Lecture 4

Thermohaline Circulation Variability

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1 A brief introduction

While the major surface oceanic currents are predominantly driven by the wind-stress, the dynamics of the deep circulation depends mostly on horizontal density gradients, established as a result of the combined effect of surface thermal and saline forcing.

The meridional inhomogeneity of radiative heating of the atmosphere produces horizontal density difference between the colder polar and the warmer equatorial sea surface temperature. The effect of this temperature gradient alone would be the generation of denser water at higher latitudes and of lighter water in tropical regions. However the excess of evaporation over precipitation towards the equator causes the mean salinity to decrease with latitude. The equation of state for seawater is approximately given by

$$\rho = \rho_0(1 - \alpha T + \beta S), \tag{1}$$

so that the thermal effect on density opposes that of salinity (α and β are the expansions coefficient of seawater).

In summary:

- temperature favors downwelling of dense water at high latitudes and upwelling at the equator;
- for salinity the opposite is true.

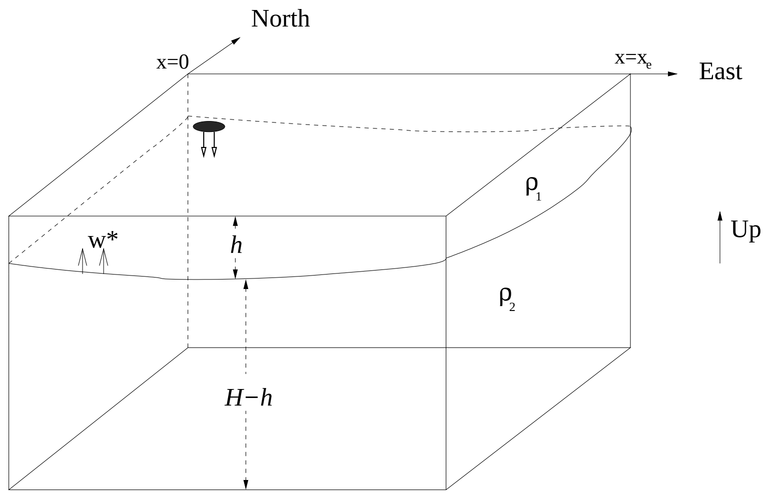
The net result of these competing effects is the establishment of a flow which extends to the deep oceanic layers, known as *thermohaline circulation*.

In the present climate, the North Atlantic deep circulation is dominated by two thermally direct cells, one with high-latitude sinking in the Northern Hemisphere and one with high-latitude sinking in the Southern Hemisphere. However, paleoclimatic data indicate that as recently as 11,000 years ago the deep circulation and the downwelling at high latitudes has been much weaker.

The two processes involved can in fact give rise to the existence of multiple steady states, with the possibility of transitions among the equilibria.

2 The Stommel-Arons model

The classical approach to the study of the buoyancy-driven circulation is Stommel-Arons model of the abyssal oceanic flow. One interpretation of the model regards the ocean as a box with a rigid lid and a two-layer approximation. The upper layer has depth h and density ρ_1 while the lower one has depth $H-h$ and density ρ_2.



HIgh-latitude convection transforms water of density ρ_1 into water of density ρ_2, and this downwelling is assumed to be localised at the poleward edge of the box. In steady state there is a velocity, w^*, at the layers' interface that compensates for this high-latitude density exchange. This upwelling is assumed to be diffuse, and is constrained by:

$$\int_{Basin} dx\, dy\, w^* = \text{Deep water production rate.}$$

The Stommel-Arons model examines the flow driven by this large scale interfacial velocity. Specifically, the steady dynamics in the lower layer obeys

$$\begin{aligned} -fv &= -p_{2x}/\rho_0 - ru && (2)\\ fu &= -p_{2y}/\rho_0 - rv && (3)\\ p_{1,2z} &= -\rho_{1,2}g && (4)\\ [(H-h)u]_x &+ [(H-h)v]_y = -w^*. && (5) \end{aligned}$$

where $r \ll f$ is the dissipation rate. We now wish to find an expression for the dynamic part of the pressure of the lower layer in terms of the layer thickness, h. Firstly, using the hydrostatic relation, we find

$$\begin{aligned} p_1 &= -\rho_1 g z + \hat{p}_1(x,y)\,,\\ p_2 &= -\rho_2 g z + \hat{p}_2(x,y)\,. \end{aligned}$$

From continuity of pressure at the interface, $z = -h$, we have

$$\hat{p}_2 = -\rho_0 g' h + \hat{p}_1 \tag{6}$$

where $g' \equiv g(\rho_2 - \rho_1)/\rho_0$ is the reduced gravity.

