Lecture 2: Energy Balance and the Troposphere

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The philosophy throughout these lectures is that in order to understand a complex system we must have a description of it at multiple levels, from a back-of-the-envelope calculation through idealized numerical models to a comprehensive simulation with all the bells and whistles. Because this is Walsh we will always try to include a back-of-the-envelope calculation and go from there, but other approaches are possible.

1   What are we trying to explain?

A schematic of the overall structure of the atmosphere and ocean is given in Fig. 1, with some pictures of the real atmosphere from observations given in Fig. 2 and Fig. 3. In Fig. 1 we sketch the troposphere, where temperature decreases with height, and the stratosphere, where temperature increases with height, and the dividing tropopause which is fairly high over the tropics (15km) and lower over polar regions (8km). We might immediately ask, what determines this structure? What determines the height of tropopause? Why is it about $10\,\mathrm{km}$, and not $100\,\mathrm{km}$ or $1\,\mathrm{km}$? And what determines the width of the tropics where the tropopause is high? And so on.

Turning to the ocean, we have, again very schematically, warm water in the upper ocean and cold water below. The layer between them, where temperature varies very fast vertically, is called the thermocline. Sometimes we make an analogy between the thermocline and the tropopause, but actually the thermocline is more like the whole troposphere because they are both characterized by large vertical temperature gradients and relatively fast dynamics. Questions for oceanographers include what determines this structure of the ocean? What is the nature of the circulation that maintains it? More specifically, what determines the depth of the thermocline?

These are the *kinds* of questions we will consider in these lectures. We'll try to answer some of them, but not all. The philosophy throughout is that in order to understand a complex system we must have a description of the system at multiple levels, from a back-of-the-envelope calculation through idealized numerical models to a comprehensive simulation with all the bells and whistles. Because this is Walsh we will always try to include a back-of-the-envelope calculation and go from there, but other approaches are possible.

Our goals in this lecture are fairly fundamental:

- Understand at an elementary level what determines the surface temperature of Earth.

- Understand the need for a troposphere, and what determines its thickness.

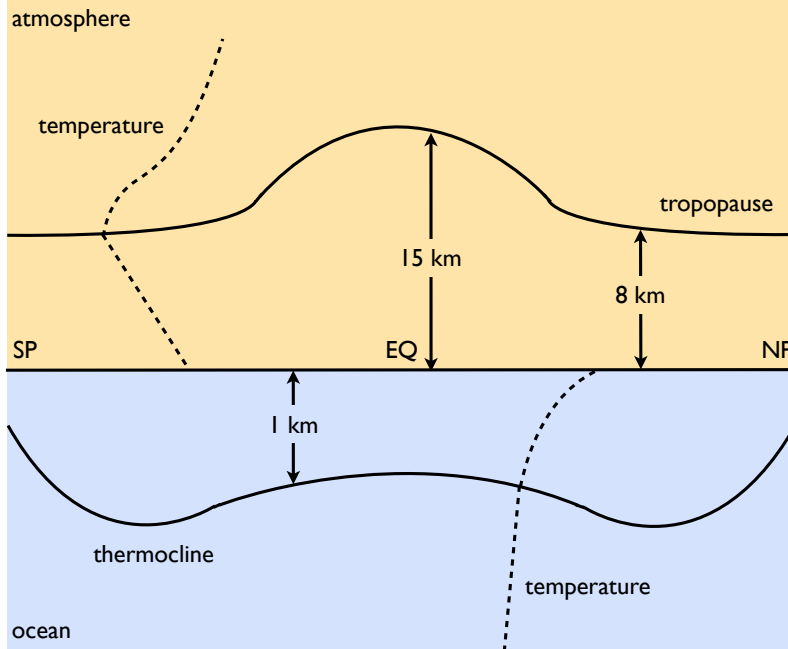To answer that we begin with a tutorial on radiation.

Figure 1: A schematic of thermal structures of ocean and atmosphere. The solid lines mark the tropopause and the base of the thermocline, and the near-vertical dashed lines are representative profiles of temperature.

2 Radiative Balance

2.1 The very basics

All macroscopic bodies except those at absolute zero (are there any?) emit thermal radiation. The black body emission per unit wavelength or per unit frequency are given by Planck's function which is, for the two cases respectively,

$$B_\lambda(T) = \frac{2\pi h c^2}{\lambda^5} \frac{1}{\exp(hc/\lambda k_B T) - 1}, \qquad B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/k_B T) - 1} \tag{1}$$

where c is the speed of light, h is the Planck constant and k_B is the Boltzmann constant. Conventions for frequency and wavelength are such that $c = \omega/k = \omega\lambda/2\pi = \nu\lambda$. Integrating either of the above expressions over wavelength or frequency, respectively, gives the Stefan–Boltzmann law

$$B(T) = \sigma T^4, \tag{2}$$

where σ is Stefan's constant,

$$\sigma = \frac{2\pi^5 k_B^4}{15 h^3 c^2} = 5.6704 \times 10^{-8} \text{ W m}^{-2} \text{K}^{-4}. \tag{3}$$

