Lecture 3: Solutions to Laplace's Tidal Equations

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1 Introduction

In this lecture we discuss assumptions involved in obtaining Laplace's Tidal Equations (LTE) from Euler's equations. We first derive an expression for the solid Earth tides. Solutions of LTE for various boundary conditions are discussed, and an energy equation for tides is presented.

Solutions to the Laplace Tidal equations for a stratified ocean are discussed in §2. We obtain expression for solid earth tide in §3. Different models of dissipation are examined in §4. Boundary conditions for LTE's are discussed in §5. In §6 the energy equation for LTE's is derived.

2 The Laplace Tidal equations for a Stratified ocean

To obtain the LTE for stratified ocean we assume pressure is hydrostatic and seek seperable solutions of the form

$$\begin{aligned} u(x,y,z,t) &= U(x,y,t)F_u(z), \\ w(x,y,z,t) &= W(x,y,t)F_w(z), \\ p(x,y,z,t) &= Z(x,y,t)F_p(z). \end{aligned}$$

The LTE for stratified ocean are

$$U_t - fV = -gZ_x \quad , \tag{1}$$

$$V_t - fU = -gZ_y \quad , \tag{2}$$

$$Z_t + D_n(U_x + V_y) = 0 \quad , \tag{3}$$

$$F_{w_{zz}} + \frac{N^2}{gD_n}F_w = 0 \quad , \tag{4}$$

where n is an index for the normal modes in the ocean. These equations are the constant depth LTE but where D_n is the equivalent depth of each mode, and $D_n \neq D_*$, rather

$$D_n = \frac{[\int_{-D_*}^{0} N(z')dz']^2}{gn^2\pi^2}$$

and $N(z)$ is the buoyancy frequency. For the zeroth mode or $n = 0, D_* \approx D$. The eigenvalue problem in (4) can be solved to determine the D_n using the following boundary conditions

$$\begin{aligned} F_w = 0 \quad &\text{at} \quad Z = -D_*, \\ F_w - D_n F_{w_z} = 0 \quad &\text{at} \quad Z = 0. \end{aligned}$$

2.1 Barotropic Solution (n = 0)

The normal mode equations described above and indexed by n an integer can be solved for specific modes. The zeroth order mode is given by $n = 0$ and is also called the barotropic or external mode. It is characterized by a solution which is depth independent. Below we will solve for the barotropic mode without rotation and no variations in the y direction ($f = 0$ and $\partial/\partial y = 0$). If we assume a plane wave solution, solving (1) - (4) gives,

$$\begin{aligned} Z_0 &= ae^{-i\sigma t + ikx} , \\ U_0 &= a\left(\frac{gk}{\sigma}\right)e^{-i\sigma t + ikx} , \\ \sigma &= \pm k\sqrt{gD_*}. \end{aligned}$$

Figure 1 shows the barotropic solution for velocity. For the semidiurnal tidal frequency, the phase velocity of the zeroth mode wave, given by $c_0 = \sqrt{gD_0} = \sigma/\kappa = 200$ m/s. Since this speed is also given by λ/T, where λ is the horizontal wavelength of the wave (distance from wave crest to wave crest), then $\lambda = 8640$ km. This wave is very fast and very long. For the case of no rotation this wave is dispersionless, but not when $f \neq 0$.

2.2 Baroclinic Solution, Mode 1 (n = 1)

The first baroclinic mode, indexed by $n = 1$ is also called the first internal mode. The rest of the modes for $n > 1$ are also internal modes and have more variation in depth. We can solve (1) - (4) for $n = 1$ without rotation and with no variations in y ($f = 0$ and $\partial/\partial y = 0$) giving the first internal mode,

$$\begin{aligned} Z_n &= ae^{-i\sigma t + ikx} , \\ F_{U_1} &= cos(\frac{\pi(z + D_*)}{D_*}) , \\ F_{W_1} &= sin(\frac{\pi(z + D_*)}{D_*}) , \\ \sigma &= \pm k\sqrt{gD_1}. \end{aligned}$$