If we integrate the continuity equation vertically over the whole box, applying the condition that there is no vertical velocity at the top and bottom we have

$$[hu_1 + (H-h)u_2]_x + [hv_1 + (H-h)v_2]_y = 0\,. \tag{7}$$

Away from the boundaries we can neglect dissipation and we use geostrophic balance in both layers. Multiplying the upper layer momentum equations by h and the lower layer momentum equations by $H - h$, and forming a vorticity equation we find

$$f\{[hu_1 + (H-h)u_2]_x + [hv_1 + (H-h)v_2]_y\} + \beta[hv_1 + (H-h)v_2] = \\ h_x(\hat{p}_2 - \hat{p}_1)_y + h_y(\hat{p}_1 - \hat{p}_2)_x\,. \tag{8}$$

Because $\hat{p}_2 - \hat{p}_1$ depends linearly on h [from (6)] the RHS of equation (8) vanishes as does the first bracketed term on the LHS [because of (7)]. Thus

$$hv_1 + (H-h)v_2 = 0 \tag{9}$$

$$hu_1 + (H-h)u_2 = 0, \tag{10}$$

and there is no vertically averaged flow. Because the interior velocities are geostrophic we must have

$$h\nabla\hat{p}_1 + (H-h)\nabla\hat{p}_2 = 0. \tag{11}$$

Finally, eliminating for $\hat{p}_1$ from (6) and integrating we have

$$\hat{p}_2 = -\frac{\rho_0 g' h^2}{2H}. \tag{12}$$

The vertically averaged lower layer equations thus satisfy:

$$f(H-h)v = P_x + r(H-h)u\,, \tag{13}$$

$$-f(H-h)u = P_y - r(H-h)v\,, \tag{14}$$

where we have defined the vertically averaged pressure in the lower layer

$$P \equiv g'(\frac{h^3}{3H} - \frac{h^2}{2}). \tag{15}$$

In the regime $r \ll f$, P obeys the potential vorticity equation [$\beta \equiv df/dy$]

$$\frac{\beta}{f^2}P_x = w^* - \nabla(r\frac{\nabla P}{f^2}). \tag{16}$$

Integrating the mass conservation equation (4) across the box from $x = 0$ to x_e, and assuming no normal flow at the boundaries we obtain the net meridional abyssal mass transport, $\psi(y)$,

$$\psi(y) \equiv \int_0^{x_e} dx\,(H-h)v = -\int_0^{x_e} dx \int_0^y dy'\,w^*(x,y'). \tag{17}$$

A relation between P and ψ is obtained integrating 14 across the width of the basin and neglecting dissipation, hence

$$f\psi(y) = P(x_e, y) - P(0, y). \tag{18}$$

In the ocean *interior* we can obtain P from (16) by neglecting dissipation and imposing $u = 0$ ($P_y = 0$) at $x = x_e$.

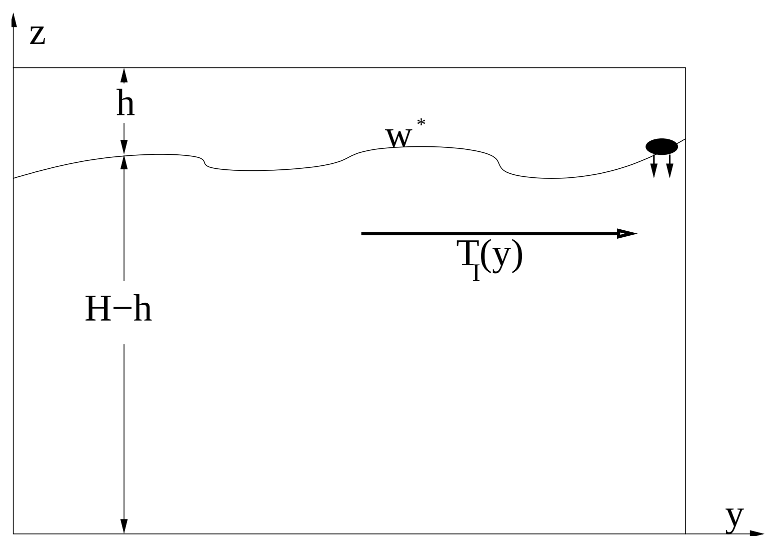
$$P_I(x, y) = -\frac{f^2}{\beta}\int_x^{x_e} dx'\, w^*(x', y) + P_0, \tag{19}$$

where P_0 is the (constant) value of P on the eastern boundary.

The *interior* mass transport, i.e. the mass transport that excludes the western boundary layer contribution is

$$\psi_I(y) \equiv f^{-1}[P_0 - P_I(0, y)] = \frac{f}{\beta}\int_0^{x_e} dx\, w^*(x, y) \geq 0.$$

A cross section shows that the interior flow is towards the source in the abyss!



