: $u'(x,z,t) = \Psi_z$, and $w'(x,z,t) = -\Psi_x$. We can then carry out the linear stability analysis by assuming: $\Psi \propto exp(ikx + \lambda t)$. Then the stream function satisfies:

$$\Psi^t_{zzz} = ik\alpha + R\alpha(\lambda\Psi_z + ikU\Psi_z ik\Psi U_z) \quad for \quad z > Y, \tag{70}$$

$$\Psi^b_{zzz} = ik + R(\lambda\Psi_z + ikU\Psi_z ik\Psi U_z) \quad for \quad z \leq Y. \tag{71}$$

At the free surface (z=1), the linearized kinematic condition gives:

$$\lambda + U(z=1)ik = -ik\Psi^t, \tag{72}$$

and the stress free condition gives:

$$\Psi^t(z=1) + U_{zz}(z=1) = 0. \tag{73}$$

The non-slip boundary condition at the bottom implies:

$$\Psi_z^b(z=0) = \Psi^b(z=0) = 0. \tag{74}$$

At the interface between the two layers, the continuation of the shear stress requires:

$$\Psi_{zz}^b(z=Y) = \frac{1}{\alpha}\Psi_{zz}^t(z=Y). \tag{75}$$

The continuation of the velocity and the stream function needs:

$$\Psi_z^b(z=Y) = \Psi_z^t(z=Y) \quad and \quad \Psi^b(z=Y) = \Psi^t(z=Y). \tag{76}$$

Do long wave expansion, the Reynolds number is given in figure 6. From this figure we find that the Reynolds number of the bi-viscous fluid approaches the Reynolds number for the Bingham fluid as the viscosity ratio approaches zero. This result is quite different from the result obtained by Hjorth ([5]), which states that the free surface flow of the bi-viscous fluid is linearly unconditionally stable as the viscosity ratio α goes to zero. The reason for this difference is probably due to the different interfacial conditions between two different viscosity region.

2.6 Improved vertical averaged model

Researches on the Newtonian fluid flowing down the inclined plane have shown that the vertical averaged model can not give the correct stability criterion. The limitations of this model exist in the rustic character of the averaging method and the lack of freedom in the description of the hydrodynamic fields. The improved model has been derived by combining a gradient expansion to weighted residual techniques with polynomials as test functions ([12]; [13]). Based on the understanding of the free surface Bingham flow and the pseudo-plug assumption, we can use this technique to improve the vertical average model for the free surface Bingham flow.

We separate the fluid into two regions: the pseudo-plug region and the yield region. In the pseudo-plug region ($Y \le z < h$), the velocity is the same as the plug velocity $u = U(x,t)$, which is guaranteed by the pseudo-plug theory. In the yielded region($0 \le z < Y$), the boundary layer equations are:

$$R(\partial_x u + u\partial_x u + w\partial_z u) - u_{zz} = 1 - \partial_x h, \tag{77}$$

$$u_x + w_z = 0, \tag{78}$$

$$u_z|_{z=Y} = 0, \tag{79}$$

$$u|_0 = w|_0 = 0. \tag{80}$$

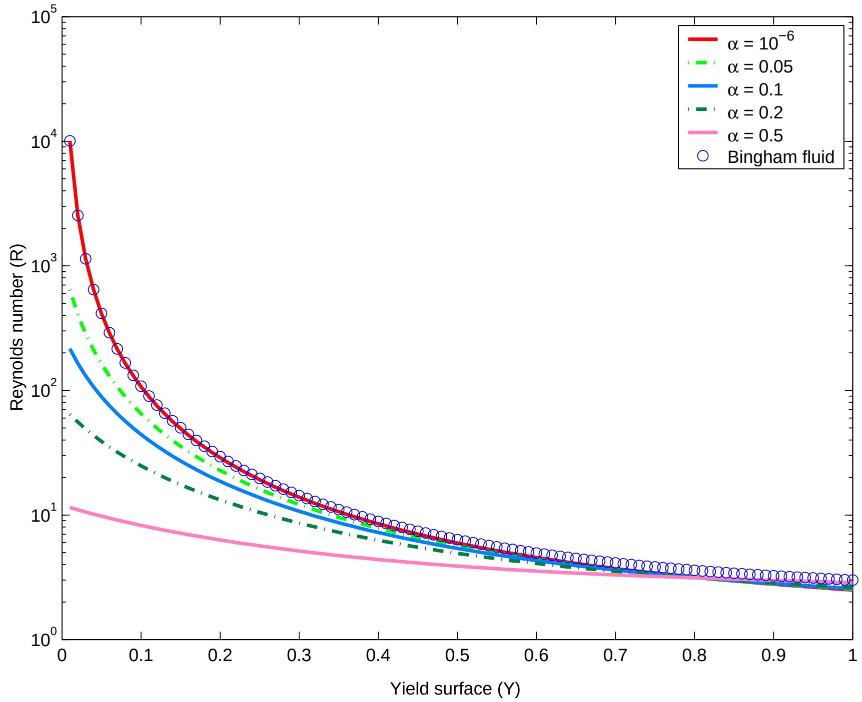

Figure 6: Compare the critical Reynolds number for the long wave expansion and the bi-viscous mode.