The mode one solution is shown in figure 1. For the baroclinic modes, the phase velocity and horizontal wavelength are given by

$$\begin{aligned} c_n &= \frac{N_0 D_*}{n\pi} = \frac{(1.45 \times 10^{-3})(4000m)}{n\pi} = \frac{1.85 \text{ m/s}}{n}, \\ \lambda_{tidal} &= \frac{80 \text{ km}}{n}. \end{aligned}$$

So the lower modes travel more quickly, and have larger horizontal scales. The M_2 internal tide is found primarily as a mode 1 tide throughout the world's oceans, while higher order modes tend to dissipate nearer their source. Due to recent (last decade) improvements, the M_2 internal tide can now be seen by satellite altimetry. The ocean surface displacement due to internal tides is given by

$$\frac{w_{\text{free surface}}}{w_{\text{interior maximum}}} \approx \frac{N_0^2 D_*}{gn\pi} \approx 3 \times 10^{-4},$$

which means that for an internal tide displacement of isopycnals on the order of 100m (which is quite large but not impossible), the surface expression would be about 30cm, easily resolved by satellite altimetry measurements which have accuracy on the order of a few centimeters.

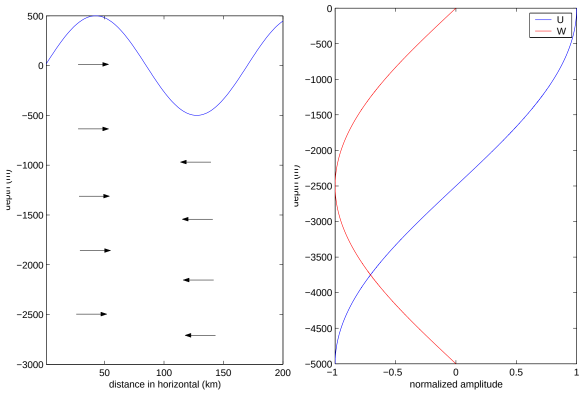

Figure 1: (a) Barotropic or depth independent solution for u velocity. Wave amplitude is greatly exaggerated. (b) Mode 1 solution for u and w velocity.

2.3 Numerical Solution to LTE

The complete problem that we would like to solve numerically to estimate the tides are the stratified linear equations,

$$u_t - fv = -\frac{p_x}{\rho_0} + \Gamma_x, \tag{5}$$

$$v_t + fu = -\frac{p_y}{\rho_0} + \Gamma_y, \tag{6}$$

$$\zeta_t + (uD)_x + (vD)_y = 0, \tag{7}$$

where Γ is full tide generating potential and $D(x, y)$ is the bottom topography. The domain is defined by the coasts of continents, ocean bottom and free surface. However, in order to solve this system of equations one must resolve short horizontal scales due to bottom topography where the bottom boundary condition on w is $w = -\mathbf{u} \cdot \nabla D$. Very few current modes are capable of this, though some have begun to resolve mode 1 in their simulations.

Instead the Laplace tidal equations for u and v may be substituted, of (5)-(7).

$$u_t - fv = -g\zeta_x + \Gamma_x, \tag{8}$$
$$v_t + fu = -g\zeta_y + \Gamma_y, \tag{9}$$

where ζ is the free surface and a smoother bottom topography is substituted,

$$w = -\mathbf{u} \cdot \nabla D_{smooth}. \tag{10}$$

However, because ζ is no longer a function of z while p in (5)-(7) was, we cannot determine $u(z)$ and (8)-(9) will only give the barotropic solution.

In the literature, the TGP is usually neglected and instead the barotropic tides and stratification are specified, which allows the simplification of (5)-(7) as

$$u_t - fv = -\frac{p_x}{\rho_0}, \tag{11}$$
$$v_t + fu = -\frac{p_y}{\rho_0}. \tag{12}$$

From this, the internal tides result from a single scattering of the barotropic tide by bottom relief $w_{int} = u_B \cdot \nabla D(x, y)$. In particular, if we decomposed D into low- and high-passed components,

$$D(x, y) = D_{lo}(x, y) + D_{hi}(x, y). \tag{13}$$

Then the ζ equation and bottom boundary conditions become

$$\zeta_{0t} + \nabla \cdot (\mathbf{u}_B D_{lo}(x, y)) = 0,$$
$$w_{int} = \mathbf{u}_B \cdot \nabla D_{hi}(x, y).$$

And the internal tide results from the bottom topography. However, this neglects multiple scattering from the topography and does not apply when the bottom slope is greater than the characteristic slope of internal waves. Currently, numerical models like the Princeton Ocean Model (POM) solve the (5)-(7). An example of numerically solved tides is shown in figure 2. In this paper, all tidal constituents were solved for using a hydrodynamic model and data assimilation from tide gauges and altimetry [1].

