GFD 07 Boundary layers:
Internal boundary layers in the ocean circulation.

We have limited attention until now to boundary layers that are located on boundaries of the fluid.

Boundary layers can occur within a fluid as well.

Two examples:

1) Internal boundary layer in the oceanic thermocline

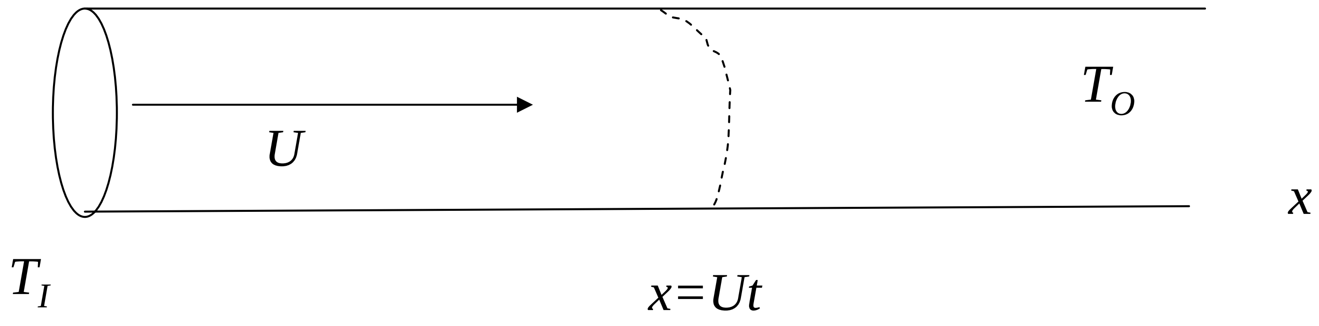
2) The equatorial undercurrent (EUC).

Model problem

Consider flow in a pipe. For simplicity assume the flow is one dimensional and has constant velocity *U*

At the entrance to the pipe, the temperature, considered a tracer is held to a value, T_I.

And that initially, and so at large distances from the entrance, the temperature is T_O



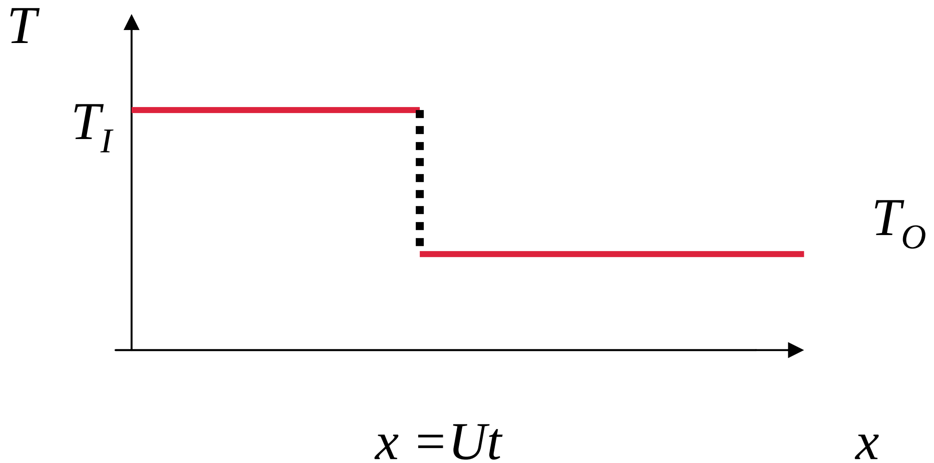
Model Equation

$$\frac{\partial T}{\partial t} + U \frac{\partial T}{\partial x} = \kappa \frac{\partial^2 T}{\partial x^2}$$

For small κ we might ignore the diffusion term leading to the *discontinuous* solution:

$$T = T_I, \qquad x - Ut \le 0,$$

$$T = T_O, \qquad x - Ut > 0$$



The transition zone

New coordinates

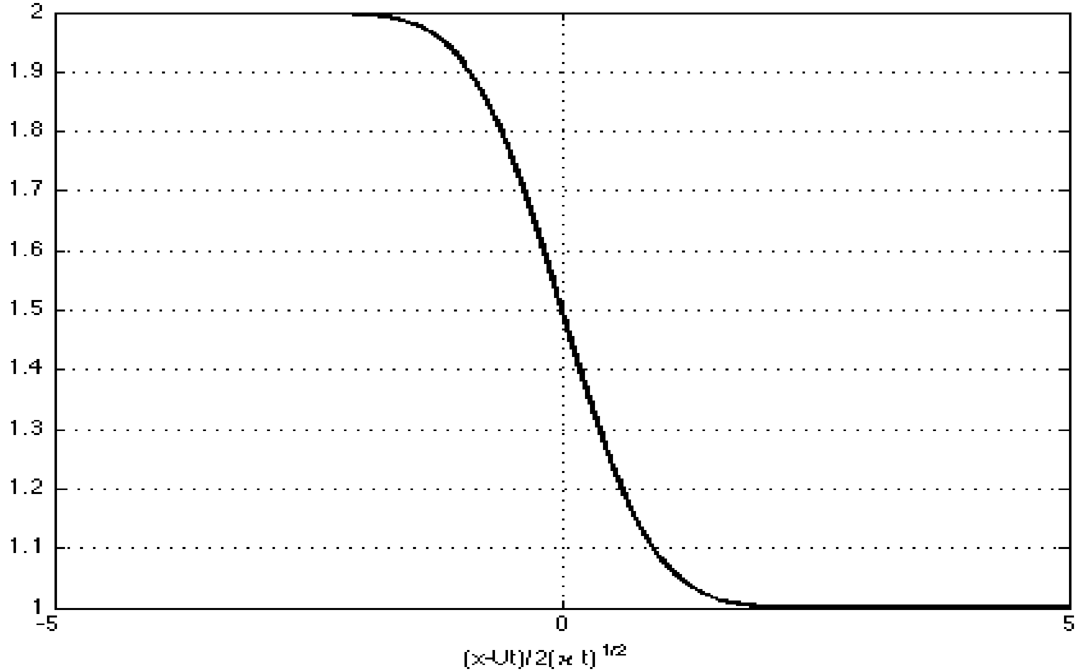
$$\xi = x - Ut$$
$$\tau = t$$

$$-\infty < \xi < \infty$$

In frame moving with the temperature front

$$\frac{\partial T}{\partial \tau} = \kappa \frac{\partial^2 T}{\partial \xi^2} \longrightarrow T = \frac{T_O - T_I}{2} \operatorname{erf}\left(\xi \big/ 2\sqrt{\kappa\tau} \right) + \frac{T_O + T_I}{2}$$

The internal boundary layer



Thickness of layer $\delta : \sqrt{\kappa t}$

The ventilated thermocline

We are going to apply these ideas to the oceanic thermocline in the sub tropical gyres where the wind stress produces a *downward* Ekman pumping.

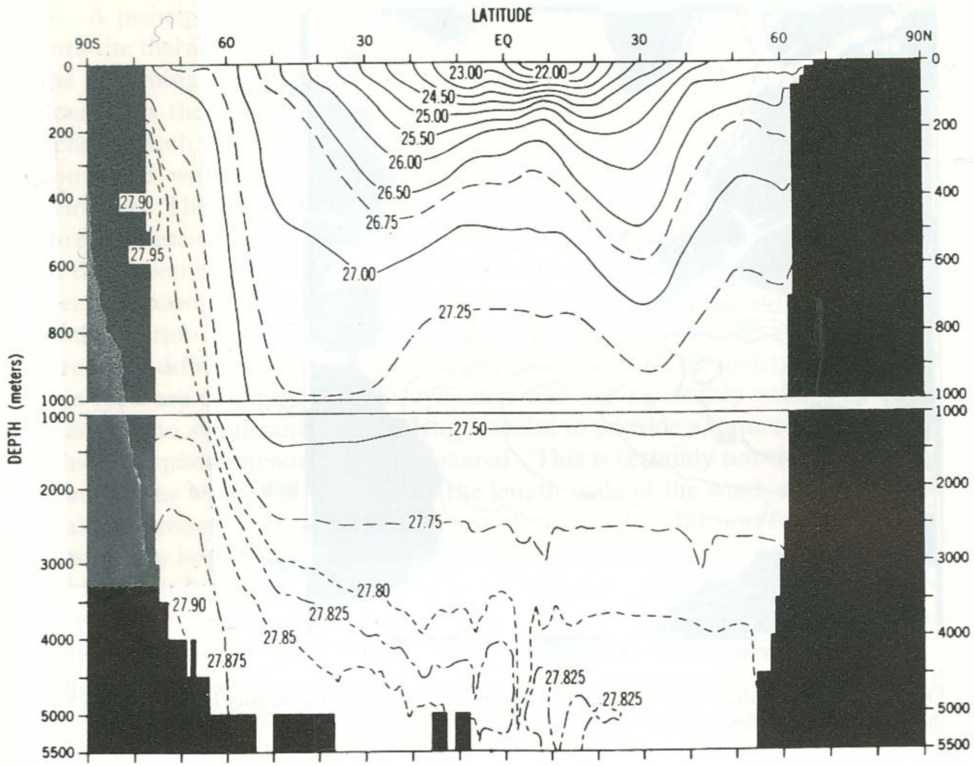


Fig. 4.1.2. Zonally averaged potential density field for the Pacific Ocean. Note the change of vertical scale at 1000 m, reflecting the decrease in gradient below this depth. (From Levitus 1982)

In analogy with the pipe, the bowl shaped region is where fluid enters the thermocline and is pumped downward carrying the surface density distribution to depth.