The maximum of Planck's function occurs at a wavelength $\lambda_m = b/T$ where $b = 2.898 \times 10^{-3}$ m K . This is Wien's displacement law, and it means that the higher the temperature
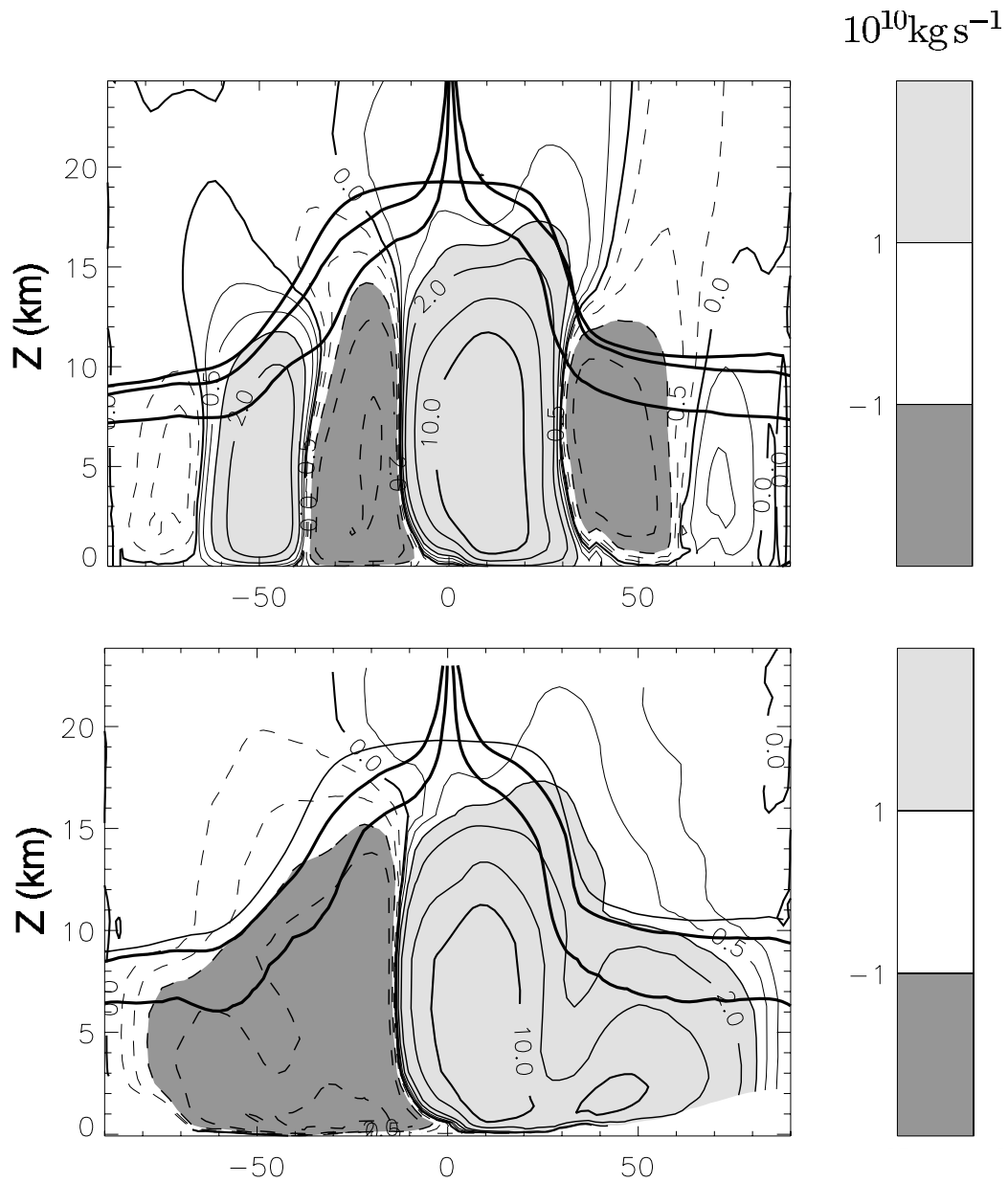
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Figure 2: Overturning circulation of the atmosphere during a Northern Hemisphere winter. The contours and shading indicate an overturning streamfunction, rising just south of the equator. The top plot shows a conventional Eulerian average and the bottom plot is a residual circulation. Three measures of the tropopause are indicated with the more nearly horizontal solid lines.
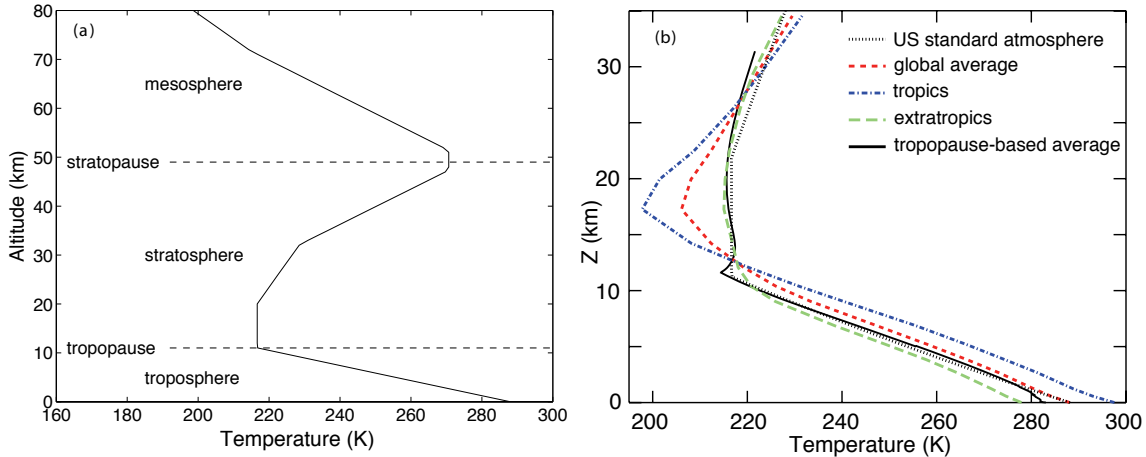
Figure 3: Temperature profiles in the atmosphere. On the left is the 'US standard atmosphere' and on the right are some observed profiles.