3 Solid Earth Tide

It has long been known that Earth's crust yields elastically to the tidal forces of the moon and sun. If we consider earth to be an incompressible elastic solid, then we can write equations for the deformation of the Earth as the following

$$-p_x + \mu\nabla^2 u = 0, \tag{14}$$
$$-p_z + \mu\nabla^2 w = 0. \tag{15}$$

where, p is pressure, (u, v) are the velocity and μ is viscosity of earth (see figure 3). Using the following boundary conditions,

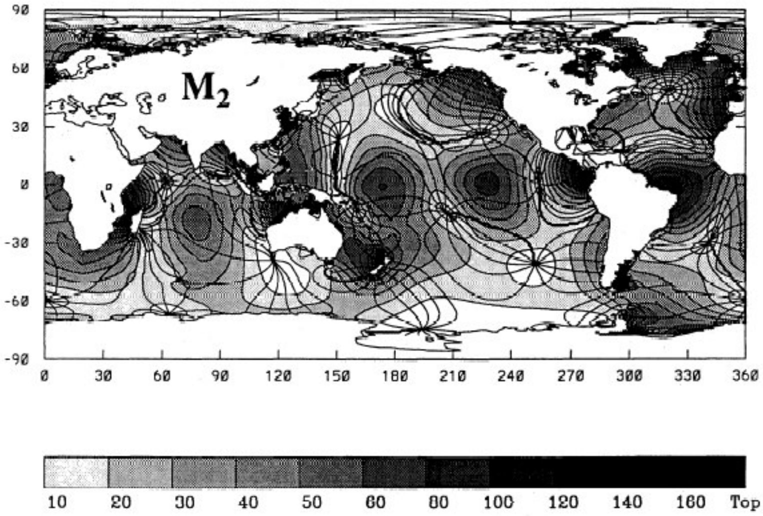

Figure 2: Cotidal map of the M_2 component. Coamplitude lines are drawn following the scaling indicated below the map. Units are in centimeters. Cophase lines are drawn with an interval of 30°, with the 0° phase as a larger drawing, referred to the passage of the astronomical forcing at Greenwich meridian [1].

$$u, w \to 0 \ as \ z \to -\infty,$$
$$\tau_{xz} = 2\mu(u_z + w_x) = 0 \ at \ z = 0,$$
$$\tau_{zz} = -p + 2\mu w_z = -\text{load} \ \equiv -\rho_w g a e^{ikx},$$

where term $\rho_w g a e^{ikx}$ gives the loading on earth surface due to ocean tide, a is the tidal amplitude and k is its horizontal wavenumber, we solve (14) and (15) with the above b.c.'s for a load of $\rho_w g a e^{ikx}$ to get an Earth surface wave displacement of

$$h\rho_w g a e^{ikx}, \tag{16}$$

where h is called "Love Number". In this case, $h = \frac{1}{2\mu k}$.

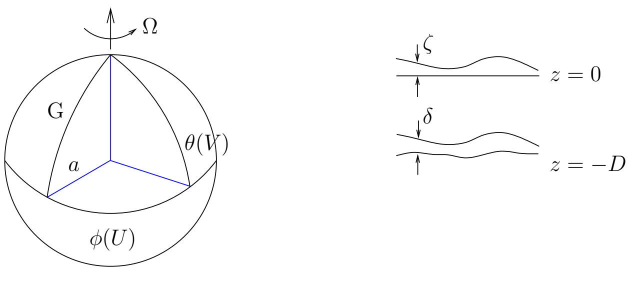

Figure 3: Solid earth tide, ζ is geocentric surface tide and δ is geocentric solid earth tide.