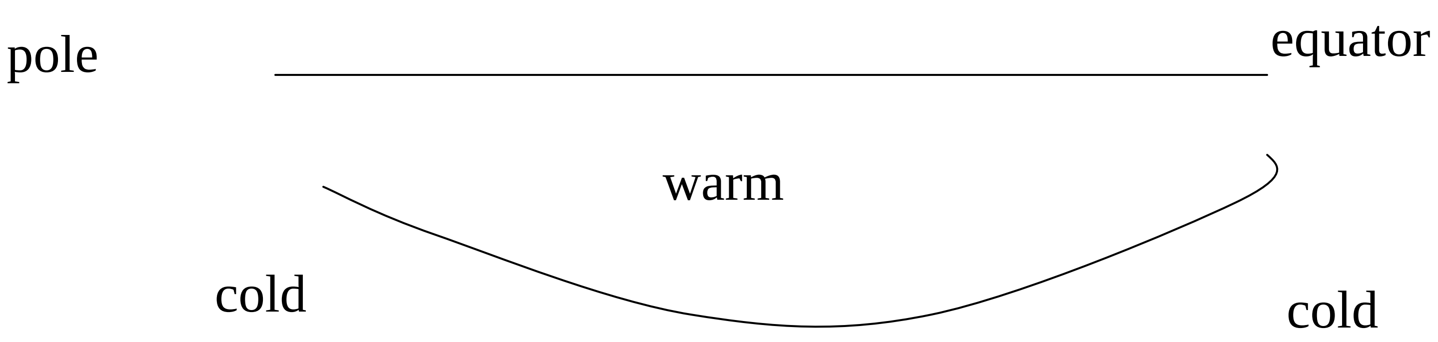
Some of the fluid flows to the equator, a region requiring special dynamics.

The deep water beneath the thermocline is water of polar origin that slowly upwells to establish a temperature contrast with the thermocline.

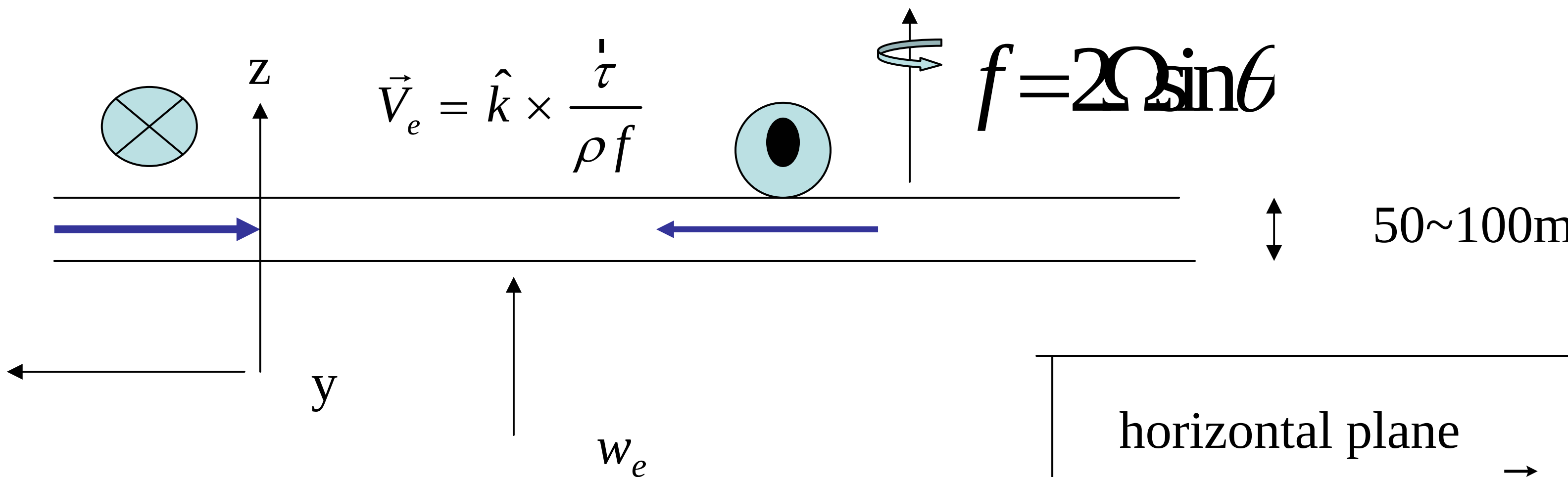
Two principal questions (at least)

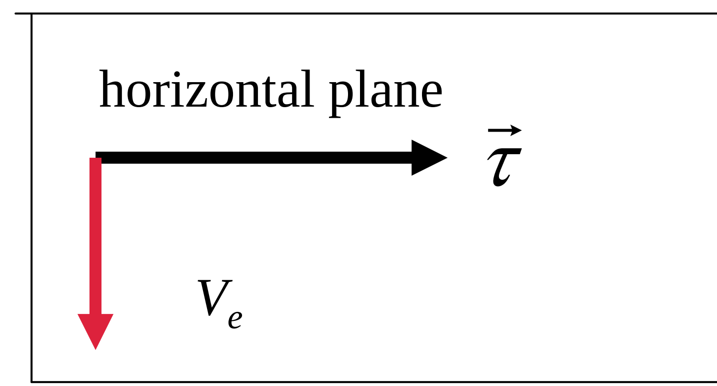
1) Why does the surface density forcing extend *only* to about 1km?

2) Why does the bowl become shallow at *low* latitudes?



Ekman Pumping



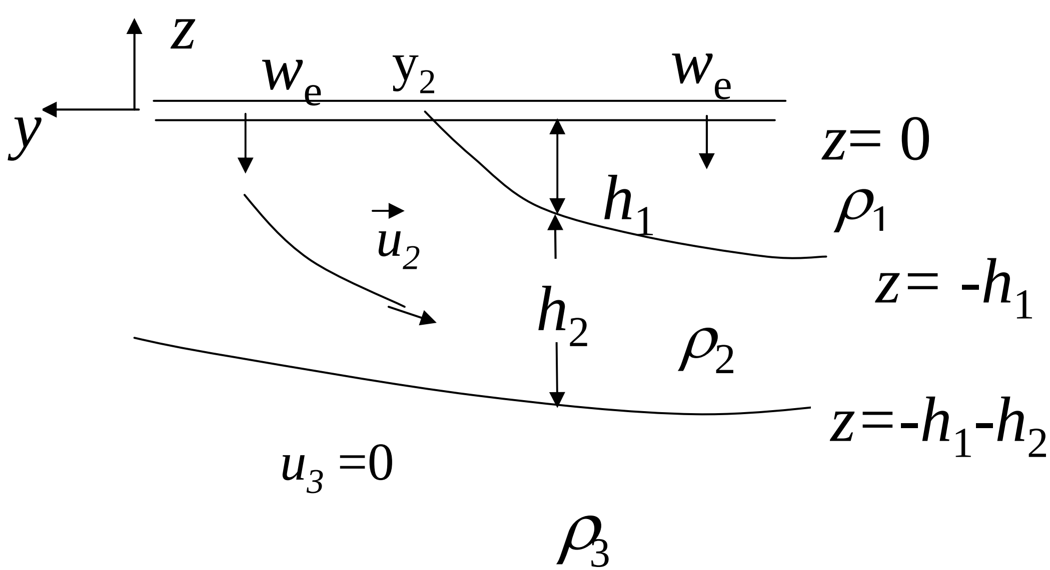


$$w_e = \hat{k}\cdot\nabla\times\frac{\vec{\tau}}{\rho f}$$

This drives the entire wind driven circulation. W_e is O(10^{-4} cm/sec)

The bowl

We first need to explain the structure of the bowl.
(why is it *shallow* at the equator?). The LPS model:



Ref Luyten, J.R., J. Pedlosky, and H. Stommel. 1983 The ventilated thermocline. *J. Phys. Ocean*. **13**, 292-309.

Planetary geostrophy

$U/\beta L^2$ <<1 for large scales (greater than about 50-100 km)

$$f = 2\Omega \sin\theta$$

$$\rho_n f u_n = -\frac{\partial p_n}{\partial y},$$

$$\rho_n f v_n = \frac{\partial p_n}{\partial x},$$

$$\beta v_n = f\frac{\partial w_n}{\partial z} \qquad \beta = {df}/{dy}$$

$$\frac{\partial u_n}{\partial x} + \frac{\partial v_n}{\partial y} + \frac{\partial w_n}{\partial z} = 0.$$

Integrating over all moving layers:

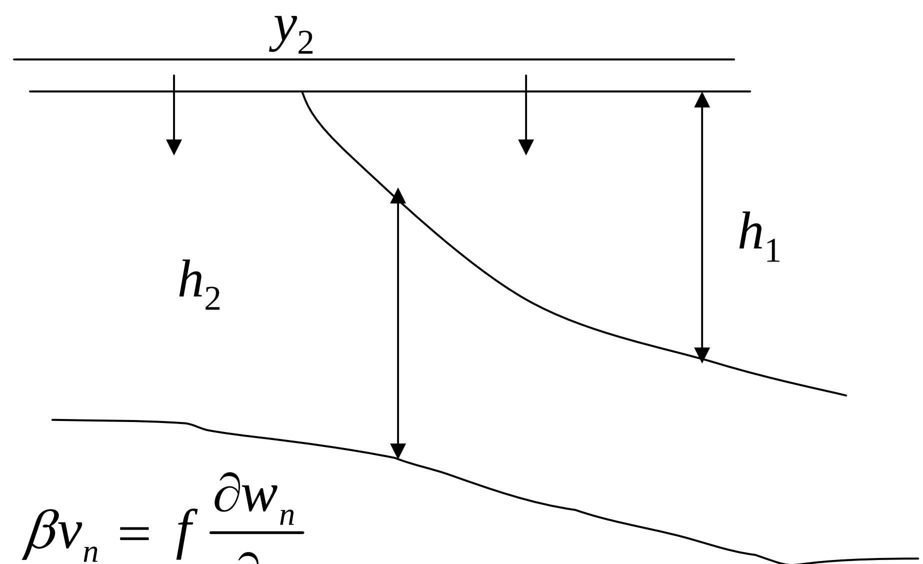
$$\rho_n g = -\frac{\partial p_n}{\partial z}$$

$$\beta \sum_n v_n h_n = f w_e$$

Continuity of w_n

Layer equation for mass conservation

$$(u_2h_2)_x + (v_2h_2)_y = -w_e \quad y > y_2$$
$$= 0 \quad y < y_2$$



$$(u_1h_1)_x + (v_1h_1)_y = -w_e \quad y < y_2$$

from integrating over each layer

$$\beta v_n = f\frac{\partial w_n}{\partial z}$$

$$\beta v_n = -f(u_{nx} + v_{ny})$$

$$= \frac{f}{h_n}\vec{u}_n \cdot \nabla h_n + f\frac{w_e}{h_n} \quad \begin{cases} y \geq y_2, n = 2 \\ y \leq y_2, n = 1 \\ else \quad 0 \end{cases}$$

yields the potential vorticity equation for each layer

Potential vorticity equations

$$u_2 \frac{\partial}{\partial x}\left(\frac{f}{h_2}\right) + v_2 \frac{\partial}{\partial y}\left(\frac{f}{h_2}\right) = \frac{f}{h_2{}^2} w_e \Theta(y - y_2),$$

$$u_1 \frac{\partial}{\partial x}\left(\frac{f}{h_1}\right) + v_1 \frac{\partial}{\partial y}\left(\frac{f}{h_1}\right) = \frac{f}{h_1{}^2} w_e \Theta(y_2 - y)$$

$\frac{f}{h_n}$ is the potential vorticity of layer n

$$\begin{aligned} \Theta(x) &= 1, \quad x > 0 \\ &= 0, \quad x < 0 \end{aligned}$$

Is Heavyside fnc.