Thus there must be a flow in the western boundary layer, which returns the flow towards the source. Near the western boundary we rescale x such that it becomes small, of the same order of magnitude as r/β. Then, near the boundary we have

$$\frac{\beta}{f^2}P_x = -\frac{r}{f^2}P_{xx}.$$

The solution for the whole box then is

$$P = P_I + \underbrace{A(y)\exp(-\beta x/r)}_{\text{Boundary layer correction}}.$$

$A(y)$ is determined by mass conservation. If we take the continuity equation (4) and integrate it across the whole of the E–W direction and from $y' = 0$ to y then we obtain again equation 17. Substituting P gives

$$\int_0^{x_e} dx P_x = \int_0^{x_e} dx \int_0^y dy' w^*(x, y') \,. \tag{20}$$

The integral of the interior part of the streamfuntion P is just the interior mass transport (equation 19) and thus

$$\psi_I - A(y) = -\int_0^{x_e} dx \int_0^y dy' w^*(x, y') \tag{21}$$

which gives solution for $A(y)$

$$A(y) = f \int_0^{x_e} dx \, [f w^*(x, y)/\beta + \int_0^y dy' \, w^*(x, y')] \,.$$

It is useful to divide the transport into an interior part, ψ_I and a boundary contribution, ψ_{WB}, so that

$$\begin{aligned} \psi(y) &= f^{-1}[\overbrace{P(x_e, y) - P_I(0, y)}^{\psi_I} + \overbrace{P_I(0, y) - P(0, y)}^{\psi_{WB}}] \\ \psi_{WB} &= -A(y) \leq 0. \end{aligned}$$

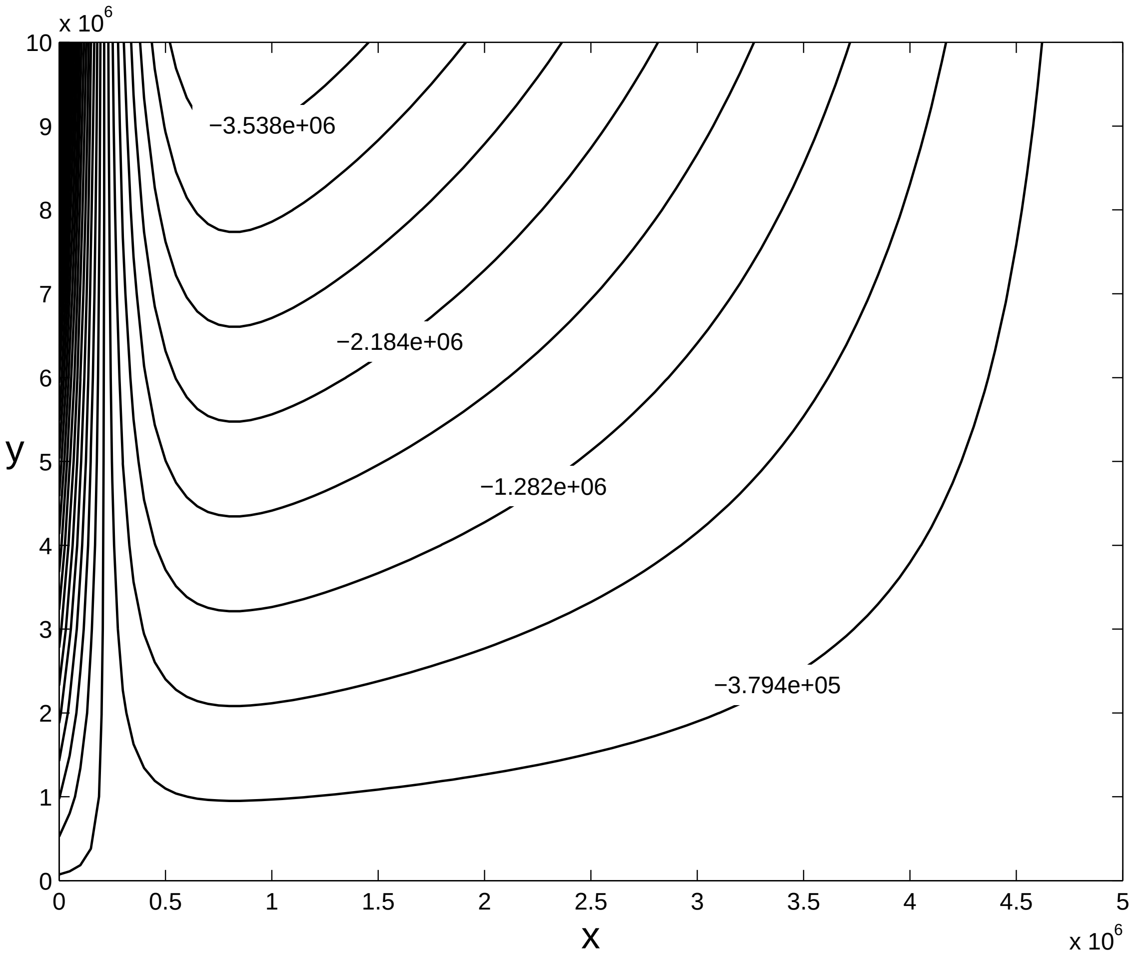
This clearly shows that the transport in the western boundary is negative (equatorward) if w^* is positive, so that ψ_{WB} is away from the high-latitude convection source.