the shorter the wavelength at which emission predominantly occurs. For the Sun, at $T \approx 6000\,\mathrm{K}$, $\lambda_m = 5 \times 10^{-7}$ m, which is in the visible range; solar radiation is also sometimes called shortwave. For Earth, at $T = 280\,\mathrm{K}$, $\lambda_m = 1 \times 10^{-5}$ m, which is in the so-called infra-red, sometimes called longwave. The radiation itself is in units of W m^{-2}, and so is a flux of energy. The radiation reaching Earth from the Sun has an intensity of $S^* = 1366\,\mathrm{W}$ m^{-2}, varying by about $1\,\mathrm{W}$ m^{-2} over the 11-year sunspot cycle.

2.2   Earth's global energy budget

The simplest model that gives the temperature of the Earth is to suppose that the incoming solar radiation is balanced by an outgoing flux of infra-red radiation at a single temperature so that

$$S_0(1 - \alpha) = \sigma T^4, \tag{4}$$

where $S_0 = S^*/4 = 342\,\mathrm{W}$ m^{-2} and α is the Earth's albedo, the fraction of solar radiation reflected and measurements show that $\alpha \approx 0.3$. The resulting temperature, T_e is variously called the effective emitting temperature, the radiation temperature or the bolometric temperature. Plugging in numbers we find

$$T_e = \left(\frac{342 \times 0.7}{5.67 \times 10^{-8}} \right)^{1/4} = 255\,\mathrm{K}\,. \tag{5}$$

The actual surface temperature on Earth averages $288\,\mathrm{K}$. If you think $255\,\mathrm{K}$ is a good estimate of $288\,\mathrm{K}$, you are at heart a planetary scientist. If you think it is a bad estimate, you are a climate scientist or a meteorologist.

[Needed: table of emitting temperature and actual surface temperature for all planetary bodies in the solar system.]

A simple feedback we can put into such a model is the ice-albedo feedback, whereby we
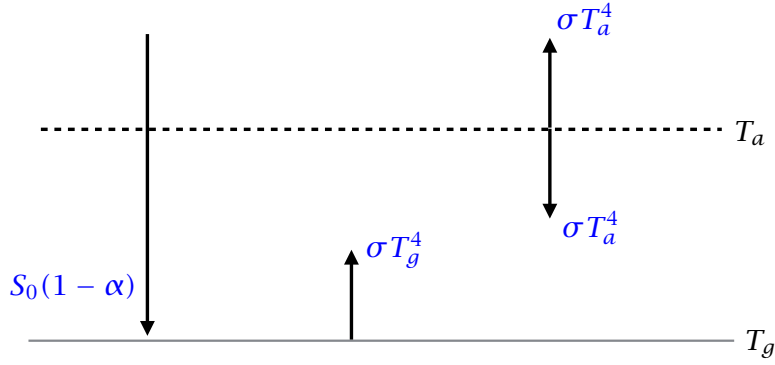
Figure 4: A simple EBM

suppose that α is a function of temperature. For example, we might suppose that

$$\alpha = \begin{cases} 0.3 & \text{for } T > T_0 \\ 0.8 & \text{for } T < T_0 \end{cases} \tag{6}$$

2.3 Effects of the atmosphere

The clear-sky atmosphere is largely transparent to solar radiation, but not to infra-red radiation. Most (but not all) of the solar radiation impinging on the atmosphere that is not reflected by clouds is thus absorbed at the Earth's surface, whereas most of the infra-red radiation emitted at the Earth's surface is absorbed by the atmosphere.

Given this, the next simplest model is to suppose there is an absorbing atmosphere above the surface, as illustrated in Fig. 4. If it is in equilibrium then the energy balance equations are:

$$\text{Top:} \qquad S_0(1 - \alpha) = \sigma T_a^4, \tag{7}$$
$$\text{Surface:} \qquad S_0(1 - \alpha) + \sigma T_a^4 = \sigma T_g^4. \tag{8}$$

(From these we also see the atmospheric balance, $2\sigma T_a^4 = \sigma t_g^4$.) The solution is

$$T_a = \left(\frac{(1-\alpha)S_0}{\sigma} \right)^{1/4}, \qquad T_g = 2^{1/4} T_a. \tag{9}$$