Geostrophy and hydrostatic relation with layer 3 at rest.

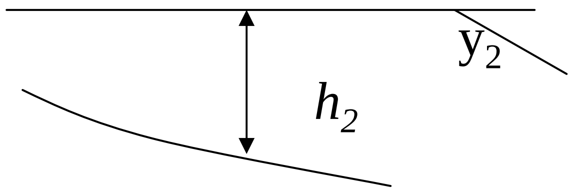
$$fu_2 = -\gamma_2 \frac{\partial h}{\partial y},$$

$$fv_2 = \gamma_2 \frac{\partial h}{\partial x}.$$

$$\boxed{h = h_1 + h_2,} \qquad \gamma_2 = \frac{\rho_3 - \rho_2}{\rho_o} g$$

Single moving layer region, $y > y_2$

$$\beta v_2 h_2 = f w_e \qquad v_2 = \frac{\gamma_2}{f}\frac{\partial h}{\partial x}, \qquad h_2 = h, \quad h_1 = 0$$

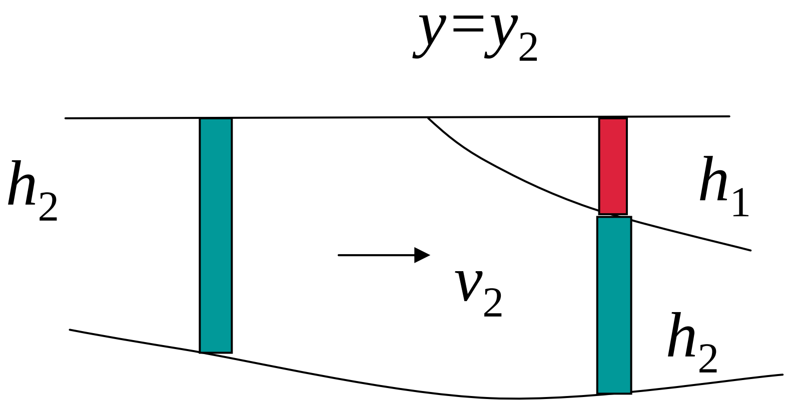


$$\frac{\partial {h_2}^2}{\partial x} = \frac{2 f^2}{\beta \gamma_2} w_e$$

Integrate to eastern boundary $x = x_e$, where u_2 =0. We will satisfy that bc by taking h_2 =0 there (not necessary, it need only be a constant but it suffices for our purposes)

$$ {h_2}^2 = -\frac{2 f^2}{\beta\gamma_2}\int_x^{x_e} w_e(x',y)\,dx' \qquad y \geq y_2$$

The process of subduction



Fluid in layer 2 is driven southward by Ekman pumping.

At $y = y_2$ it *subducts* beneath layer 1.

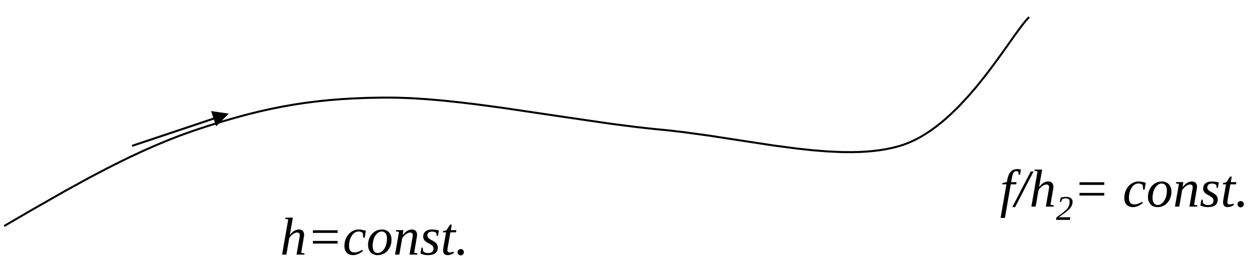
It is then no longer driven directly by the Ekman pumping which directly forces layer 1 in that region south of y_2

Conservation of potential vorticity

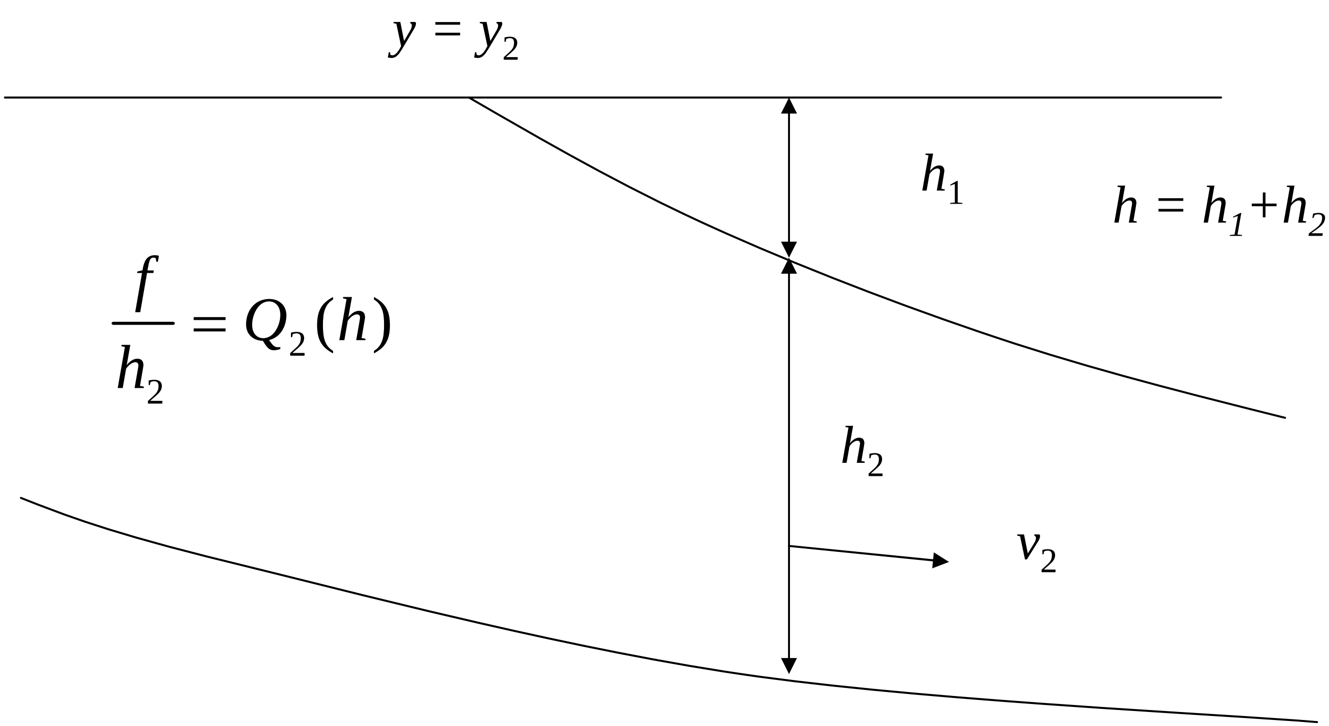
Streamlines in layer 2 are coincident with pressure field. (Geostrophy). In layer 2 this means the streamlines are lines of constant $h=h_1+h_2,$ For $y<y_2$

$$\vec{u}_2 \cdot \nabla \frac{f}{h_2} = 0 \quad \longrightarrow \quad \boxed{\frac{f}{h_2} = Q_2(h)}$$

Potential vorticity is an arbitrary function of streamline.



The determination of the function Q_2

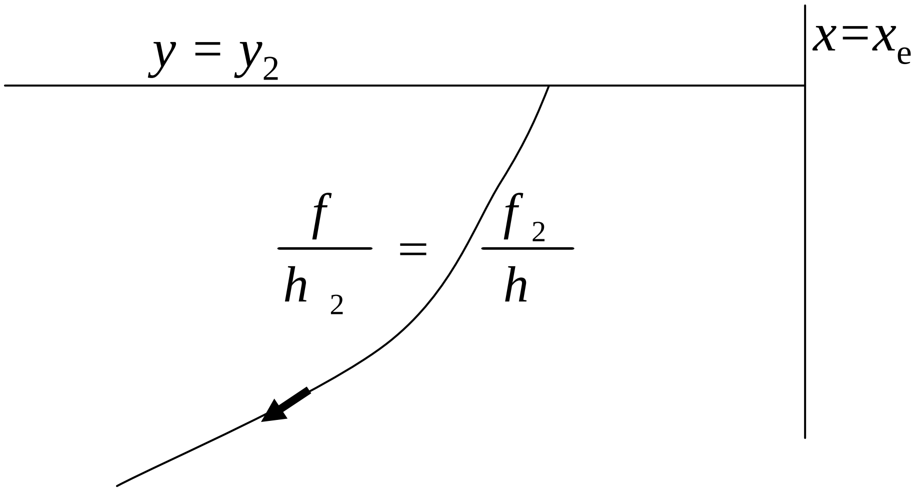


$$\frac{f}{h_2} = Q_2(h)$$

By definition at $y = y_2$, $h_2 = h$ so on that line

$$\frac{f}{h_2} = \frac{f_2}{h}$$

The conserved relation



On streamline h = const. the relationship established at the outcrop line is maintained and is valid for all points reached by streamlines emanating from the outcrop line.