In the particular case when w^* is independent of y we have, $[f = \beta y]$,

$$\psi_{WB} = -2y \int_0^{x_e} dx \, w^*(x) = \mathbf{2}\psi(y).$$

The western boundary layer transports *twice* the zonally averaged transport: the flow from the convection source and the interior flow which is towards the source.

A contour of P for uniform w^* shows an interior poleward flow and an equatorward western boundary current.



3 What determines w^*?

It is clear that the circulation in the Stommel-Arons model depends crucially on the interior upwelling, w^*, at the interface of the two layers, i.e. at the base of the thermocline. To estimate what determines the interior upwelling we use scaling arguments, which are confirmed by more detailed laminar calculations (e.g. Vallis, 2000).

The scenario is one where a bounded ocean is driven by surface buoyancy fluxes only, which are transmitted downward through diffusion. No time-dependent instabilities are considered. In this case, the interior vertical velocity satisfies the approximate balance:

$$w^* \rho_z \approx \kappa \rho_{zz}.$$

The thickness of the thermocline, (i.e. of the upper layer in Stommel-Arons model) is diffusive:

$$h = O(\frac{\kappa}{w^*}) \quad w^* = O(\frac{hV}{L}).$$

In the presence of walls that confine the flow to the East and West, a large scale East-West pressure gradient can be maintaned, so that we can assume that v is geostrophic and hydrostatic, i.e.

$$f v_z \sim g\rho_x/\rho_0.$$

We thus arrive at the following estimate for the depth of the thermocline:

$$h^3 = O(\frac{\kappa f L^2 \rho_0}{\Delta\rho g}).$$

For fixed surface density, $\Delta\rho$ is independent of κ, and therefore the depth of the thermocline satisfies

$$h = O\left(\frac{\kappa f L^2 \rho_0}{\Delta\rho g}\right)^{1/3} \sim \kappa^{1/3}, \quad w^* \sim \kappa^{2/3}.$$

For fixed surface flux, we estimate the *horizontal* density difference to be $\Delta\rho = O(Fh/\kappa)$:

$$h^4 = O(\frac{\kappa^2 f L^2 \rho_0}{Fg}).$$

In this case

$$h \sim \kappa^{1/2}, \quad w^* \sim \kappa^{1/2}.$$

These scalings have been confirmed by non-eddy-resolving numerical simulations of the primitive equations (Vallis, 2000 and Huang et al. 1994): density gradients are confined to a thin diffusive layer, while the abyssal layer is essentially homogeneous.

Essential to this scaling is the existence of an East-West pressure gradient that maintains a geostrophically balanced meridional flow.

It is therefore interesting to enquire what happens when such a balance fails because there are no boundaries at the East and West that can support a pressure difference.

4 Thermohaline flow in a reentrant geometry

In the next section we will discuss the thermohaline circulation in the specific case of a channel unbounded in the East-West directions, limited in latitude and with periodic boundary conditions at the ends. Because of the absence of meridional walls, this model could describe the circulation in the Antartic Circumpolar region.

We assume that the horizontal flow obeys the steady two dimensional equations of motion, and we neglect nonlinear advective terms:

$$\begin{aligned} -fv &= -\frac{p_x}{\rho_0} - ru \\ fu &= -\frac{p_y}{\rho_0} - rv \end{aligned} \tag{22}$$

The shape of the basin imposes periodic E-W boundary conditions for all fields so that the longitudinally averaged pressure gradient in the x-direction must vanish, i.e.:

$$\overline{p_x} = 0. \tag{23}$$

This restriction prevents the system from reaching a steady geostrophic balance and does not allow an efficient meridional transport of water. We can in fact consider the zonally averaged momentum balance:

$$\begin{aligned} -f\overline{v} &= -\frac{\overline{p_x}}{\rho_0} - r\overline{u} \\ f\overline{u} &= -\frac{\overline{p_y}}{\rho_0} - r\overline{v} \end{aligned} \tag{24}$$

and solve for the meridional velocity:

$$\overline{v} = -\frac{r}{f^2 + r^2}\frac{\overline{p_y}}{\rho_0}. \tag{25}$$

The spreading of warm water from the equator towards the polar regions is achieved only because of friction. For weak drag, $r \ll f$, the meridional flow is small.

For x-independent buoyancy forcing, and excluding the spontaneous generation of x-dependent instabilities we have $v = \overline{v}$. Thus the flow is two-dimensional and it is described by a streamfunction ψ:

$$\begin{aligned} \overline{v} &= -\psi_z \\ \overline{w} &= \psi_y \end{aligned} \tag{26}$$

hence:

$$\begin{aligned} \overline{p_z} &= -\overline{\rho} g \\ \psi_{zz} &= -\frac{rg}{f^2 + r^2}\frac{\overline{\rho_y}}{\rho_0} \end{aligned} \tag{27}$$

With density to be determined by temperature and salinity as stated in equation (1), the flow is then governed by the two evolution equations:

$$\begin{aligned}\overline{T_t} + J(\psi, \overline{T}) &= \kappa\overline{T_{zz}} + \nu\overline{T_{yy}} \\ \overline{S_t} + J(\psi, \overline{S}) &= \kappa\overline{S_{zz}} + \nu\overline{S_{yy}}\end{aligned} \tag{28}$$

where $J(A, B) = \partial_x A \partial_y B - \partial_y A \partial_x B$.