So that $T_a = 255\,\text{K}$ (as it has to be) and $T_g = 303\,\text{K}$, This is now too warm. One solution is to suppose the atmosphere has a finite emissivity, ϵ_a (which is less than one). This is getting ad hoc, but it will allow us to illustrate a nice effect. Thus,

$$\text{Top:} \qquad S_0(1 - \alpha) = \epsilon_a \sigma T_a^4 + (1 - \epsilon_a)\sigma T_g^4, \tag{10}$$
$$\text{Surface:} \qquad S_0(1 - \alpha) + \epsilon_a \sigma T_a^4 = \sigma T_g^4 + F. \tag{11}$$
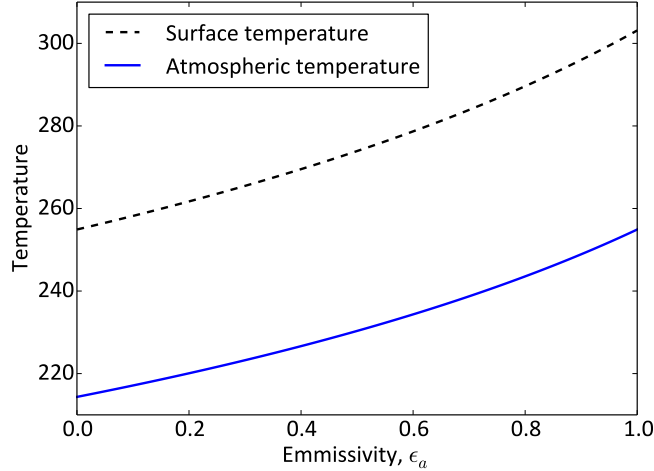
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Figure 5: Temperature as a function of emissivity in the EBM

where we also introduce a flux F from surface to atmosphere. The solution for the surface temperature is

$$\sigma T_g^4 = \frac{S_0(1-\alpha) - F/2}{1 - \epsilon_a/2} \tag{12}$$

which, for $F = 0$ and $\epsilon = 0.77$, gives $T_g = 288\,\text{K}$. The surface temperature obviously increases with ϵ_a as expected (Fig. 5).

3 Water Vapour Feedback

3.1 Saturation vapour pressure

The two main greenhouse gases are water vapour and carbon dioxide. Carbon dioxide is well mixed and is not volatile (it does not condense at Earthy temperatures). Its value is determined by geological and anthropogenic processes, and we can suppose its value to be specifiable. Water vapour levels are determined by the relative humidity of the atmosphere and, above all else, by the temperature through the Clausius–Clapeyron relation. This states that the saturation vapour pressure of water, e_s, or indeed of nearly any condensing material, varies as

$$\frac{\mathrm{d}e_s}{\mathrm{d}T} = \frac{L}{T(\rho_g^{-1} - \rho_c^{-1})} \approx \frac{L}{R_w T^2} e_s, \tag{13a,b}$$

where the second expression follows if $\rho_g \ll \rho_c$ (the density of the gas phase is much less than that of the condensed phase) and using the ideal gas law. The parameter L is the 'latent heat of condensation' and R_w is the gas constant for the gas in question, which for us is water. If L is constant (not a quantitatively good assumption, but good enough for now) we get

$$e_s(T) = e_{s0} \exp\left[\frac{L_s}{R_s}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]. \tag{14}$$
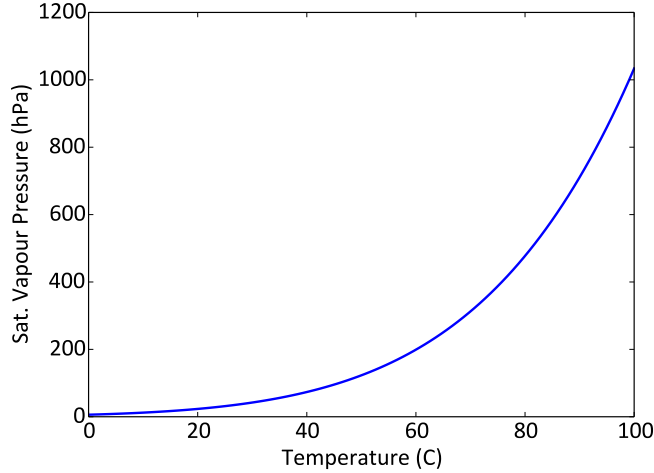
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Figure 6: Saturation vapour pressure as a function of temperature

Evidently, saturation vapour pressure is a strongly increasing function of temperature. A liquid will boil when the temperature is sufficiently high that the saturation vapour pressure equals the ambient pressure, and for water at sea-level this occurs at $100° \text{C}$. A good, semi-empirical approximation for saturation water vapour pressure is the Tetens–Bolton formula,

$$e_s = 6.112 \exp \left(\frac{17.67 * T_c}{T_c + 243.3} \right) \tag{15}$$

where T_c is temperature in Celsius and pressure is in hecto-Pascals (the same as millibars). This is actually a better approximation than (14) because it includes the variation of L with T. In any case, the main point is that water vapour content in the atmosphere increases fairly rapidly with temperature, at about 7% K $^{-1}$ (Fig. 6).