The layer thicknesses in the 2-layer region

$$\frac{f}{h_2} = \frac{f_2}{h}$$

$$h_2 = \frac{f}{f_2} h ,$$

$$h_1 = \left(1 - \frac{f}{f_2} \right) h$$

From geostrophy

$$u_1 = -\frac{1}{f} \frac{\partial}{\partial y} \left(\gamma_2 h + \gamma_1 h_1 \right), \qquad f u_2 = -\gamma_2 \frac{\partial h}{\partial y},$$

$$v_1 = \frac{1}{f} \frac{\partial}{\partial x} (\gamma_2 h + \gamma_1 h_1) \qquad f v_2 = \gamma_2 \frac{\partial h}{\partial x}.$$

$$\gamma_1 = g \frac{\rho_2 - \rho_1}{\rho_O} \qquad h = h_1 + h_2, \qquad \gamma_2 = \frac{\rho_3 - \rho_2}{\rho_o} g$$

The Sverdrup relation

$$\beta \sum_n v_n h_n = f w_e$$ With geostrophy, yields,

$$\frac{\partial}{\partial x}\left(h^2 + \frac{\gamma_1}{\gamma_2} h_1^{\ 2} \right) = \frac{2f^2}{\beta\gamma_2} w_e(x,y)$$

$$h^2 + \frac{\gamma_1}{\gamma_2} h_1^{\ 2} = -\frac{2f^2}{\beta\gamma_2} \int_x^{x_e} w_e(x',y)\,dx'$$

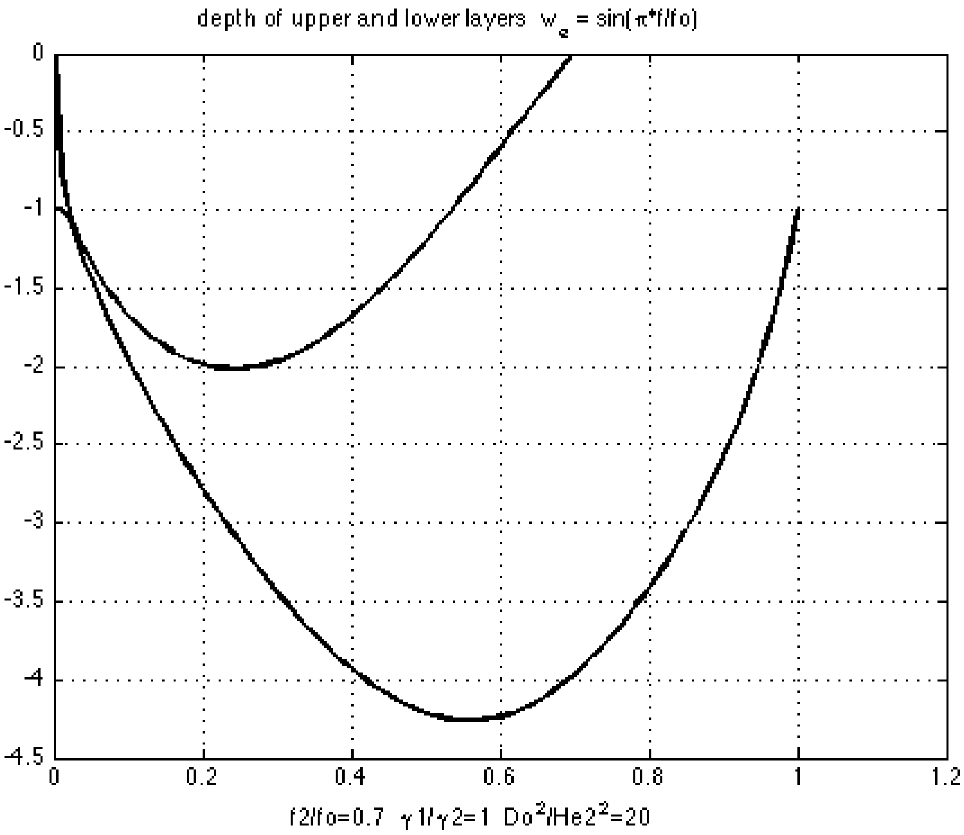
Then with pv conservation

$$h_1 = \left(1 - \frac{f}{f_2} \right) h$$

$$\boxed{h = \frac{\left(D_o^{\ 2}\right)^{1/2}}{\left(1 + \frac{\gamma_1}{\gamma_2}\left[1 - f/f_2\right]^2\right)^{1/2}}}$$

$$D_o^2 = -2\frac{f^2}{\gamma_2\beta} \int_x^{x_e} w_e(x'.y)\,dx' \geq 0$$

The thermocline bowl

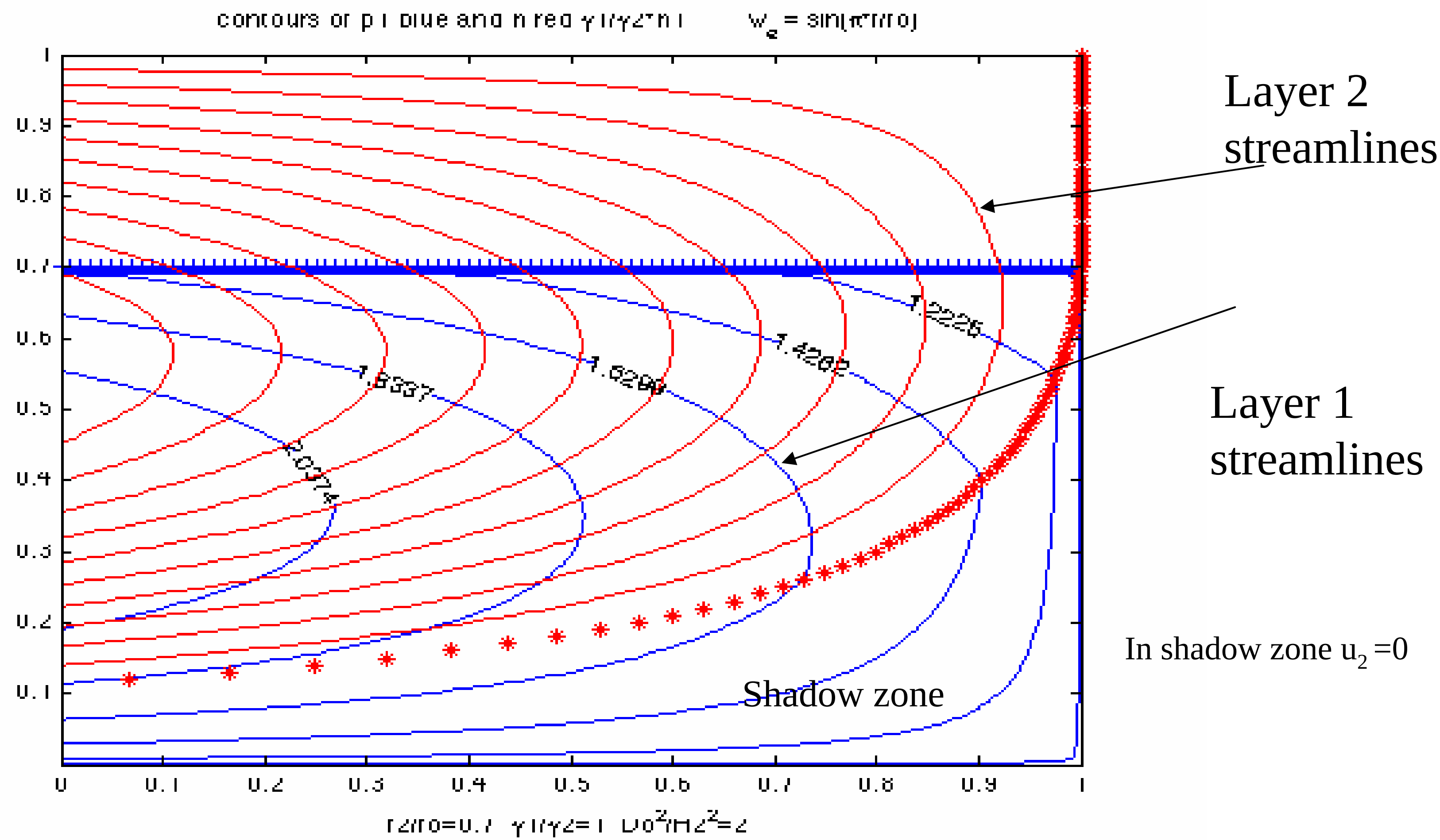


$$D_o^2 = (x_e - x)\frac{2}{\rho \gamma_2}\left\{\frac{\partial}{\partial y}\frac{\tau f}{\beta} - \tau\right\}$$

Layer thicknesses remain finite as latitude goes to zero.

$y \longrightarrow$

The horizontal circulation
(with shadow zone)



In shadow zone $u_2 = 0$

Further references

Rhines, P.B. and W.R.Young, 1982. A theory of the wind-driven circulation.I.Mid-Ocean gyres. *J. Marine Res.* **40** (Supplement) 559-596

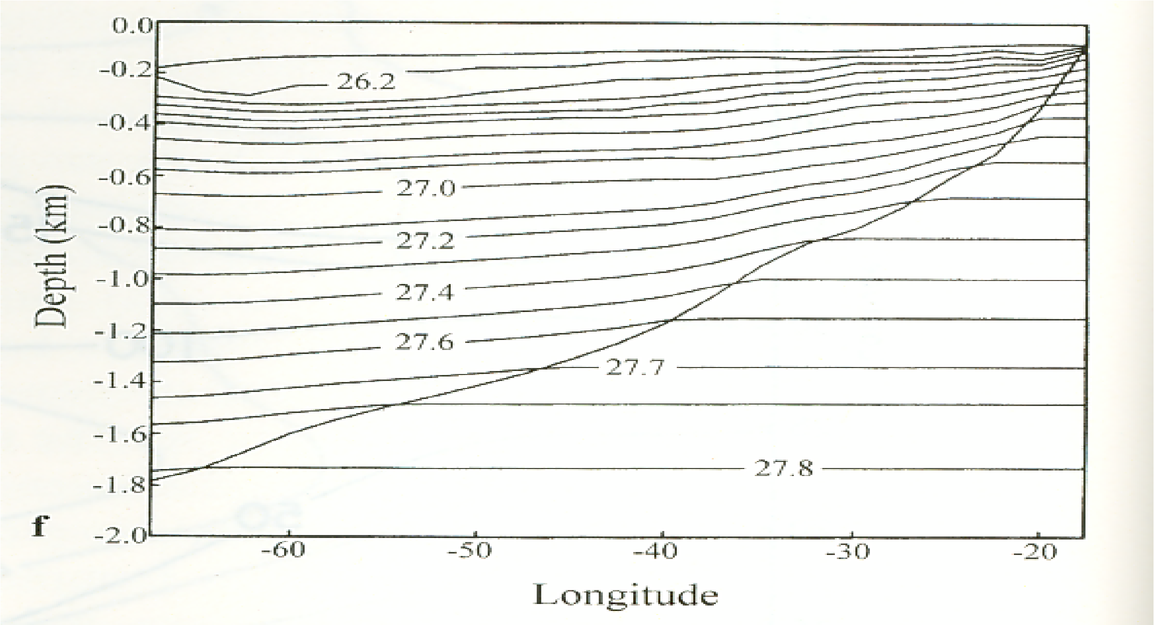
Pedlosky, J. and W.R. Young 1983 Ventilation, potential vorticity homogenization and the structure of the ocean circulation. *J.Phys.Ocean.* **13**, 2020-2037

Pedlosky, J. *Ocean Circulation Theory*. 1998. *Springer Verlag*. pp 453

Huang, R.X. (1989) On the three dimensional structure of the wind-driven thermocline in the North Atlantic. *Dyn. Atmos. and Oceans*, **15**, 117-159.