We can then write the tide generating potential in spherical harmonic (n) decomposition including the solid Earth tide as,

$$\Gamma_n = U_n + k_n U_n + q\alpha_n \zeta_{on} + k'_n q\alpha_n \zeta_{on}, \tag{17}$$

where U_n is obtained from astronomy, $k_n U_n$ is earth yielding to astronomical potential, $q\alpha_n\zeta_{on}$ is potential of tidal shell and $k'_n q\alpha_n\zeta_{on}$ is earth yielding to tidal potential.

Proceeding as above, solid earth tide is given as,

$$\delta_n = h_n \frac{U_n}{g} + h'_n \alpha_n \zeta_{on}. \tag{18}$$

k_n, k'_n, h_n and h'_n are all "Love numbers" similar to h in (16). From (17) and (18) we find

$$\left(\frac{\Gamma}{g} - \delta\right)_n = \left(\frac{1 + k_n - h_n}{g}\right) U_n + (1 + k'_n - h'_n)\alpha_n \zeta_{on}. \tag{19}$$

And upon summing up this series we get Farrell Green's function [2]

$$\sum_n (1 + k'_n - h'_n)\alpha_n \zeta_{on} = \int\int d\theta' d\phi' G_F(\theta', \phi'|\theta, \phi)\zeta_0(\theta', \phi'). \tag{20}$$

Now taking into account the solid Earth tide, we can rewrite Euler's equations from Lecture 2 as

$$\zeta_0 = \zeta - \delta \ , \tag{21}$$

$$u_t - (2\Omega\, sin\theta)v = -\frac{g(\zeta_0 - (\Gamma/g - \delta))_\phi}{a\, cos\theta} + \frac{F^\phi}{\rho D} \ , \tag{22}$$

$$v_t + (2\Omega\, sin\theta)u = -\frac{g(\zeta_0 - (\Gamma/g - \delta))_\theta}{a} + \frac{F^\theta}{\rho D} \ , \tag{23}$$

$$\begin{aligned}(\zeta_{ot} - \delta_t) + \frac{1}{a\, cos\theta}[(uD)_\phi + (vD\, cos\theta)_\theta] = &-\sum_n \left(\frac{1 + k_n - h_n}{g}\right) U_n \\ &- \int\int d\theta' d\phi' G_F(\theta', \phi'|\theta, \phi)\zeta_0(\theta', \phi') \ ,\end{aligned} \tag{24}$$

where U is mostly U_2, $k_2 \approx 0.29$ and $h_2 \approx 0.59$.

4 Dissipation Models

It is not easy to estimate the dissipation terms (F^θ and F^ϕ) in (22) and (23). This dissipation is mainly due to bottom drag and internal tides. If we model it as bottom drag, we get

$$\underline{F} = -\rho C_D |\underline{u}| \underline{u}, \tag{25}$$

where $C_D \approx 0.0025$ known from direct measurement in shallow water. The direct effect of (25) is that most of dissipation is limited to shallow seas where u and C_D are large. However, global tidal computations are mostly confined to deep-water zones for practical reasons (shallow water tides require much finer grid-spacing). So dissipation can only be properly represented by radiation of energy out of the model into bounding seas.

There are two main empirical models used to get an expression for dissipation in deep oceans. These are,

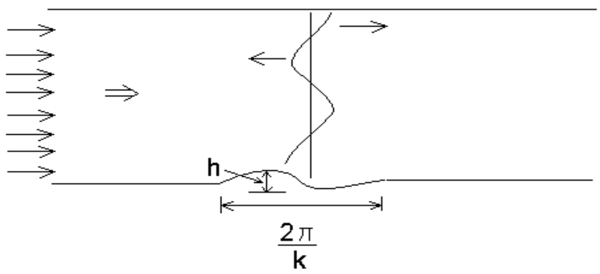

Figure 4: Jayne & Laurent model, dissipation due to barotropic tide scattering into internal tides over rough topography.

Jayne & Laurent In their runs of the Hallberg Isopycnal Model (HIM), the dissipation term is modeled as,

$$\underline{F} = -\frac{1}{2}\rho k h^2 N_b \underline{u}. \tag{26}$$

This model is based on assumption that dissipation occurs due to barotropic tide scattering into internal tides due to the rough bottom topography. h represents the height of bottom topography whose dominant horizontal wavenumber is k (see figure 4).