From the continuity equation: $u_x + w_z = 0$ and the non-slip boundary condition $u|_0 = w|_0 = 0$, we can replace w by $w = -\int_0^z u_x dz$ so that the only remaining dynamical variable is $u(x,z,t)$. Expanding $u(x,z,t)$ in the following form:

$$u(x,z,t) = a_j(x,t) f_j(\bar{z}), \tag{81}$$

where $\bar{z}$ is defined as: $\bar{z} = \frac{z}{Y}$. Both Y and expansion coefficients a_j are supposed to be slowly varying function of time t and the stream-wise coordinate x. The base function $f_j(\bar{z})$ can be chosen to be:

$$f_j(\bar{z}) = \bar{z}^{j+1} - \frac{j+1}{j+2}\bar{z}^{j+2}. \tag{82}$$

which fulfills the boundary condition: $f_j(0) = f_j'(1) = 0$. It is easily observed that the vertical velocity profile used in VAM is merely proportional to $f_0(\bar{z})$. It can be shown that the consistent first order model can be obtained by considering a reduced set of test functions comprising monomials up to degree 6 included.

Inserting the truncated expansion $u(x,z,t) = \sum_{j=0}^{4} a_j(x,t) f_j(\bar{z})$ into the momentum equation along the x-direction, and neglecting all terms in $a_j (j > 0)$ involving derivatives with respect to x and t, we will have:

$$
\begin{aligned}
0 &= \frac{1}{RY^2}(a_0 - 2a_1) - \frac{1}{R} + \frac{\partial_x h}{R}, && (83)\\
0 &= \frac{1}{RY^2}(4a_1 - 6a_2) + \partial_t a_0 - \frac{a_0}{Y}\partial_t Y, && (84)\\
0 &= \frac{1}{RY^2}(9a_2 - 12a_3) - \frac{1}{2}\partial_t a_0 + \frac{a_0}{Y}\partial_t Y + \frac{1}{2}a_0\partial_x a_0 - \frac{a_0^2}{2Y}\partial_x Y, && (85)\\
0 &= \frac{1}{RY^2}(16a_3 - 20a_4) - \frac{1}{3}a_0\partial_x a_0 + \frac{2a_0^2}{3Y}\partial_x Y, && (86)\\
0 &= \frac{1}{RY^2}25a_4 + \frac{1}{6}(\frac{1}{2}a_0\partial_x a_0 - \frac{a_0^2}{Y}\partial Y). && (87)
\end{aligned}
$$

Eliminating a_1, a_2, a_3 and a_4 by inserting their expression into the first equation, we arrive:

$$a_0 = Y^2 - \frac{R}{3}Y^2\partial_t a_0 + \frac{RY}{6}a_0\partial_t Y - \frac{RY^2}{10}a_0\partial_x a_0 + \frac{RY}{30}a_0^2\partial_x Y - Y^2\partial_x h. \quad (88)$$

Now we need to obtain the plug velocity: $u(z = Y) = U(x, t)$. The plug velocity can be expressed as:

$$U(x, t) = \sum_j a_j(x, t) f_j(\bar{z} = 1). \quad (89)$$

Substituting the expansion coefficient $a_j(x, t)$ into this equation, thus we have:

$$U(x, t) = \frac{1}{2}a_0 - \frac{1}{45}RY^2 a_0\partial_x a_0 + \frac{1}{360}RY a_0^2\partial_x Y - \frac{1}{24}RY^2\partial_t a_0. \quad (90)$$

Also, we use p to denote the local instantaneous flow rate: $p = \int_0^Y u dz = Y\int_0^1 u d\bar{z}$. The value for p can be calculated out and expressed as:

$$P(x, t) = \frac{1}{3}a_0 - \frac{3}{280}a_0 Y^2 R\partial_x a_0 + \frac{1}{504}a_0^2 YR\partial_x Y - \frac{1}{45}\partial_t a_0 RY^2 + \frac{1}{360}a_0 RY\partial_t Y. \quad (91)$$