The “continuous”model

Adding more layers one approaches a high resolution finite difference form of the solution of the continuous model in *z* as Huang (1989) has done.



The equatorial inertial boundary layer

$$D_o^2 = (x_e - x)\frac{2}{\rho\gamma_2}\left\{\frac{\partial\tau}{\partial y}\frac{f}{\beta} - \tau\right\}$$

Dominates as f goes to zero

So the layer thicknesses remain finite as y and f go to zero at the equator

$$h^2 + \frac{\gamma_1}{\gamma_2}h_1^2 = D_o^2$$

But

$$v_2 = \frac{\gamma_2}{f}\frac{\partial h}{\partial x}$$

becomes singular at equator

$$v_1 h_1 + v_2 h_2 = \frac{\tau}{\rho_o f}$$

"geostrophic" transport equal and opposite to Ekman transport. Need to remove the singularity in v

Equations of motion in the equatorial region. Layer model.

$$u_n \frac{\partial v_n}{\partial x} + v_n \frac{\partial v_n}{\partial y} + \beta y u_n = -\frac{1}{\rho_o}\frac{\partial p_n}{\partial y}$$

$$f = \beta y$$

$$u_n \frac{\partial u_n}{\partial x} + v_n \frac{\partial u_n}{\partial y} - \beta y v_n = -\frac{1}{\rho_o}\frac{\partial p_n}{\partial x},$$

$$\frac{\partial}{\partial x}(u_n h_n) + \frac{\partial}{\partial y}(v_n h_n) = 0$$

Scaling

$$x = Lx' \qquad (u,v) = U(u', \frac{l}{L}v')$$

$$y = l y' \qquad p = \rho_o \beta l^2 U p'$$

$$h = Hh'$$

Pierre Welander

Pierre Welander
the young scientist

Pierre Welander
enjoying retirement

Veronis: *J.Marine Res.*1997,**55**, i-vii

Equations of motion
Non dimensional

dropping primes

$$\frac{U}{\beta\lambda^2}\frac{\lambda^2}{L^2}\left(u_n\frac{\partial v_n}{\partial x}+v_n\frac{\partial v_n}{\partial y}\right)+yu_n=-\frac{\partial p_n}{\partial y}$$

$$\frac{U}{\beta\lambda^2}\left(u_n\frac{\partial u_n}{\partial x}+v_n\frac{\partial u_n}{\partial y}\right)-yv_n=-\frac{1}{\rho_o}\frac{\partial p_n}{\partial x},$$

As we approach the equator need to keep advective terms in second equation to heal singularity in v_n

$$\frac{\partial}{\partial x}(u_n h_n)+\frac{\partial}{\partial y}(v_n h_n)=0$$

$$U=\beta\lambda^2$$

Note zonal velocity remains in geostrophic balance

Pressure-depth scaling

From geostrophy of u and the hydrostatic balance.

$$p = O(\rho_o U \beta \ell^2) = \rho_o \beta^2 \ell^4 = O(\gamma_2 H)$$

so

$$H = \frac{U\beta \ell^2}{\gamma_2} = \frac{\beta^2 \ell^4}{\gamma_2}$$

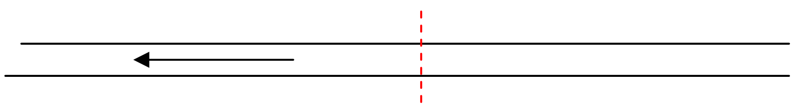
$$D_o^2 = (x_e - x)\frac{2}{\rho\gamma_2}\left\{\frac{\partial \tau}{\partial y}\frac{f}{\beta} - \tau\right\}$$

Matching to the ventilated thermocline solution in the matching region as we leave the equatorial zone.

$$\longrightarrow \qquad H^2 = \frac{\tau_o L}{\rho_o \gamma_2}$$

Further scaling

From the balance of transport between the Ekman layer and the equatorial thermocline:



y = 0

$$V_e = \tau / \rho f \qquad f = \beta \ell$$

With geostrophic balance for *U*

$$V_g H = U \frac{\ell}{L} H = -V_e$$

$$\rho f U \ell = p = \rho \gamma H$$

$$\therefore$$

$$\tau L = \rho \gamma H^2$$

Work potential energy balance

Scaling results

$$\ell = \left(\frac{\gamma_2 \tau_o L}{\rho_o \beta^4} \right)^{1/8}$$ ← 200 km << L

$$H = \left(\frac{\tau_o L}{\gamma_2 \rho_o} \right)^{1/2}$$ ← 100 meters

$$U = \left(\frac{\gamma_2 \tau_o L}{\rho_o} \right)^{1/4}$$ ← 1 meter/second

Pedlosky, J. An inertial theory of the equatorial undercurrent. *J. Phys. Ocean.* **17**, 1978-1985

Scaled boundary layer equations (1)

$$-(y+\zeta_n)v_n = -\frac{\partial B_n}{\partial x}, \qquad \zeta_n = -\frac{\partial u_n}{\partial y},$$

Relative vorticity dominated by $-\dfrac{\partial u}{\partial y}$

$$B_n = p_n + \frac{1}{2}u_n^{\ 2},$$

Stream function for layers beneath the surface.

$$\boxed{h_n\vec{u}_n = \hat{k}\times\nabla\psi_n,}$$

thus

$$q_n\frac{\partial\psi_n}{\partial x} = \frac{\partial B_n}{\partial x}$$

where

$$q_n = \frac{y - \partial u_n/\partial y}{h_n}$$

Scaled boundary layer equations (2)

$$yu_n = -\frac{\partial p_n}{\partial y}$$

Zonal geostrophy

Equivalent to $$q_n \frac{\partial \psi_n}{\partial y} = \frac{\partial B_n}{\partial y}$$

$$q_n \nabla \psi_n = \nabla B_n$$

Conservation of B_n and q_n

from $$q_n \nabla \psi_n = \nabla B_n$$

$$\longrightarrow \quad q_n \overset{=0}{\boxed{\cancel{\vec{u}_n \cdot \nabla \psi_n}}} = \vec{u}_n \cdot \nabla B_n = 0$$

$$\nabla q_n \times \nabla \psi_n = 0 \quad \longrightarrow \quad \vec{u}_n \cdot \nabla q_n = 0$$

$$\boxed{q_n = Q_n(B_n)}$$

The equation of motion for layer 2

$p_2 = h, \quad p_1 = h + \Gamma_{12} h_1,$ Hydrostatic relation (n.d.)

$$\Gamma_{12} = \frac{\gamma_1}{\gamma_2}$$

conservation of pv and B_2

$$\frac{y - \partial u_2 / \partial y}{h_2} = Q_2\left(h + \frac{1}{2}u_2^{\,2}\right)$$

geostrophic balance for u_2

$$\frac{\partial h}{\partial y} = -y u_2$$

The determination of Q_2
The link to mid-latitudes.

For large y (bdy layer coordinate) must merge with mid latitude dynamics

$$q_2 \approx \frac{y}{h_2}, \; B_2 \approx h$$

From the ventilated thermocline solution

$$\frac{y}{h_2} = Q_2(h) = \frac{y_2}{h}$$

thus $$\boxed{Q_2(B_2) = \frac{y_2}{B_2}}$$

$$\longrightarrow \quad \boxed{\frac{y - \partial u_2 / \partial y}{h_2} = \frac{y_2}{h + \frac{1}{2} u_2{}^2}}$$

The boundary layer differential equations

$$\frac{\partial u_2}{\partial y} = y - \frac{y_2 h_2}{h + \frac{1}{2} u_2^{\ 2}},$$

ODE s only in y

$$\frac{\partial h}{\partial y} = -y u_2$$

need a relation between h and h_2
but Sverdrup relation no longer valid.

$$h_2 = h - h_1$$

Two closures: both sketchy

let y_n>>1 be the northern latitude where the solution merges with the mid-latitude solution.

One closure assumes: $h_1(x,y) = h_1(x,y_n)$ for all y.

The second assumes

$$h_1(x,y) = h_1(x,y_n) + \{h(x,y_n) - h(x,y)\}/\Gamma_{12}$$

or upper layer *pressure gradient* independent of y

Boundary condition at the equator

Fluid does not cross equator (pv conserved).

On y =0 B_2 =const. = B_o

and B_o = $h(0,y_n)$

$$(u_2 h_2) = -\frac{1}{q_2}\frac{\partial B_2}{\partial y} = -\frac{\partial}{\partial y}\left(\frac{B_2^2}{2y_2}\right)$$

$$\left.\int_0^{y_n} u_2 h_2 \, dy\right]_{x=0} = \frac{B_o^2 - h^2(0,y_n)}{2y_2}$$

Solutions

Solutions obtained by a shooting method. Starting with *h* from the mid-latitude solution, guess a starting u_2 at $y=y_n$and integrate to the equation and try to "hit" B_o. Then adjust guess for u_2.