The boundary conditions at the top of the layer for the two variables are very different. Sea surface temperature can be thought as adapting instantaneously to variations in heat flux, giving rise to a prescribed distribution of *temperature* with respect to latitude. Instead, surface salinity plays a minor role in the balance between evaporation and precipitation, so that the surface *salinity flux* is imposed by the atmosphere. Thus, we impose the following boundary conditions at the surface $z = 0$ and at the bottom of the sea $z = -H$:

$$\begin{aligned}\bar{T} &= \Delta T \Theta(y), \quad \kappa\bar{S}_z = F\mathcal{F}(y) \ \text{ at } \ z = 0 \\ \kappa\bar{T}_z &= \kappa\bar{S}_z = 0 \ \text{ at } \ z = -H.\end{aligned}$$

We now adimensionalize the set of equations (28), choosing the following scalings for lenghts, temperature and salinity:

$$z = H\,\zeta, \quad y = L\,\eta, \quad T = \Delta T\,\theta, \quad S = \frac{\alpha\Delta T}{\beta}\,\sigma \tag{29}$$

while for density, stream function and time, we nondimensionalize (27) using:

$$\rho = \rho_0\alpha\Delta T\,\pi, \quad \psi = \frac{H^2 rg\alpha\Delta T}{f^2 L}\,\phi, \quad t = \frac{\kappa}{H^2}\epsilon^2\,\tau. \tag{30}$$

Substituting the non-dimensional variables into the governing equations, we obtain:

$$\begin{aligned}\phi_{\zeta\zeta} &= (\theta - \sigma)_\eta \\ \epsilon^2\theta_\tau + \epsilon J(\psi, \theta) &= \theta_{\zeta\zeta} + \delta\theta_{\eta\eta} \\ \epsilon^2\sigma_\tau + \epsilon J(\psi, \sigma) &= \sigma_{\zeta\zeta} + \delta\sigma_{\eta\eta}\end{aligned} \tag{31}$$

with boundary conditions:

$$\begin{aligned}\phi &= 0; \ \ \theta_\zeta = \sigma_\zeta = 0 \ \text{ at } \ \zeta = 0 \\ \phi &= 0; \ \ \theta = \Theta(\eta) \ \text{ at } \ \zeta = 1 \\ \sigma_\zeta &= R\mathcal{F}(\eta) \ \ \text{ at } \ \zeta = 1\end{aligned} \tag{32}$$

There are three parameters governing the behavior, defined as:

$$\overbrace{\epsilon \equiv \frac{rgH^3\alpha\Delta T}{\kappa f^2 L^2}}^{\text{Rayleigh-Ekman \#}}, \quad \overbrace{R \equiv \frac{\beta F H}{\kappa\alpha\Delta T}}^{\text{density ratio}}, \quad \delta \equiv \frac{\nu H^2}{\kappa L^2}.$$

- ϵ is the product of the Rayleigh number and the Ekman number square;
- R expresses the ratio between temperature and salinity contributions to density variation;
- δ weights the importance of meridional to vertical diffusivities for T and S.

For weak drag $\epsilon << 1$ and we can simplify the analysis by expanding the three variables above in power series of ϵ:

$$\begin{aligned}\phi &= \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + 0(\epsilon^3)\\ \theta &= \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + 0(\epsilon^3)\\ \sigma &= \sigma_0 + \epsilon\sigma_1 + \epsilon^2\sigma_2 + 0(\epsilon^3)\end{aligned}$$

We further assume that the density ratio is small, specifically

$$R = O(\epsilon^2);\ \delta = O(\epsilon^2) \tag{33}$$

As a preliminary observation, we note that with this ordering of the parameter R the forcing (32) on the surface salinity flux enters only at order ϵ^2:

$$\sigma_{0\zeta} + \epsilon\sigma_{1\zeta} + \epsilon^2\sigma_{2\zeta} = \epsilon^2 R\mathcal{F}(\eta), \tag{34}$$

nevertheless, σ is $O(1)$.

Solving for the various orders in ϵ we get a hierarchy of equations. Starting from the leading order, $O(1)$, the temperature and salinity equations are:

$$\sigma_{0\zeta\zeta} = \theta_{0\zeta\zeta} = 0,$$

with boundary conditions:

$$\begin{aligned}\sigma_{0\zeta} = \theta_{0\zeta} = 0 \quad &\text{at} \quad \zeta = 0\\ \sigma_{0\zeta} = 0, \quad \theta_0 = \Theta(y) \quad &\text{at} \quad \zeta = 1\end{aligned}$$

Thus the two fields are vertically homogeneous at leading order:

$$\sigma_0 = \sigma_0(\eta,\tau), \theta_0 = \theta_0(\eta,\tau). \tag{35}$$

Furthermore, because of the fixed temperature boundary condition, the temperature at leading order is determined and $\theta_0 = \Theta(y)$. However, the leading order salinity is determined by the balance at higher orders.