Integrating the kinematic boundary condition across the pseudo-plug region gives:

$$h_t + \frac{\partial}{\partial x}(U(x, t)(h - Y)) + \frac{\partial}{\partial x}P = 0. \quad (92)$$

Also we can integrate the momentum equation in x-direction across the pseudo-plug region, we then have:

$$R(U_t + UU_x) = 1 - h_x - \frac{sgn(B)}{h - Y}. \quad (93)$$

Plus the equations for a_0, p and U, we then have a set of five equations. These five equations close the system. To test this set of equations, we can do linear stability analysis and comparing the critical Reynolds number with that obtained from the long wave expansion for the boundary layer model. The equilibrium is:

$$h = 1, Y = 1 - B, U = \frac{1}{2}(1 - B)^2, a_0 = (1 - B)^2, P = \frac{a_0}{3}Y. \quad (94)$$

Apply small perturbations: $h = h + h'$, $Y = Y + Y'$, $U = U + U'$, $p = p + p'$, and $a_0 = a_0 + a_0'$. Do normal mode expansion for these perturbations: $h', U', Y', P', a_0' \propto exp(\lambda t + ikx)$. Then the perturbed equations will be:

$$\lambda h + ikBU(x,t) + \frac{1}{2}(1-B)^2 ik(h-Y) + ikp = 0, \tag{95}$$

$$R(\lambda U + \frac{1}{2}(1-B)^2 ikU) = -ikh - \frac{h-Y}{B}, \tag{96}$$

$$U = \frac{a_0}{2} - \frac{1}{45}R(1-B)^4 ika_0 + \frac{R}{360}(1-B)^5 ikY - \frac{R}{24}(1-B)^2\lambda a_0, \tag{97}$$

$$\begin{aligned} a_0 &= 2(1-B)Y - \frac{R(1-B)^2}{3}\lambda a_0 + \frac{R(1-B)^3}{6}\lambda Y \\ &- \frac{R(1-B)^4}{10}ika_0 + \frac{R(1-B)^5}{30}ikY - (1-B)^2 ikh, \end{aligned} \tag{98}$$

$$P = \frac{1}{3}a_0 Y - \frac{3RY}{280}(1-B)^4 ika_0 + \frac{RY}{504}(1-B)^5 ik - \frac{R}{45}(1-B)^2\lambda a_0 + \frac{\lambda RY}{360}(1-B)^3. \tag{99}$$

For the improved vertical average model, the Reynolds number R in the long wave length limit is very close to that obtained from the long wavelength expansion for the boundary layer model (See figure 7).

3 Channel flow of Bingham fluid

In this section, we will study the channel flow of Bingham fluid.

3.1 Govening equations

For the Bingham fluid flowing through a channel with width L, the momentum equation can be written as[1]:

$$\rho(u_t + uu_x + wu_z) = -p_x + \partial_x\tau_{xx} + \partial_z\tau_{zx}, \tag{100}$$
$$\rho(w_t + uw_x + ww_z) = -p_z + \partial_z\tau_{xz} + \partial_z\tau_{zz}, \tag{101}$$

with the continuity equation: $u_x + w_z = 0$, with the boundary condition: $w = u = 0$ on $z = L, -L$. We scale the problem in the following way:

$$u \sim Uu;\ w \sim Uw;\ x \sim Lx;\ z \sim Lz; p \sim \rho U^2 p. \tag{102}$$

Define the Bingham number as: $B = \frac{\tau_Y L}{\nu U}$, and the dimensionless viscosity coefficient ν as: $\nu = \frac{\mu}{\rho U L}$. Also we separate the background pressure gradient from the evolving pressure gradient:

$$p = \nu x + p'(x,z,t). \tag{103}$$

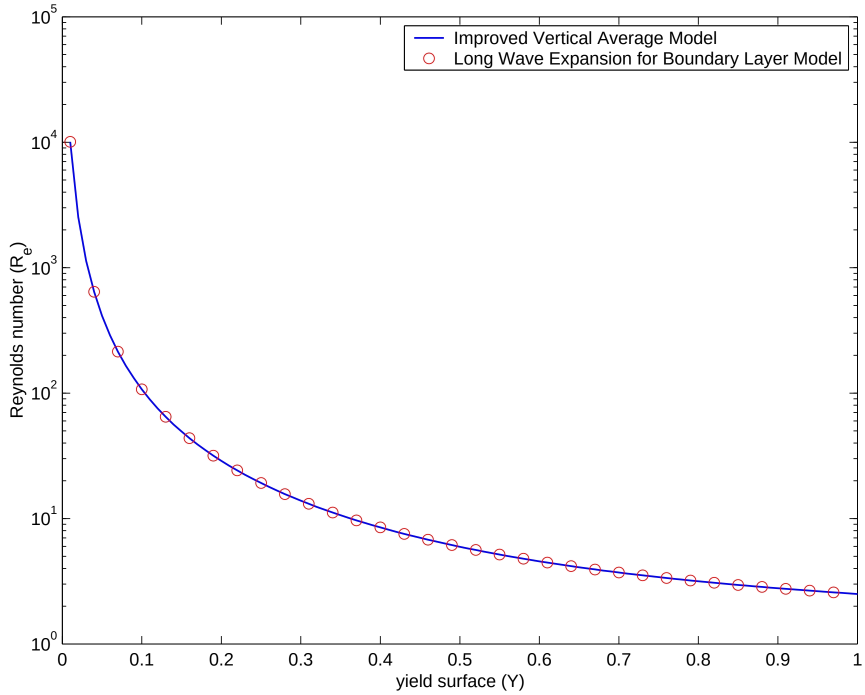

Figure 7: Compare the critical Reynolds number for the improved vertical averaged model.

Droping the primes, the momentum equation can be written as:

$$\rho(u_t + uu_x + wu_z) = \nu - p_x + \nu(\partial_x \tau_{xx} + \partial_z \tau_{zx}), \tag{104}$$

$$\rho(w_t + uw_x + ww_z) = -p_z + \nu(\partial_z \tau_{xz} + \partial_z \tau_{zz}), \tag{105}$$

with,

$$\begin{cases} \tau_{xz} &= (1 + \frac{B}{|U_z|})U_z \quad if \quad \tau > B, \\ \tau_{xz} &= 0 \quad if \quad \tau < B. \end{cases} \tag{106}$$

The equilibrium state will be $w = p = 0$ and $u = U(z)$:

$$U(z) = -\frac{1}{2}(1 - z^2) - B(1 - |z|) \qquad \text{for} \quad |z| > B, \tag{107}$$

$$U(z) = (1 - B)^2/2 \qquad \text{for} \quad |z| < B \tag{108}$$

Do linear stability analysis:

$$(u, w, p) = (U, 0, 0) + (\Psi_y, ik\Psi, p)exp(ikx + \lambda t). \tag{109}$$

Then we will get the Orr-Sommerfeld equation;

$$(\frac{ikU}{\nu} + \frac{\lambda}{\nu})(\Psi_{yy} - k^2\Psi) - U''\Psi = (\partial_y^2 - k^2)^2\Psi - 4Bk^2\partial_y(\frac{\Psi_y}{|U'|}). \tag{110}$$

The boundary conditions are $\Psi = \Psi_y = 0$ at $y = -1, 1$.

3.2 $k \ll 1$ mode

For $k \ll 1$ mode, we can do long wave expansion and the leading order term for outside of the plug region is:

$$\psi_{yy} - \frac{\lambda}{\nu}\Psi = 0, \tag{111}$$

with the boundary condition: $\psi_y = 0$ on $y = 1$. For the region inside the pseudo-plug region, we have:

$$\psi_{yy} = 0. \tag{112}$$

We can solve these equations analytically, and get the eigen-value and eigen-functions for odd mode and even mode respectively. The odd mode in stream function Ψ is corresponding to the 'sausage' mode of the flow, the even mode in stream function Ψ is corresponding to the 'kink' mode of the flow. For the odd mode in stream function, we have: $\Psi = 0$ on $y = 0$. The eigen-value satisfies:

$$\lambda = -\frac{n^2\pi^2}{(1-B)^2}\nu. \tag{113}$$

For the even mode, we have: $\psi_y = 0$ on $y = 0$. And the eigen-value satisfies:

$$-B\frac{\lambda}{\nu}tanh(\sqrt{\frac{\lambda}{\nu}}(1-B)) = 1. \tag{114}$$

The above equations has no real solution for real λ, and the flow is linearly unconditional stable.