3.2 Radiative feedback and runaway greenhouse

Returning now to the EBM of the previous section, if we differentiate (12) we obtain

$$\frac{4 \, \mathrm{d}T_g}{T_g} = \frac{\mathrm{d}\epsilon_a}{2 - \epsilon_a}. \tag{16}$$

Now, ϵ_a may vary both because we add CO_2 (which we will denote as c) and because water vapour content may changes, so we write

$$\mathrm{d}\epsilon_a = \mathcal{A} \, \mathrm{d}c + \mathcal{B} \, \mathrm{d}e_s \tag{17}$$

where $\mathcal{A}$ and $\mathcal{B}$ are quantities that reflect the radiative properties of CO_2 and water vapour. If the main reason water vapour changes is because of the change in saturation vapour pressure with temperature then, using (13b) and (17), (16) becomes

$$(8 - 4\epsilon_a)\frac{\mathrm{d}T_g}{T_g} = \mathcal{A} \, \mathrm{d}c + \mathcal{B} \, \mathrm{d}e_s = \mathcal{A}, \mathrm{d}c + \frac{\mathcal{B}L}{R_w T^2} e_s \, \mathrm{d}T_g. \tag{18}$$

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or

$$\left(\frac{8 - 4\epsilon_a}{T_g} - \frac{\mathcal{B}Le_s}{R_w T_g^2}\right) \mathrm{d}T_g = \mathcal{A}\,\mathrm{d}c \tag{19}$$

Note that changes in atmospheric temperature is proportional to changes in surface temperature. Thus

$$\frac{\mathrm{d}T_g}{\mathrm{d}c} = \frac{\mathcal{A}T_g}{8 - 4\epsilon_a}\left(\frac{1}{1 - y}\right) \quad \text{where} \quad y = \frac{\mathcal{B}e_s L}{R_w T_g(8 - 4\epsilon_a)}. \tag{20}$$

This is a rather interesting equation. It is not to be believed at a quantitative level, but it is perhaps the simplest model that captures in a physically plausible way the greenhouse-gas effects of both water vapour and CO_2 . The following is apparent:

- Adding carbon dioxide to the atmosphere causes temperature to go up (because $4\epsilon_a < 8$), providing $y < 1$, so the model delineates between forcing and feedback.

- The feedback is captured by the terms involving y, and it can be larger than the direct effect depending on the size of $\mathcal{B}$.

- As $y \to 1$ the feedback becomes very large, and this is called the *runaway greenhouse effect*. As the temperature increases the water vapour content increases, temperature further increases and so on.

- There is no a priori reason why y should be less than unity. For example, it will be large if the temperature is high, and so if e_s is high. It seems then that that T_g will *decrease* as c increases!

The last item seems totally unphysical, and to see what is going on we need to construct an explicit model of the greenhouse effect with water vapour feedback. We will do that soon but it will be easier if we must look in a bit more detail about radiation.

4 Radiative Transfer in a Grey Atmosphere

4.1 Assumptions

Radiative intensity, I is the radiative flux per solid angle and when dealing with radiation in three-dimensional problems we have to deal with directionality. We also have to deal with the dependence of absorption on wavelength, and with scattering. In dealing with radiation in the Earth's atmosphere we will make a number of main simplifications.

1. We can have completely separate treatments of solar and infra-red radiation.

2. Much of the time we can assume there is no solar absorption in the atmosphere. This is not quantitatively true but if it were the case, most of the atmosphere would be about the same.

3. We integrate over solid angles in the upward pointing hemisphere and again in the downward pointing atmosphere, so that we have two streams of radiation.

4. We'll integrate over wavelength in the infra-red and assume that a single emissivity suffices.

5. There is no scattering of infra-red radiation.

4.2   Equations of radiative transfer

Consider a monochromatic beam of radiation passing through a gas, and suppose for a moment the gas does not emit any radiation but only absorbs it. For a thin layer of gas the change in intensity of the beam is then

$$\mathrm{d}I = -I\mathrm{d}\tau \tag{21}$$

where τ is the *optical depth.* The equation may be regarded as a definition of optical depth — it is the fraction of the incoming radiation absorbed — with the difficulty then arising in relating it to the physical properties of the gas. Eq. (21) can be formally integrated to give $I = I_0 \exp(-\tau)$, where the factor $T = \exp(-\tau)$ is the *transmittance* of the layer. The optical depth of two layers is the sum of their optical depths and the total transmittance is the product of the two transmittances.

The optical depth of a gas is related both to the amount of gas and to its properties, and for a thin layer of gas of thickness $\mathrm{d}s$ we can write

$$\mathrm{d}\tau = k_A \rho \mathrm{d}s \tag{22}$$

where k_A is the mass absorption coefficient. In general the optical depth will depend on the wavelength but we shall assume it does not; that is, the atmosphere is *grey.* In the atmosphere if the pressure is hydrostatic then, in the vertical direction, $\mathrm{d}\tau = k_A \rho \, \mathrm{d}z = k_A \mathrm{d}p$ so that

$$\tau(p_1, p_2) = k_A(p_1 - p_2) \tag{23}$$

In fact the mass absorption coefficient increases with pressure so that in the atmosphere a somewhat better approximation is to write

$$\tau \approx \tau_r \frac{(p_1 - p_2)(p_1 + p_2)/2}{p_r^2} \tag{24}$$

where p_r is a reference pressure and τ_r is a reference optical depth, a function of the properties of the gas in question.

The slab of gas will also emit radiation, so taking this into account (21) becomes

$$\mathrm{d}I = (B - I)\mathrm{d}\tau \tag{25}$$

This is known as the Schwarzschild equation and it applies at each wavelength, but if we assume τ is not a function of wavelength and we integrate over all wavelengths then $B = \sigma T^4$. (You can either take this to be obvious or do a bit of algebra involving integrations over solid angles to convince yourself, or consult a radiation book like Goody or Petty or Pierrehumbert.) In terrestrial applications we assume that (25) applies in the infra-red, and do a separate calculation for solar radiation.