Arbic In his runs of the HIM for the dissipation term we have,

$$\begin{aligned}\underline{F} &= -p'_{IW}\nabla h \\ &= \rho(\nabla\chi \cdot \underline{u})\underline{\nabla} h,\end{aligned} \tag{27}$$

where,

$$\chi = \frac{N_b\sqrt{\sigma^2 - f^2}}{2\pi\sigma}\int\int \frac{h(x')}{|x - x'|}dx'dy' \tag{28}$$

for tide of frequency σ. N_b is buoyancy freqency, f is the Coriolis parameter and $h(x)$ is bottom topography. Arbic's model is based on the assumption that tidal dissipation can be calculated by finding the pressure drop in tidal currents across topographic features at the bottom.

5 Boundary Conditions

Laplace tidal equations have never been solved well enough so as to remove tides from altimetry without data assimilation. Different methods use different boundary conditions for solution of LTE at numerical coast. Some of the main boundary conditions in use are,

1. $\underline{u} \cdot \hat{h} = 0$ at the numerical coast.
   This is the most commonly used boundary condition. This boundary condition represents a no-energy-flux coast. It is important to have correct information regarding dissipation if this boundary is used since all energy must be dissipated within the system. Numerical schemes need to resolve $-\rho C_D \underline{u}|\underline{u}|$ well which requires high resolution in shallow waters.

2. $\zeta = \zeta_{obs}$ at the numerical coast.
   This boundary condition allows a energy flux ($\rho g D|\underline{u}| \cdot \hat{h}\zeta$) through the coast. The scheme is less sensitive to the details of the dissipation model used, and is less sensitive to the discretization used. However, this system can still respond resonantly. Another problem with this scheme is that observed tidal data is not easily available along all coasts.

3. $\zeta = c\ (\underline{u} \cdot \hat{h})$ at the numerical coast.
   This boundary allows energy to be dissipated at coast. In this boundary the parameter c can be adjusted so as to get results to match the observed tidal results. This leads to an energy flux of $< \rho g D c \zeta^2 >$ flowing out of the coast.

6 Energetics

If we ignore the solid Earth tide, we can derive equations of energy and perhaps estimate dissipation due to the tides as a residual. Starting with Laplace's tidal equations,

$$\begin{aligned} u_t - fv &= -g(\zeta - \Gamma/g)_x + F^x/\rho D \qquad && \times \rho u D \\ v_t + fu &= -g(\zeta - \Gamma/g)_y + F^y/\rho D && \times \rho v D \\ \zeta_t + (uD)_x + (vD)_y &= 0 && \times \rho g \zeta \end{aligned}$$

Then multiplying by the terms at right, adding the three equations together and assuming that ρ, g and D are constant, we arrive at

$$\frac{1}{2}\rho D(u^2+v^2)_t + \frac{1}{g}\rho g(\zeta^2)_t + \nabla\cdot(\rho g \zeta u D) = \rho\zeta_t\Gamma + \nabla\cdot(\rho u D \Gamma) + u\cdot F \tag{29}$$

$$KE_t + PE_t + \nabla \cdot Eflux = \begin{array}{c}\textit{Fluid crossing}\\ \textit{equipotentials}\\ \textit{vertically}\end{array} + \begin{array}{c}\textit{Fluid crossing}\\ \textit{equipotentials}\\ \textit{horizontally}\end{array} + \begin{array}{c}\textit{Work by}\\ \textit{dissipative}\\ \textit{forces}\end{array} \tag{30}$$

where KE_t is the time derivative of kinetic energy, PE_t is the time derivative of potential energy, and $Eflux$ is energy flux.

Energy Averaged Over One Tidal Period, Integrated Over the Ocean

It is convenient to consider the energy as averaged over one tidal period. For a periodic tide, let $< \cdot >$ denote the average over one period. This will simplify the above equations, since

$$< KE_t > = < PE_t > = 0. \tag{31}$$

Then from (29) we are left with

$$\nabla\cdot < P > = < W_t > + < u \cdot F >, \tag{32}$$

where P is energy flux and W_t is the working by potential vertical and horizontal forces.