3.3 k finite mode

For finite k, we have $\psi_{yy} - \psi_{xy} \approx 0$, and $\psi_{xy} \approx 0$, which implies:

$$\psi = \psi_y = \psi_{yy} = 0, \tag{115}$$

inside the pseudo-plug. It reveals a very interesting phenomena of the flow: the plug in this case bahaves like a solid and seperate the flow completely. In fact, the pertubation in top part of the flow will not be transport to the bottom part of the flow! Both odd and even mode will exist and have the same eigenvalue, which is the same as the predicted by the regularized mode.([1])

4 Convection of Bingham fluid

Put the Bingham fluid between two plates with fixed temperatures: T_0 and T_1, with $T_0 > T_1$. The momentum equation will be:

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}, \tag{116}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g + \alpha g(T - T_0) + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z}. \tag{117}$$

the temperature equation is:

$$\frac{DT}{\partial t} = \kappa \nabla^2 T, \tag{118}$$

with the continuity equation: $\nabla \cdot \boldsymbol{u} = 0$ and the following constitutive equation:

$$\begin{cases} \tau_{ij} &= (\mu + \frac{\tau_Y}{\dot{\gamma}})\dot{\gamma}_{ij} \quad if \quad \tau > \tau_Y, \\ \dot{\gamma}_{ij} &= 0 \qquad if \quad \tau < \tau_Y, \end{cases} \tag{119}$$

where τ_Y is the yield stress of the Bingham fluid, and ν is the viscosity. we scale the problem in the following way:

$$x \sim d, t \sim \frac{\kappa}{d^2}, u \sim \frac{d}{\kappa}, T \sim \frac{1}{\beta d}, p \sim \frac{d^2}{\rho\kappa}, \tag{120}$$

where d is the spacing between two plates, and β is the inverse temperature gradient. Define the following dimensionless numbers: Rayleigh number R_a:

$$R_a = \frac{\alpha\beta g d^4}{\kappa\mu}, \tag{121}$$

and define the Prandtle number Pr:

$$P_r = \frac{\mu}{\kappa}, \tag{122}$$

and define the Bingham number: B

$$B = \frac{\tau_Y \rho\kappa}{d^2}, \tag{123}$$

Then the scaled equations will be:

$$\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + P_r\frac{\partial \tau_{xx}}{\partial x} + P_r\frac{\partial \tau_{xz}}{\partial z}, \tag{124}$$

$$\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + P_r R_a T + P_r\frac{\partial \tau_{xz}}{\partial x} + P_r\frac{\partial \tau_{zz}}{\partial z}. \tag{125}$$

the temperature equation is:

$$\frac{DT}{\partial t} - w = \kappa \nabla^2 T \tag{126}$$

with the continuity equation: $\nabla \cdot \boldsymbol{u} = 0$ and the following constitutive equation:

$$\begin{cases} \tau_{ij} &= (1 + \frac{B}{\dot{\gamma}})\dot{\gamma}_{ij} \quad if \quad \tau > B, \\ \dot{\gamma}_{ij} &= 0 \qquad if \quad \tau < B. \end{cases} \tag{127}$$

We have the none-slip boundary condition at the top and the bottom surface: $u = w = 0$, also the temperature at the top is 1, and the temperature at the bottom is 0. At the equilibrium, the fluid has no motion: $u^* = w^* = 0$ and no stress: $\tau^*_{ij} = 0$. The equilibrium temperature structure will be: $T^* = 1 - z$ and the equilibrium pressure profile will be: $p^* = z - \frac{z^2}{2}$. Apply the small perturbations: u', w', θ, p', and drop the primes, then the perturbed momemtum equation will be:

$$\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + P_r\frac{\partial \tau_{xx}}{\partial x} + P_r\frac{\partial \tau_{xz}}{\partial z}, \tag{128}$$

$$\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + P_r R_a \theta + P_r\frac{\partial \tau_{xz}}{\partial x} + P_r\frac{\partial \tau_{zz}}{\partial z}. \tag{129}$$

We conduct the linear stability analysis by assuming $u, w, \theta, p \propto exp(ikx + \lambda t)$. Since the fluid is heated from the below, the fluid is at the true equilibrium state in the beginning. The initial stress in the fluid is zero. Therefore, adding infinitesimal perturbation will not make the stress in the fluid larger than the yield stress (see equation 127). It can be easily shown that the only solution is $u = 0, w = 0$. Therefore, the fluid is linearly unconditional stable under the infinitesimal perturbation.

5 Conclusion

For Bingham fluid in the laboratory and in the nature, the fluid is likely to relaxed to a pseudo-plug state. In this case, the stress in the pseudo-plug is slightly above the yield stress and the fluid can be linearly unstable. However, for the thermal convection of the Bingham fluid, since the initial stress in the fluid is zero, adding infinitesimal perturbation will not make the stress in the fluid larger than the yield stress. The thermal convection of the Bingham fluid will be linearly unconditional stable.

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