With the first closure

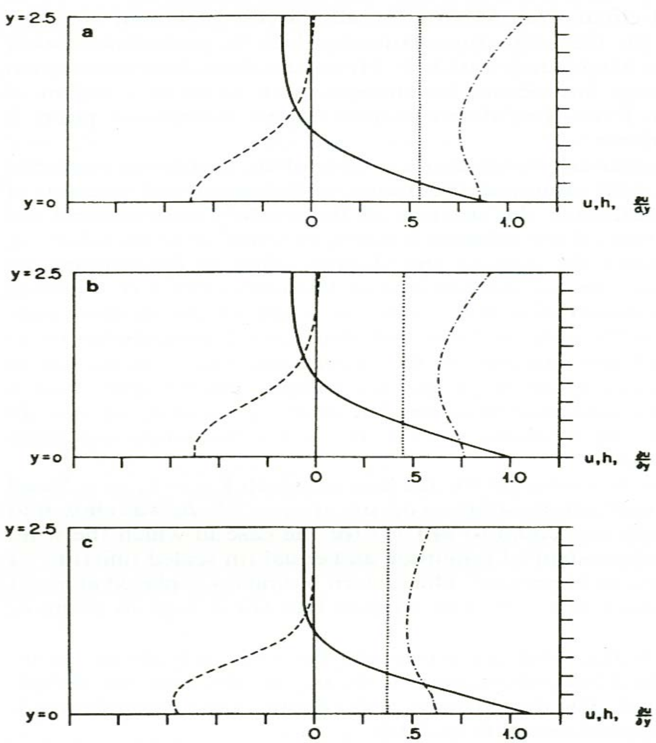
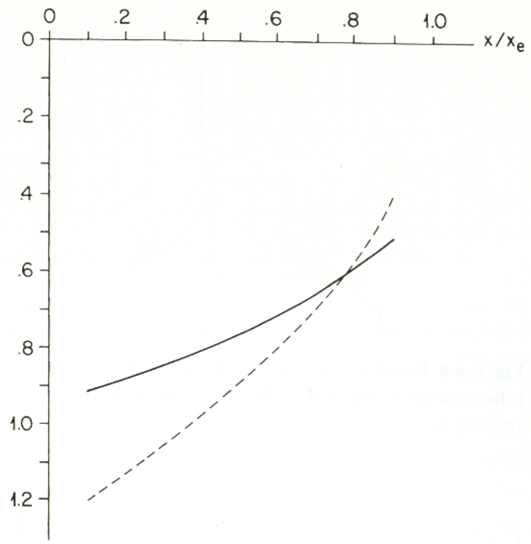


Fig. 6.4.2. Solutions of (6.4.26) for u_2 (*solid line*), $\partial u_2/\partial y$ (*dashed line*) and h (*dash-dotted*). In this case the wind stress is a constant and $\Gamma_{12} = 1$. The three panels correspond to profiles at $x = 0.25$, 0.50, and 0.75, respectively. $B_0 = 1.265$ and $y_2 = 5$. (From Pedlosky 1987)

The equatorial thermocline

Fig. 6.4.3. Base of the moving layer representing the core of the undercurrent shown as a *solid line* at the equator and as a *dashed line* in the matching region at $y = y_n$. (From Pedlosky 1987)



Results with second closure

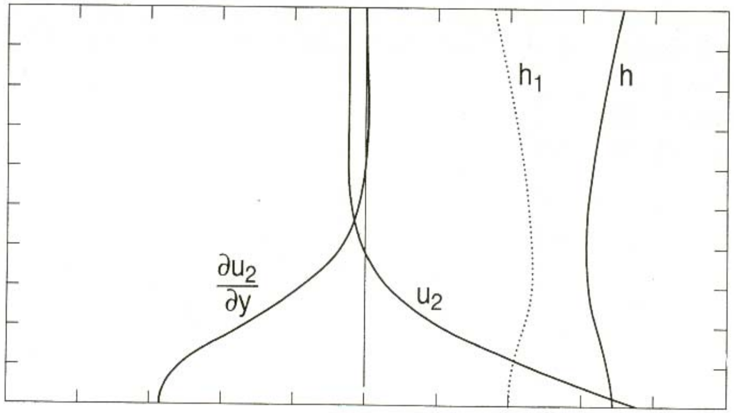


Fig. 6.4.4. Profiles of u_2, $\partial u_2/\partial y$, h, and h_1 for the case in which (6.4.19) is used. The parameters are otherwise as in Fig. 6.4.2. The calculation is at $x = 0.5$. The maximum velocity of the eastward velocity is now 0.910

The link to mid-latitudes

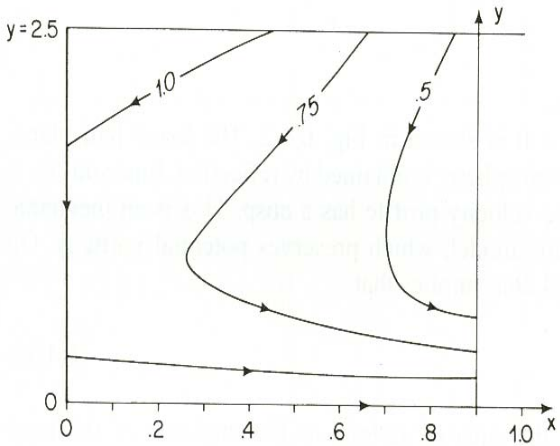
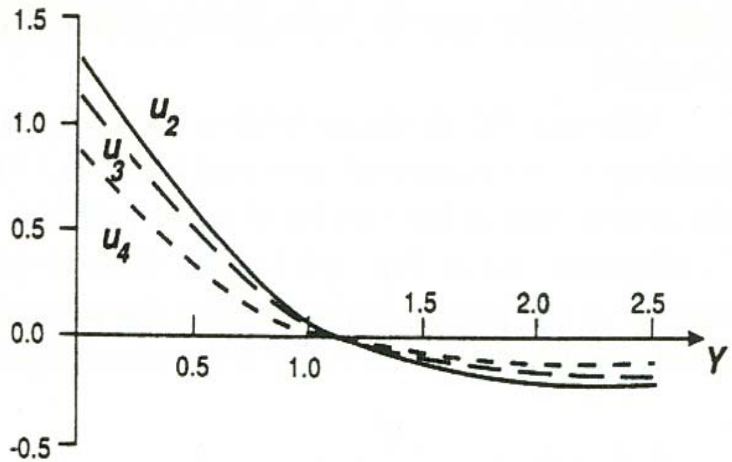


Fig. 6.4.5. Several lines of constant B_2, i.e., streamlines for the flow calculated in Fig. 6.4.2. (From Pedlosky 1987)

A multi-layer model

Fig. 6.4.6. Results of a four-layer model showing the monotonic decrease of the velocity with depth in the undercurrent solution. (Courtesy of R. Samelson, pers. comm.)



EUC observations

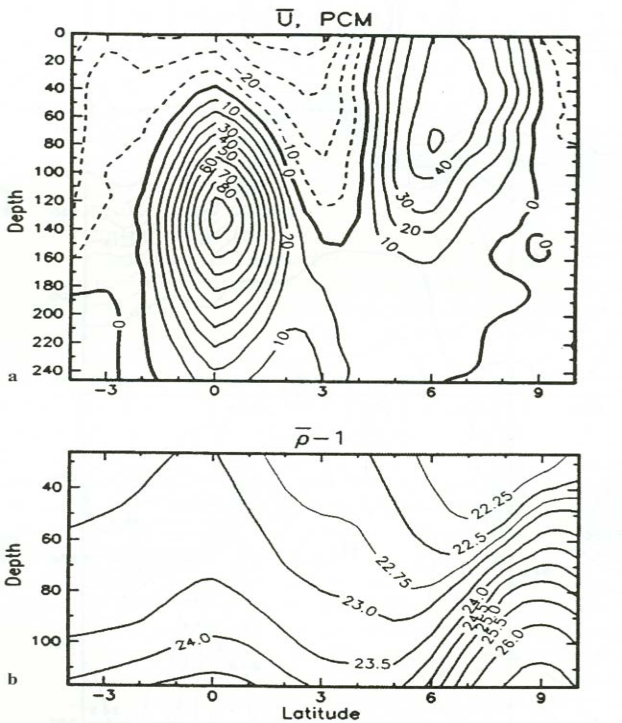


Fig. 6.1.2. a Contours of zonal velocity in the EUC measured dire
current meter in the Pacific at the same longitude as Fig. 6.1.1. N
between the actual and geostrophic velocities. **b** The density field m
Note that the meridional density gradient vanishes at the equator. (From Johnson and Luther 1994)

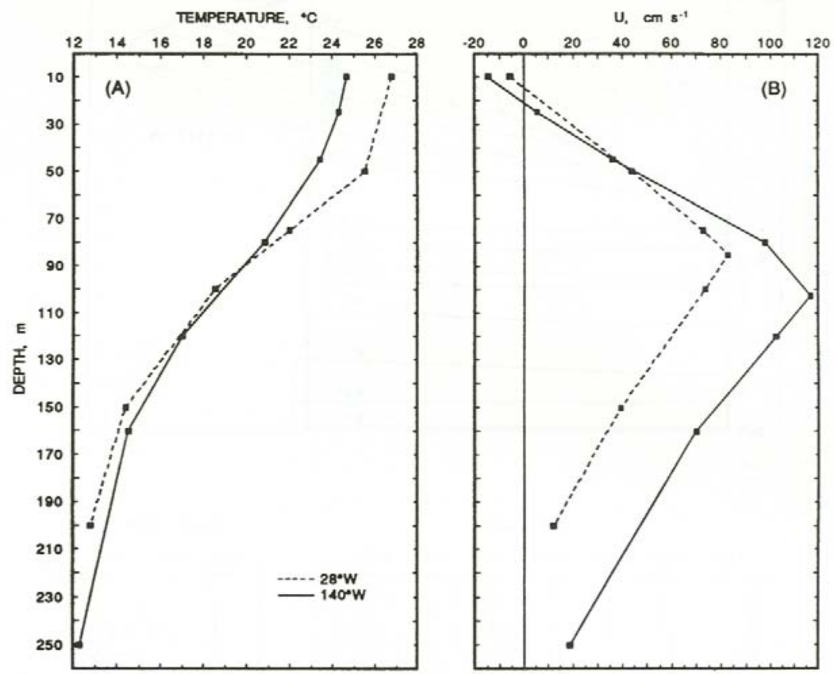


Fig. 6.1.3. Temperature and zonal velocity profiles from the Atlantic and Pacific oceans. In each case the measurements represent 2-year means. (From Halpern and Weisberg 1989)