Now, in the atmosphere under two-stream approximation in the atmosphere we have upward, U, and downward, D, radiation and we write

$$-\frac{\mathrm{d}U}{\mathrm{d}\tau} = B - U, \qquad \frac{\mathrm{d}D}{\mathrm{d}\tau} = B - D. \tag{26a,b}$$

The *convention* we have chosen here is that τ increases downwards. This is convenient for atmospheric applications, for then we have $\tau = 0$ at the top of the atmosphere, but it is not mandated. We could choose it the other way and flip the signs of the right-hand sides and no physical result depends on this choice, or on the origin of τ. We will use these equations for the infra-red radiation and in what follows assume that solar radiation is all absorbed at the surface.

4.3 Solutions

Formal Solution

Consider the generic equation for radiation travelling in the direction of increasing τ or decreasing τ, B and U respectively

$$\frac{\mathrm{d}D}{\mathrm{d}\tau} = B - D, \qquad \frac{\mathrm{d}U}{\mathrm{d}\tau} = U - B. \tag{27}$$

Multiplying by the integrating factors $\exp(\tau)$ and $\exp(-\tau)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(D\mathrm{e}^{\tau}) = B\mathrm{e}^{\tau}, \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}(U\mathrm{e}^{-\tau}) = -B\mathrm{e}^{-\tau} \tag{28}$$

Integrating between $\tau = 0$ and τ' we obtain

$$D(\tau')\mathrm{e}^{\tau'} - D(0) = \int_0^{\tau'} B(\tau)\mathrm{e}^{\tau}\,\mathrm{d}\tau, \qquad U(\tau')\mathrm{e}^{-\tau'} - U(0) = -\int_0^{\tau'} B(\tau)\mathrm{e}^{-\tau}\,\mathrm{d}\tau \tag{29}$$

or

$$D(\tau') = \mathrm{e}^{-\tau'}\left[D(0) - \int_0^{\tau'} B(\tau)\mathrm{e}^{\tau}\,\mathrm{d}\tau\right], \qquad U(0) = U(\tau')\mathrm{e}^{-\tau'} + \int_0^{\tau'} B(\tau)\mathrm{e}^{-\tau}\,\mathrm{d}\tau \tag{30}$$

The first term in each solution is the attenuation of incoming radiation and the second is the cumulative emission. There are other ways to write the solution, but in general the solution of radiative problems can be written only in the form of integrals. Nevertheless, in some important special cases we can get a local solution as below.

Radiative equilibrium in planetary atmospheres

Consider an atmosphere with net incoming solar radiation S_{net} and suppose the planet is in radiative equilibrium with the incoming solar balanced by outgoing infra-red. The radiative transfer equations are thus to be solved with the boundary conditions that

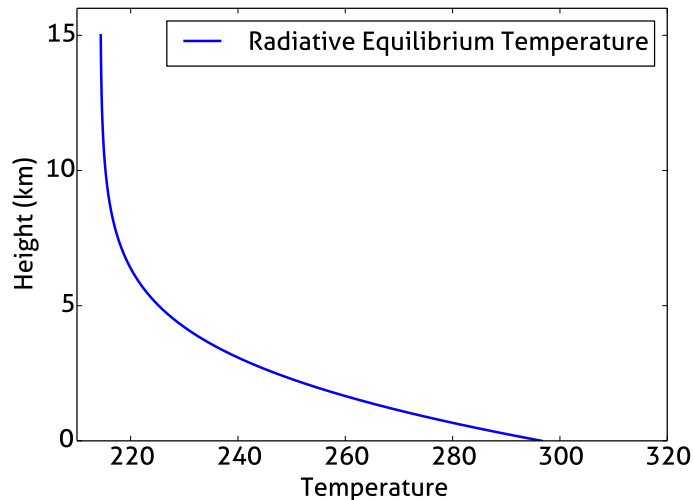$$D = 0, \ U = U_t \quad \text{at} \quad \tau = 0, \tag{31}$$

Figure 7: Radiative equilibrium temperature (solid curve) calculated using (36), with an optical depth of $\tau_0 = 8/3$, $H_a = 2\,\text{km}$ and a net incoming solar radiation of $239\,\text{W m}^{-2}$.

where $U_t = S_\text{net}$ is the net outgoing long-wave radiation (OLR) at the top of the atmosphere. There are still too many variables as we don't know B, but we can obtain a *radiative equilibrium* solution if we assume there is no longwave heating in the column. The heating is proportional to the divergence of the net flux, so that if this is presumed zero then $\partial(U - D)/\partial z = 0$ so that

$$\frac{\partial(U - D)}{\partial \tau} = 0. \tag{32}$$

Let us rewrite (26) as

$$\frac{\partial}{\partial \tau}(U - D) = U + D - 2B, \tag{33a}$$

$$\frac{\partial}{\partial \tau}(U + D) = U - D. \tag{33b}$$

A solution of these equations that satisfies the boundary conditions is

$$D = \frac{\tau}{2}U_t, \qquad U = \left(1 + \frac{\tau}{2}\right)U_t, \qquad B = \left(\frac{1 + \tau}{2}\right)U_t. \tag{34}$$

where U_t is the outgoing longwave radiation at the top of the atmosphere. The only thing remaining is to relate τ to z, and a simple recipe that is similar to (23) is to suppose that τ has an exponential profile.

$$\tau(z) = \tau_0 \exp(-z/H_a) \tag{35}$$

where typical values are $\tau_0 \approx 4$ and $H_a \approx 2\,\text{km}$. The temperature then goes like

$$T^4 = U_t \left(\frac{1 + \tau_0 e^{-z/H_a}}{2\sigma}\right), \tag{36}$$

as illustrated in Fig. 7. Note the following aspects of the solution.