We can also now determine the leading order streamfunction, which satisfies

$$\phi_{0\zeta\zeta} = (\theta_0 - \sigma_0)_\eta$$

and the condition $\phi_0 = 0$ at the two boundaries. Integrating vertically we find:

$$\phi_0 = \frac{1}{2}\zeta(\zeta+1)(\theta_0 - \sigma_0)_\eta \tag{36}$$

At next order, $O(\epsilon)$, the salinity equation is:

$$-\phi_{0\zeta}\sigma_{0\eta} = \sigma_{1\zeta\zeta}$$

which, when integrated vertically, gives:

$$-\phi_0\sigma_{0\eta} = \sigma_{1\zeta}, \tag{37}$$

because the meridional flow turns lateral gradients into stratification. Because both the top and bottom boundary conditions for σ_1 are automatically satisfied, we must proceed to the next order to determine σ_0.

The evolution equation for σ_0 is obtained by vertically averaging the evolution equation of the salinity at $O(\epsilon^2)$ which is given by:

$$\partial_\tau\sigma_0 + \partial_\eta(\phi_0\sigma_{1\zeta}) - \partial_\zeta(\phi_0\sigma_{1\eta} + \phi_1\sigma_{0\eta}) = \sigma_{2\eta\eta} + \frac{\delta}{\epsilon^2}\sigma_{0\eta\eta}.$$

We thus obtain:

$$\partial_\tau\overline{\sigma_0} + \partial_\eta\int_0^1 d\zeta(\phi_0\sigma_{1\zeta}) = \frac{R}{\epsilon^2}\mathcal{F}(\eta) + \frac{\delta}{\epsilon^2}\sigma_{0\eta\eta}.$$

Here we have used the result that the third term on the left hand side vanishes and we have applied the surface condition $\sigma_{2\zeta}|_{\zeta=0} = \frac{R}{\epsilon^2}\mathcal{F}(\eta))$. If we substitue the expression for $\sigma_{1\zeta}$ obtained from (37) we find:

$$\partial_\tau\sigma_0 - \partial_\eta\int_0^1 d\zeta(\phi_0^2\sigma_{0\eta}) = \frac{R}{\epsilon^2}\mathcal{F}(\eta) + \frac{\delta}{\epsilon^2}\sigma_{0\eta\eta}.$$

Finally, using the expression (36) for ϕ_0 we get:

$$\partial_\tau\sigma_0 = \frac{1}{120}\partial_\eta[(\theta_0 - \sigma_0)_\eta]^2\sigma_{0\eta} + \frac{R}{\epsilon^2}\mathcal{F}(\eta) + \frac{\delta}{\epsilon^2}\sigma_{0\eta\eta} \tag{38}$$

with $\theta_0 = \Theta(\eta)$.

It is also useful to write the dimensional forms of the equations, which are given by

$$\psi = \frac{rg}{2(f^2+r^2)}z(z+H)(\alpha T_{0y} - \beta S_{0y}),$$

$$T_0 = \Delta T\Theta(y),$$

$$S_{0t} = \frac{\epsilon^2 L^4\kappa}{120H^2}\left[\frac{(\beta S_{0y} - \alpha T_{0y})^2}{(\alpha\Delta T)^2}S_{0y}\right]_y + \frac{F}{H}\mathcal{F} + \nu S_{yy}. \tag{39}$$

The meridional circulation transports salt downgradient with a nonlinear "diffusivity" proportional to ρ_y^2.

For $r << f$, ψ is independent of κ and the density field is almost vertically homogeneous. Thus the qualitative picture of the circulation in a channel, in the limit where the friction is very small, is very different than that obtained in the presence of meridional walls. The circulation is also accompanied by a large east-west velocity, which is in thermal wind balance, which does not influence the meridional circulation.

5 The two-box model approximation

In this section a two-box approximation of (39) is considered, since this reduction illustrates the qualitative properties of the full partial differential equation. This approximation also leads to a model which is very similar to that original proposed by Stommel (1961) in the limit of rapid temperature relaxation.

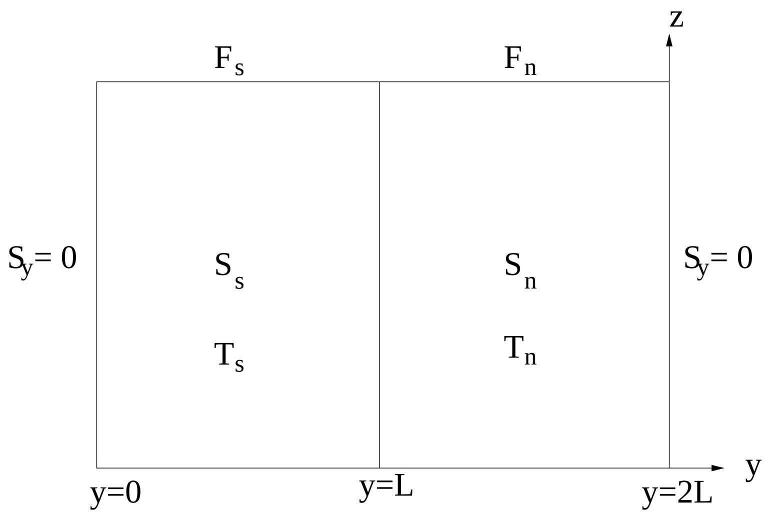


Figure 1: The box-model approximation to (39).