Numerical calculations

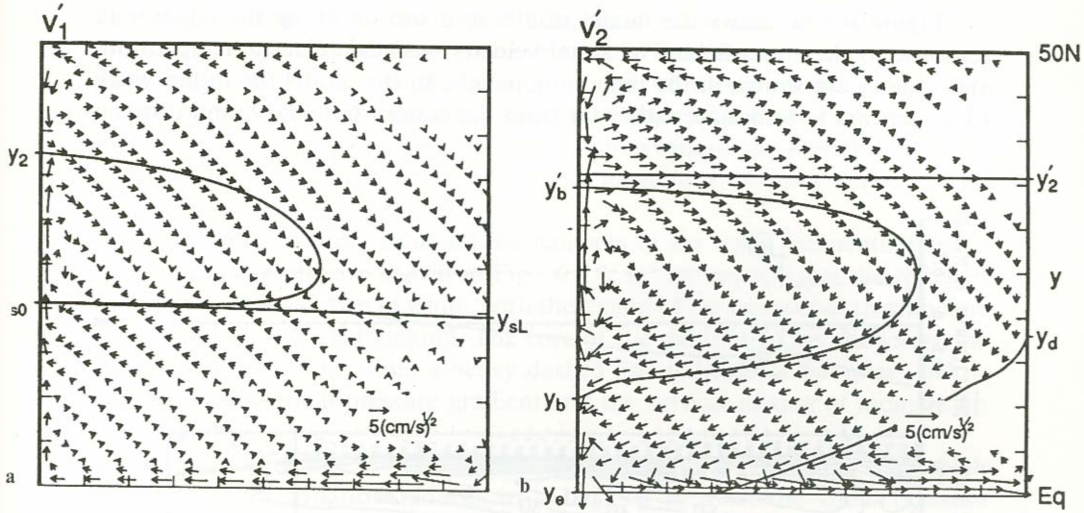
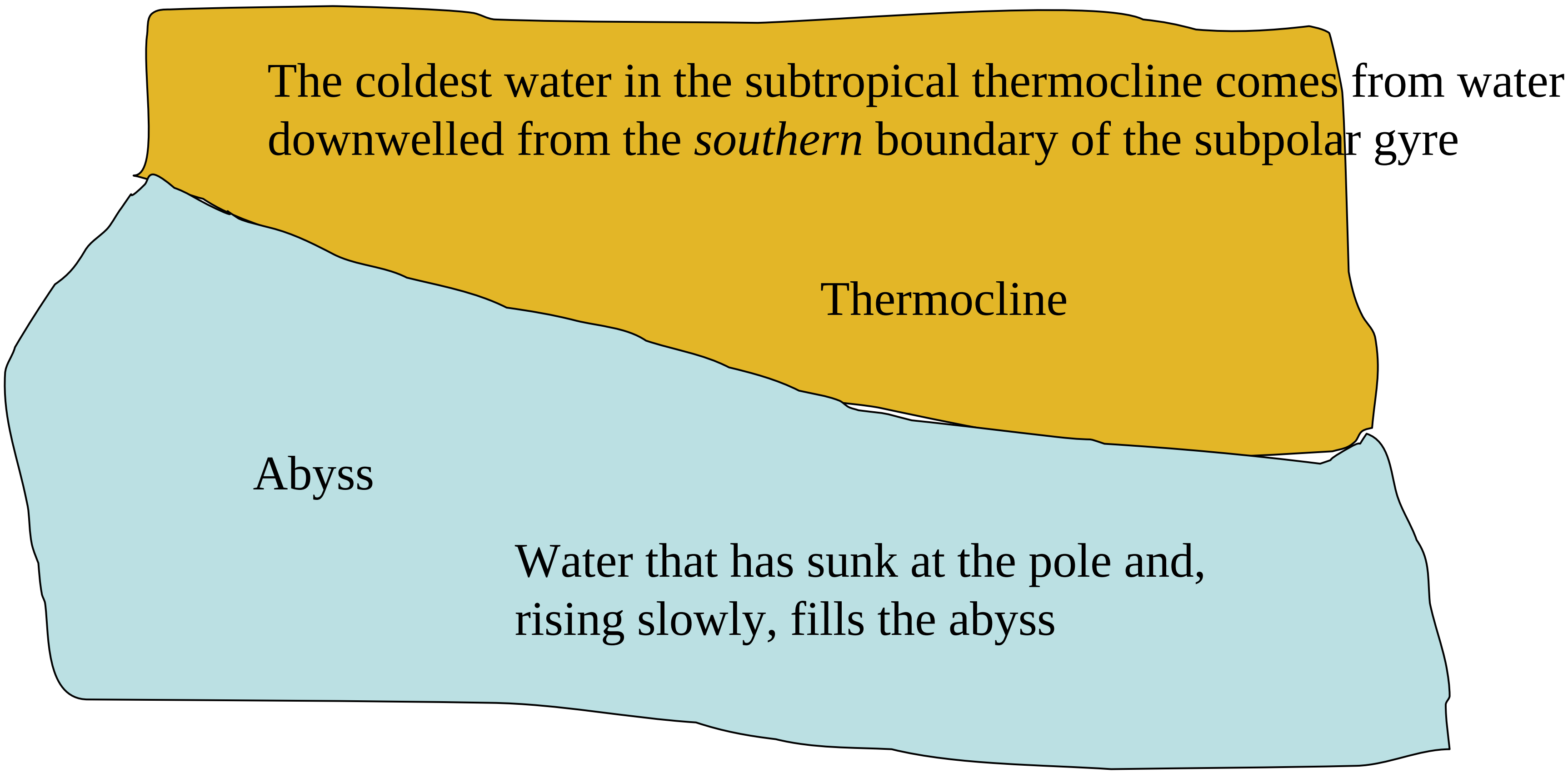


Fig. 6.7.2a,b. As in Fig. 6.7.1 except that the parameters of the calculation yield a shadow zone boundary which strikes the equator within the basin. In this case the EUC is fed from the subtropical gyre through the interior as well as the western boundary current. (From McCreary and Lu 1994)

McCreary, J.P. Jr. and P. Lu 1994. The interaction between the subtropical and equatorial circulations: The subtropical cell. *J.Phys. Ocean.*, **24**, 466-497

An Internal boundary layer in the thermocline



The diffusive internal layer refs:

To smooth out the temperature difference between the abyss and the thermocline a diffusive layer might be expected as anticipated by Welander

Welander, P. 1971 The thermocline problem, *Phil. Trans. R. Soc. Lond. A.* **270**, 415-421

Salmon, R. 1990. The thermocline as an “internal boundary layer”. *J. Marine. Res.*, **48**, 437-469.

Samelson, R.M. and G.K. Vallis, 1997. Large-scale circulation with small diapycnal diffusion: The two thermocline limit. *J. Marine Res.* , **55**, 223-275.

Boundary layer equations

Following Samelson and Vallis (1997)

$$-fv = -\frac{\partial p}{\partial x} - \varepsilon u,$$

$$b = -\frac{\rho - \rho_o}{\rho_o} g$$

$$fu = -\frac{\partial p}{\partial y} - \varepsilon v,$$

$$b_t + ub_x + vb_y + wb_z = \kappa_v b_{zz} + \kappa_H \nabla^2 b - \lambda \nabla^4 b$$

$$\frac{\partial p}{\partial z} = b,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The Result of the S&V calculation

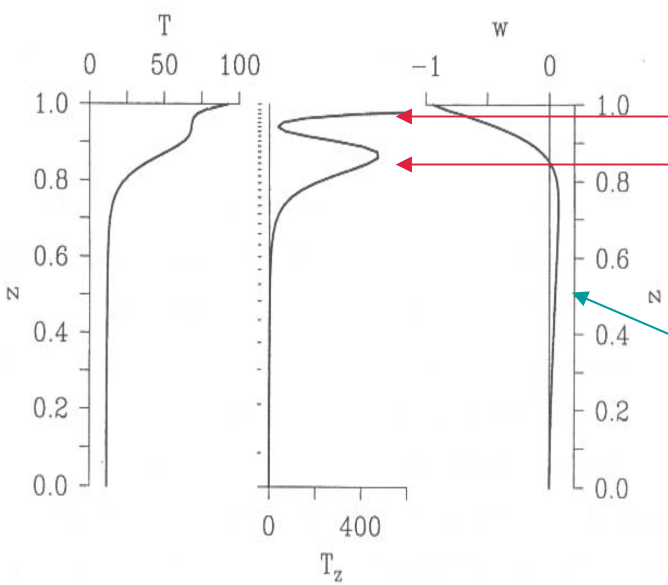


Figure 4. Vertical profiles of T (left panel), T_z (center), and w (right) at the center of the domain, $(x, y) = (0.5, 0.5)$, for the solution in Figure 2.

Double structure to thermocline and an interior maximum in T_z

Upper maximum is the ventilated thermocline contribution.Note deep *positive* w in the abyss with a zero at the base of the thermocline

Scaling the internal thermocline (1)

from $$\frac{\partial w}{\partial z} = \frac{\beta}{f} v$$

since w is 0 at the base of the adiabatic, ventilated thermocline.

$$W = \frac{\beta}{f} U \delta$$

The horizontal gradient of buoyancy is determined by the slope of the isopycnals in the ventilated thermocline solution

From thermal wind $$u_z = -b_y / f$$

$$\Delta b / L = f U / D_a$$

D_a is the vertical scale of the adiabatic thermocline

$$D_a^2 : \frac{f^2 W_e L}{\beta \Delta b}$$

Scaling the internal thermocline (2)

In the diffusive region of the internal thermocline, vertical diffusion balances vertical advection.