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1. Temperature increases rapidly with height near the ground.

2. The upper atmosphere is nearly isothermal.

3. The temperature at the top of the atmosphere, T_t is given by

$$\sigma T_t^4 = \frac{U_t}{2} \tag{37}$$

Thus, if we define the emitting temperature, T_e, to be such that $\sigma T_e^4 = U_t$, then $T_t = T_e/2^{1/4}$. Note also $B_t/U_t = 1/2$.

In fact, the temperature gradient near the ground varies so rapidly it is likely to be convectively unstable, which we come to in the next lecture. Also, note that we do not need to impose a temperature boundary condition at the ground; in fact there is no ground in this problem! — but what happens if we add one? That is, suppose that we declare that there is a black surface at some height, say $z = 0$, and we require that the atmosphere remain in radiative equilibrium. What temperature does that surface have to be?

From (34) the upward irradiance and temperature at any height z are related by

$$U = \left(\frac{2+\tau}{1+\tau}\right)\sigma T^4. \tag{38}$$

At $z = 0$ the surface will have to supply upwards radiation equal to that given by (38), and therefore its temperature, T_g is given by

$$\sigma T_g^4 = \left(\frac{2+\tau}{1+\tau}\right)\sigma T_s^4, \tag{39}$$

where T_s is the temperature of the fluid adjacent to the ground (the 'surface temperature'). That is, $T_g > T_s$ and there is a temperature discontinuity at the ground. Sometimes in very still conditions a very rapid change of temperature near the ground can in fact be observed, but usually the presence of conduction and convection will ensure that T_g and T_s are equal.

In the limit in which $\tau = 0$ in the upper atmosphere (let us prematurely call this the 'stratosphere') then we see that

$$D = 0, \qquad U = U_t, \qquad B = \frac{U_t}{2}. \tag{40}$$

That is, the atmosphere is isothermal, there is no downwelling irradiance and the upward flux is constant. The stratospheric temperature, T_{st} and the emitting temperature are related by

$$T_{st} = \frac{T_e}{2^{1/4}}. \tag{41}$$

Summary Points

To sum up, what have we found?

1. If we suppose the atmosphere is grey, and we know how optical depth varies with height, then if the atmosphere is in longwave radiative equilibrium we can construct an explicit solution for the temperature as a function of height.

2. The temperature will typically decrease very rapidly in height away from the surface. So much so it is likely to be convectively unstable, as we discuss in the next lecture.

3. The radiative equilibrium temperature does not care or know whether a surface (i.e., the ground) is present. If a surface *is* present, and we require that radiate equilibrium still hold, the temperature of the ground must be higher than the temperature of the air adjacent to it. This is because the ground must supply the same amount of radiation as would be supplied by an infinite layer of air below that level. Thus, there is a temperature discontinuity at the ground, which in reality would normally be wiped out by convection.

5 An explicit model of the Runaway Greenhouse Effect

We now come back to the greenhouse effect and construct an explicit model of runaway greenhouse. (The term 'runaway greenhouse' was coined by Ingersoll (1969).) Suppose the atmosphere is in radiative equilibrium. From the derivations above, we can relate the surface and ground temperatures to the incoming solar radiation through the relation

$$T_s^4 = \frac{T_e^4}{2}(1 + \tau_0), \qquad T_g^4 = T_e^4(1 + \frac{\tau_0}{2}) \tag{42}$$

Thus, if $\tau_0 = 1.254$ then, for $T_e = 255\,\mathrm{K}$ we find $T_g = 288\,\mathrm{K}$ and $T_s = 262\,\mathrm{K}$. The assumption of radiative equilibrium and the ensuing temperature discontinuity are unrealistic but the model will illustrate and important point. We'll construct a more realistic model in the next lecture.

Suppose that we let τ_0 be a function of temperature, increasing with the saturation vapor pressure at the surface. Thus, let

$$\tau_0 = A + Be_s(T_g) \tag{43}$$

where A and B are semi-empirical constants, and e_s is the saturation vapor pressure as given by the solution of the Clausius–Clapeyron equation, (15). We will tune their values such that $T_g = 288$ when $T_e = 255$, and with some experience of hindsight we set the ratio $A/B = 8$, whence we obtain $A = 1.12$ and $B = 0.14$. The reason for such a seemingly high ratio is that a grey model is too prone to give a runaway greenhouse because of its lack of windows in the infra-red. Thus, in reality, even as temperature and water vapor content increase some infra-red radiation *can* escape from the surface.

Putting the above together, the ground temperature is solution of

$$T_g^4 = T_e^4 \left(1 + \frac{1}{2}[A + Be_s(T_g)] \right). \tag{44}$$

This algebraic equation is quite nonlinear and must be solved numerically but a few points are apparent.