The left hand box in figure (1) is the equatorial box, denoted by subscript s, while the right hand is the polar box, denoted by subscript n. The salnity is assumed to be independent of latitude and depth within each box.

F represents the surface flux of salinity, S the salinity and T the temperature. There is no meridional flux of salinity at the sides. Integrating (39) in latitude over the equatorial and the polar box the salt equation we obtain the following two equations:

$$\begin{aligned} \dot{S}_s &= [\mu S_{0y}(\beta/\alpha S_{0y} - T_{0y})^2 + \nu S_{0y}]_{y=L} + \frac{F}{H}\mathcal{F}_s, \\ \dot{S}_n &= -[\mu S_{0y}(\beta/\alpha S_{0y} - T_{0y})^2 + \nu S_{0y}]_{y=L} + \frac{F}{H}\mathcal{F}_n. \end{aligned} \tag{40}$$

Notice that we need $\mathcal{F}_s + \mathcal{F}_n = 0$, in order to conserve the mean salinity of the system.

In order to determine the salinity gradient at the latitude $y = L$, S_{0y}, we use the following differentiation rule:

$$S_{0y}|_{y=L} = \frac{S_n - S_s}{L}, \quad T_{0y} = \frac{T_n - T_s}{L}.$$

Defining

$$\sigma \equiv \frac{\beta(S_n - S_s)}{\alpha(T_n - T_s)},$$

and rescaling time, the salinity difference satisfies

$$\dot{\sigma} = -\sigma(\sigma - 1)^2 + \gamma - \lambda\sigma. \tag{41}$$

In this equation γ is a parameter expressing the ratio between N-S salt flux effect and heat temperature gradient:

$$\gamma \propto \frac{F_n - F_s}{T_n - T_s} > 0$$

while λ is proportional to the lateral diffusion, ν. We expect stationary condition to be reached for compensating temperature and salinity effects. If $T_n - T_s < 0$, $F_n < 0$ and $F_s > 0$, the equilibria will correspond to positive values of γ.

Stommel (1961) used a slightly different box-model, which in the limit of rapid temperature relaxation is:

$$\dot{\sigma} = -\sigma|\sigma - 1| + \gamma - \lambda\sigma.$$

Both systems will reach a steady state, minimum of a potential, V because

$$\dot{\sigma} = -\frac{\partial V(\sigma)}{\partial \sigma} \;\Rightarrow\; V_\sigma \dot{\sigma} = \dot{V} = -(V_\sigma)^2 \leq 0.$$

The potential, V, associated with (41) is a function of σ given by

$$V(\sigma) = \frac{1}{4}\sigma^4 - \frac{2}{3}\sigma^3 + \left(\lambda + \frac{1}{2}\right)\sigma^2 - \gamma\sigma.$$

Depending on γ and λ, V can have one or two minima as illustrated in the following figure. Equilibria are associated with extrema of the potential, V: minima are stable and maxima are unstable.

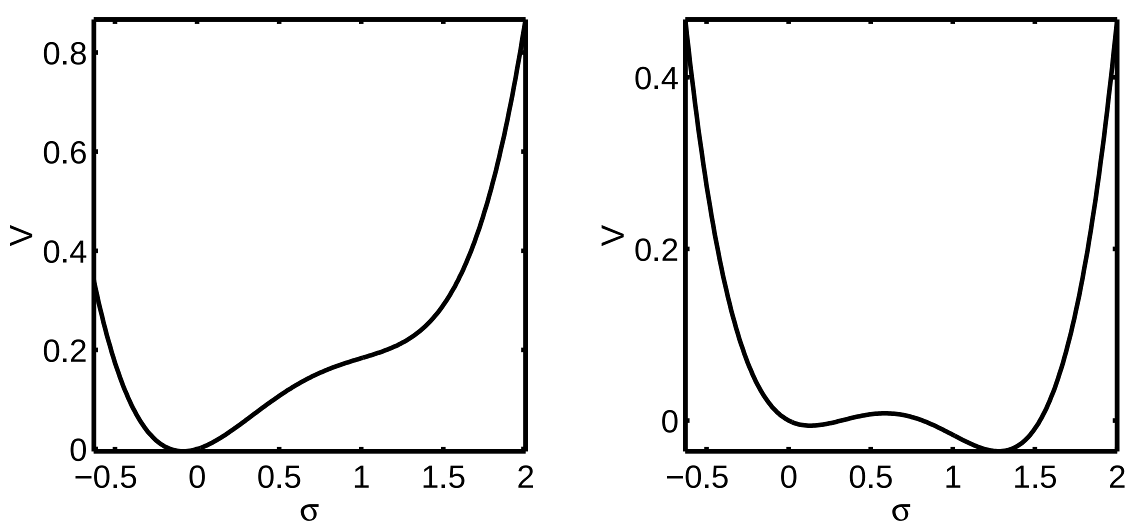


Figure 2: The potential V for two different values of λ.