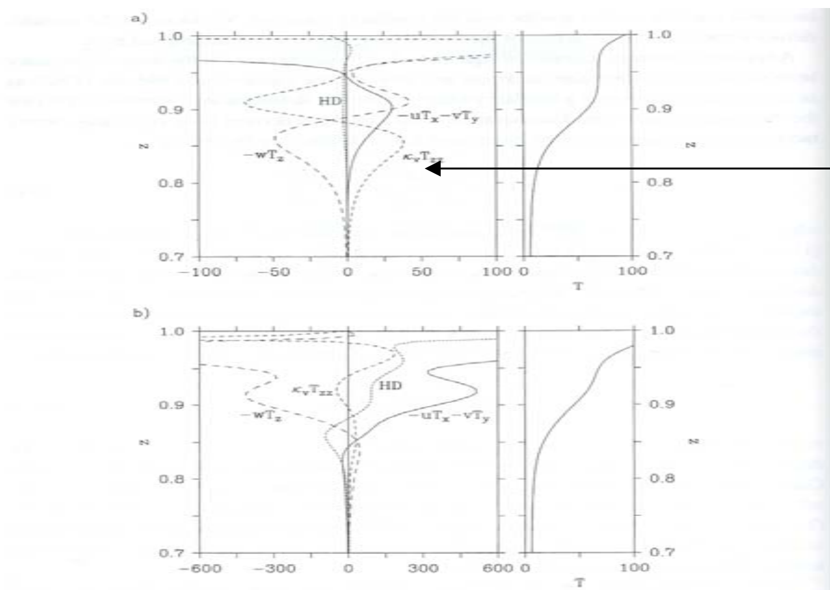


Figure 18. Vertical profiles of terms in the thermodynamic equation for $0.7 < z < 1$ for the solution in Figure 11 ($\kappa_h = 0.002$) with $\kappa_v = 0.003$ at (a) the center of the domain, $(x, y) = (0.5, 0.5)$, and (b) near the western boundary, $(x, y) = (0.024, 0.5)$. The profiles for horizontal advection ($-uT_x - vT_y$), vertical advection ($-wT_z$), vertical diffusion ($\kappa_v T_{zz}$), and horizontal (Laplacian plus biharmonic) diffusion (HD) are labeled accordingly. The corresponding profiles of T are also shown (right panels). The units are $T_*/t_* = 5.4 \times 10^{-4}$ K yr^{-1}.

In the interior

$$wT_z \approx \kappa T_{zz}$$

$$W/\delta \sim \kappa_v/\delta^2$$

The internal thermocline scale

$$W = \frac{\beta}{f} U \delta \qquad \Delta b / L = f U / D_a$$

$$\longrightarrow W = \frac{\beta}{f} \frac{\Delta b}{L} D_a \delta = \frac{\kappa_v}{\delta} \qquad \delta = \left(\frac{\kappa_v f L}{\beta \Delta b D_a} \right)^{1/2}$$

$$\rightarrow W : \kappa_v^{1/2}$$

$$D_a^2 : \frac{f^2 W_e L}{\beta \Delta b}$$

finally

$$\boxed{\delta = \kappa_v^{1/2} \left(\frac{f^2 L}{\beta \Delta b W_e} \right)^{1/4}}$$

Distinction is the 1/2 power law and not 1/3 as would obtain if δ and not D_a were used in thermal wind eqn.

Scaling law from calculations

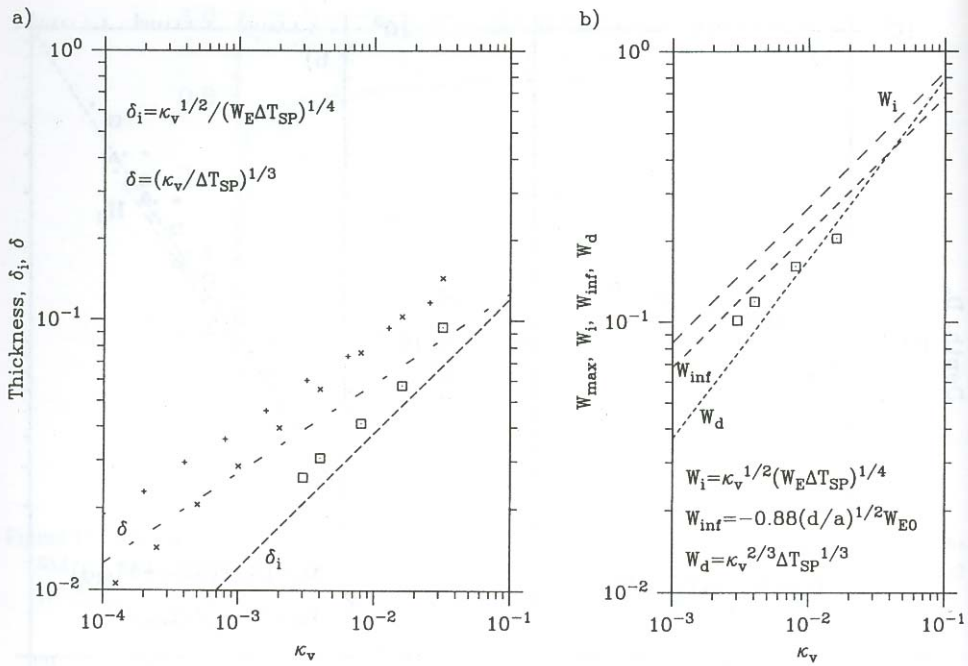


Figure 13. (a) Thickness of the internal peak of T_z (□) versus κ_v, from the profiles in Figure 11. The internal boundary layer scale δ_i and the advective-diffusive scale δ are also shown (dashed lines), along with the corresponding thicknesses from solutions of the similarity equations (6.1) (×) and (6.2) (+). (b) Maximum upward vertical velocity at $(x, y) = (0.5, 0.5)$ versus κ_v, from the solutions in Figure 11. The internal boundary layer scale W_i, the asymptotic estimate $W_{inf} = W_\infty$ from Young and Ierley (1986) for solutions of (6.1), and the advective-diffusive scale W_d are also shown (dashed lines).

Satisfies the $\kappa^{1/2}$ law.

An alternative picture

There have been calculations in which the entire thermocline is a dissipative boundary layer.

A discussion that follows some ideas of Welander and especially Salmon,R. 1990. *J.Mar.Res.* **48**, 437-469.

$$fv = \frac{1}{\rho_o}\frac{\partial p}{\partial x}$$

$$\beta v = f\frac{\partial w}{\partial z}$$

$$\longrightarrow \quad \frac{\partial p/\rho_o}{\partial x} = \frac{f^2}{\beta}\frac{\partial w}{\partial z}$$

Implies the existence of a function *M* such that:

$$\frac{p}{\rho_o} = M_z, \quad \frac{f^2 w}{\beta} = M_x$$

$$u\rho_x + v\rho_y + w\rho_z = \kappa_v \rho_{zz}$$

$$\therefore \; u = -M_{zy}/f, \; v = M_{zx}/f \;, \; g\rho/\rho_o = -M_{zz}$$

The M equation and scales

$$\frac{1}{f}\left[M_x M_{zzy} - M_y M_{zzx}\right] + \frac{\beta}{f^2} M_x M_{zzz} = \kappa_v M_{zzzz}$$

Simple case when *M=M(x,z)* then horizontal advection terms vanish. Equivalent to system:

Scales *L,U,d,g',W*

$$\beta v = f w_z,\quad v_z = -\frac{g\rho_x}{f\rho_o},\quad w\rho_z = \kappa_v \rho_{zz}$$

$$U = \frac{f}{\beta} W/d \qquad \frac{U}{d} = \frac{g'}{fL} \qquad W = \frac{\kappa_v}{d} \longrightarrow d = \left(\frac{\kappa_v f^2 L}{\beta g'}\right)^{1/3} \rightarrow W = \kappa_v^{2/3}\left(\frac{\beta g'}{f^2 L}\right)$$

This gives a thicker bl and a weaker w

Numerical calculations

From Vallis 2006 Cambridge U. Press

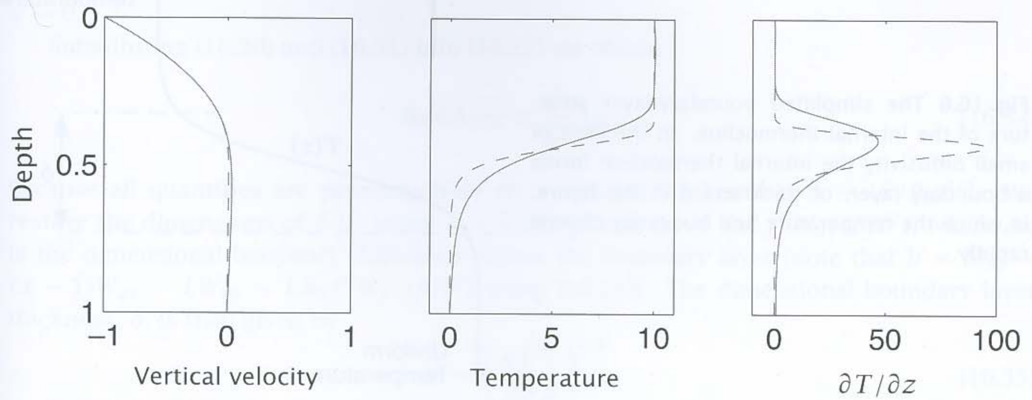


Fig. 16.4 Solution of the one-dimensional thermocline equation, (16.27), with boundary conditions (16.28), for two different values of the diffusivity: $\hat{\kappa} = 3.2 \times 10^{-3}$ (solid line) and $\hat{\kappa} = 0.4 \times 10^{-3}$ (dashed line), in the domain $0 \leq \hat{z} \leq -1$. 'Vertical velocity' is W, 'temperature' is $-W_{\hat{z}\hat{z}}$, and all units are the non-dimensional ones of the equation itself. A negative vertical velocity, $\widehat{W_E} = -1$, is imposed at the surface (representing Ekman pumping) and $B_0 = 10$. The internal boundary layer thickness increases as $\hat{\kappa}^{1/3}$, so doubling in thickness for an eightfold increase in $\hat{\kappa}$. The upwelling velocity also increases with $\hat{\kappa}$ (as $\hat{\kappa}^{2/3}$), but this is barely noticeable on the graph because the downwelling velocity, above the internal boundary layer, is much larger and almost independent of $\hat{\kappa}$. The depth of the boundary layer increases as $\widehat{W}_E^{1/2}$, so if $\widehat{W_E} = 0$ the boundary layer is at the surface, as in Fig. 16.5.