1. For any given T_e we can obtain a graphic solution by plotting T_g and $T_e^4(1 + \tau_0/2)^{1/4}$ and seeing where the two curves intersect. For a range of values of T_e we will obtain two solutions, as illustrated in Fig. 8. However, if T_e is too high there will be no intersection of the curves because the value of $T_e^4(1 + \tau_0/2)$ will always be larger than T_g.
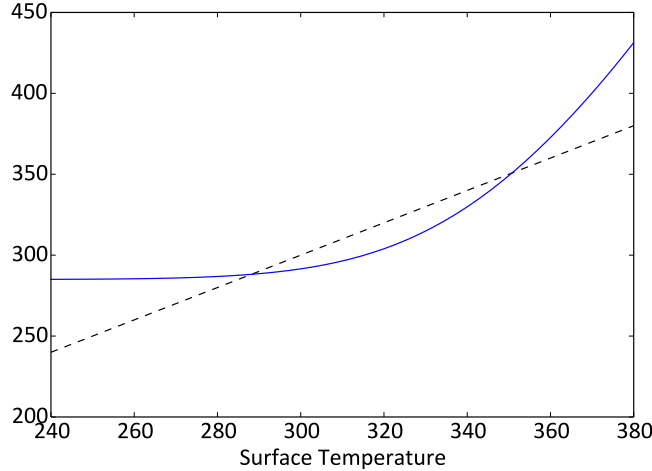
Figure 8: Graphical solution to the energy balance model (44) with $T_e = 255\,\mathrm{K}$. The dashed curve is T_g and the solid curve plots values of $T_e^4(1 + \tau_0(T_g)/2)^{1/4}$, with τ_0 given by (43). Solutions occur at $T_g \approx 288\,\mathrm{K}$ and $T_g = 350\,\mathrm{K}$.

2. If T_e increases and Be_s is much smaller than A, then a solution is found by increasing T_g.

3. If T_e increases and Be_s is suitably large then we can imagine that a solution will be found with a *lower* value of T_g.

Numerical solutions, found iteratively, are illustrated in Fig. 9, and as expected there are two branches to the solution. [A much more detailed discussion with many extensions is to be found in the report by P. Martin in this volume.] For the parameters plotted, there is no solution if $T_e > 269\,\mathrm{K}$. That is to say, if a planet obeying the model above were in an orbit such that $T_e > 269\,\mathrm{K}$ then infra-red radiation would not be able to escape from the surface, and the surface temperature would keep on rising. All the water on the planet surface would boil, and eventually the water vapor would escape to space. Such a scenario may have occurred on Venus in the past.

5.1 Stability of solutions

The upper branch of the solution plotted in Fig. 9 runs counter to our intuition, in that temperature decreases as emitting temperature increases. The situation arises because the greenhouse effect is so strong, so that an increase in emitting temperature can lead to a decrease in surface temperature if the greenhouse effect also falls considerably. However, this solution is *unstable* as we now show.

We add a time dependence to the energy balance model and write

$$C\frac{\mathrm{d}T_g}{\mathrm{d}t} = \sigma T_e^4 - \sigma \frac{T_g^4}{1 + \tau_0/2} \tag{45}$$
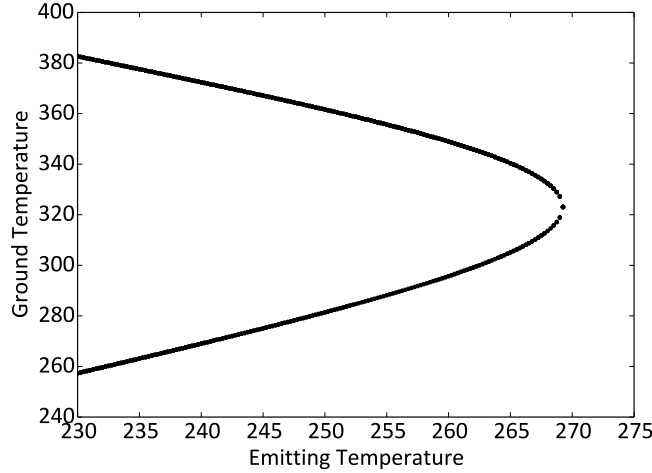
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Figure 9: Solutions the energy balance model (44) obtained numerically. Plotted are values of T_g as a function of The dashed curve is T_g and the solid curve plots values of $T_e^4(1 + \tau_0(T_g)/2)^{1/4}$, with τ_0 given by (43) .

We perturb the system about an equilibrium point and so obtain

$$C\frac{\mathrm{d}T_g'}{\mathrm{d}t} = \frac{-4\sigma T_g^3 T_g'}{1+\tau_0} + \frac{\sigma T_g^4 \tau_0'}{(1+\tau_0)^2}, \tag{46a}$$

$$= \left(\frac{T_g}{1+\tau_0/2}\frac{\mathrm{d}\tau_0}{\mathrm{d}T_g} - 4\right)T_g' \tag{46b}$$

Thus, the solution will be stable or unstable according as whether the term in brackets is negative or positive, respectively.

A tiny bit of algebra will reveal that the ratio of the two terms in brackets in (46b) precisely the same as the ratio of the gradients of the solid curve and the dashed curve at the intersection points in Fig. 8. Thus, the solution at the higher temperature (about $350\,\mathrm{K}$ in the graph) is unstable, because the gradient of the blue curve is greater than the gradient of the dashed curve. Similarly, the solution at the lower temperature ($288\,\mathrm{K}$) is stable. All of the solutions on the upper branch on Fig. 9 are therefore unstable.

References

Ingersoll, A. P., 1969. The runaway greenhouse: A history of water on Venus. *J. Atmos. Sci.*, **26**, 6, 1191–1198.