Multiple equilibria (3) are obtained for:

$$(1 - 3\lambda)^{3/2} \geq |1 - \frac{27}{2}\gamma + 9\lambda|, \tag{42}$$

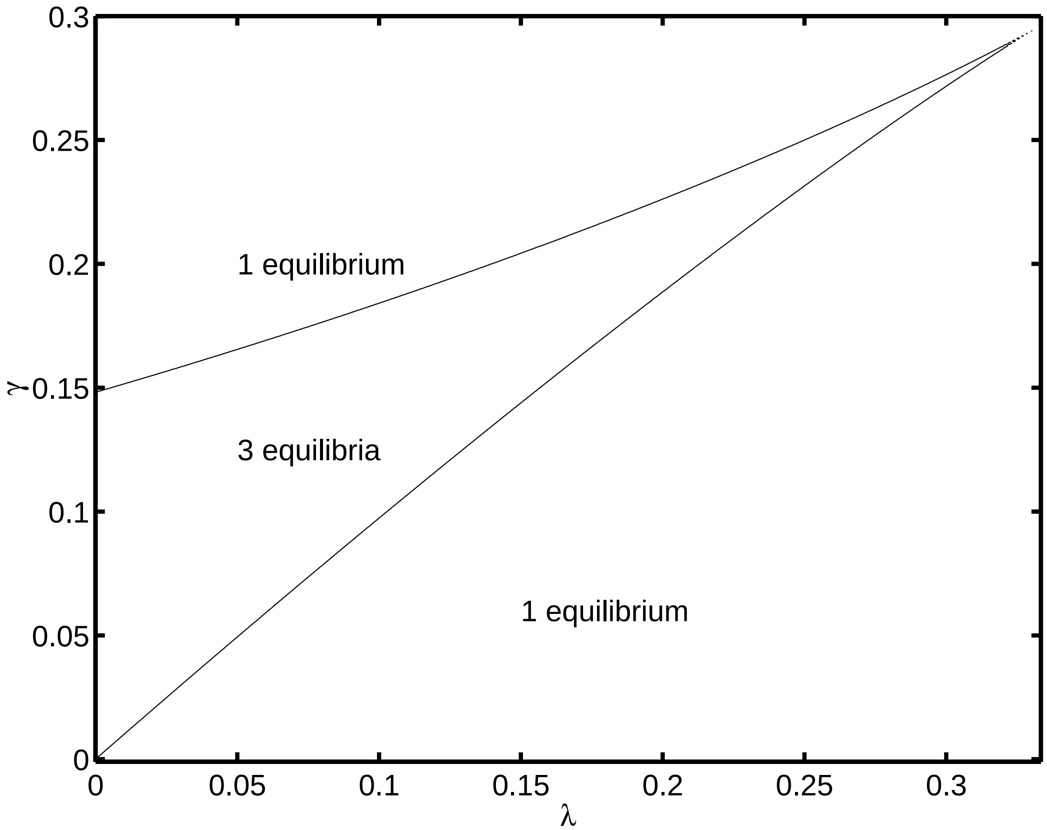


Figure 3: The region where multiple equilibria exist is bounded by the curve (refcusp:eqn) in the $\gamma - \lambda$ space.

In the limit $\lambda \ll 1$ (weak lateral diffusion) it is possible to find approximate expressions for the steady states, which are given by:

$$\sigma(\sigma - 1)^2 = \gamma \ll 1.$$

There is a thermally-driven solution with small salinity gradient:

$$\sigma_a \approx \gamma \ll 1.$$

There is a salt-compensated solution with small density gradient:

$$\sigma_c \approx 1 + \sqrt{\gamma}.$$

The third solution, $\sigma_b \approx 1 - \sqrt{\gamma}$ is unstable.

Recalling that the meridional overturning circulation is given by $\psi \propto (1-\sigma)$, the meridional circulations associated with the two stable equilibria are:

$$\begin{aligned}\psi_a &\propto 1-\gamma \\ \psi_c &\propto -\sqrt{\gamma}.\end{aligned}$$

Thus ψ_a and ψ_c have opposite sign and the haline-driven circulation, ψ_c, is much weaker than the thermally driven flow.

The deterministic model does cannot lead to time-dependent variability of the thermohaline circulation, since it only admit (multiple) fixed points. Thus, the system cannot spontaneously jump from one equilibrium to the other: all initial states to the left (right) of the potential barrier end up in the same left-(right)hand well.

Notes by Fiona Eccles and Chiara Toniolo

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