Now if we reconsider the case of the basin with no flow through its boundaries ($\vec{u}\cdot < \hat{n} > = 0$), then we further have that $\int \nabla\cdot < P > dxdy = 0$ since there can be no net energy flux into or out of the basin. Then we can integrate the remaining terms in the energy equation over the ocean basin and find that

$$\int < W_t > dxdy = \int_{ocean} \rho \zeta_t \Gamma dxdy = -\int_{ocean} < u \cdot F > dxdy. \tag{33}$$

One caveat is that the solid Earth tide is not dissipation free, i.e. the Love numbers are complex, but this equation is true provided that they are real. Now, since $-\int < u \cdot F > dxdy$ is balanced by working of fluid moving up and down. This fluid movement ζ can be measured from global altimetry, giving an estimate if dissipation of energy due to the tides.

6.1 Including the solid earth tide

If we now include the solid Earth tide, then our third equation becomes

$$(\zeta - \delta)_t + \nabla \cdot \vec{u} D = 0 \tag{34}$$

and if we follow the same procedure as before we have

$$\begin{aligned} KE_t &= -\nabla \cdot (\rho g D \vec{u}(\zeta - \Gamma/g)) + \rho g(\zeta - \Gamma/g) \nabla \cdot \vec{u} D + u \cdot F \\ &= -\nabla \cdot (\rho g D \vec{u}(\zeta - \Gamma/g)) - \rho g(\zeta - \Gamma/g)(\zeta_t - \delta_t) + \vec{u} \cdot F \\ &= \nabla \cdot \rho D u \Gamma + \rho(\zeta - \delta)_t \Gamma - \nabla \cdot \rho g D u \zeta - \rho g \zeta(\zeta - \delta)_t + u \cdot F. \end{aligned}$$

Similarly for potential energy,

$$\begin{aligned} PE &= \int_{-D+\delta}^{\zeta} \rho g z dz = \frac{1}{2} \rho g(\zeta^2 - (-D + \delta)^2), \\ PE_t &= \rho g(\delta \delta_t - \zeta \zeta_t - D \zeta_t). \end{aligned}$$

Adding these together we find that

$$\begin{aligned} KE_t + PE_t + \nabla \cdot (\rho g u D \zeta) &= \nabla \cdot \rho D u \Gamma + \rho(\zeta - \delta)_t \Gamma + u \cdot F \\ &\quad - \rho g \zeta(\zeta_t - \delta_t) + \rho g(\zeta \zeta_t - \delta \delta_t + D \delta_t) \\ &= \nabla \cdot \rho D u \Gamma + D u \Gamma + \rho(\zeta - \delta)_t \Gamma + u \cdot F + \rho g(\zeta - \delta + D)\delta_t. \end{aligned}$$

Then if we consider the observed tide only, $\zeta_0 = \zeta - \delta$, the difference between the ocean tides and the solid earth tide, we have the energy equation.

$$KE_t + PE_t + \nabla \cdot \vec{P} = W_t + u \cdot F, \tag{35}$$

with

$$KE = \frac{1}{2} \rho D(u^2 + v^2), \tag{36}$$

$$PE = \frac{1}{2} \rho g(\zeta_0^2 + 2\zeta_0 \delta + 2\delta D), \tag{37}$$

$$\vec{P} = \rho g D \vec{u}(\zeta_0 + \delta), \tag{38}$$

$$W_t = \rho \zeta_{0_t} \Gamma + \rho \nabla \cdot u D \Gamma + \rho g(\zeta_0 + D)\delta_t. \tag{39}$$

Energy Averaged Over One Tidal Period, Integrated Over the Ocean

If we again average over one tidal period, then

$$\nabla\cdot < P >=< W_t > + < u \cdot F > \tag{40}$$

Given altimeter data ζ_1 it may be possible to map $< u \cdot F >$ [3].

If we further assume that the tides are periodic as $(e^{-i\sigma t})$, then noting that in the equation for W_t that $\int_{ocean} \rho \nabla \cdot u D\Gamma = 0$ and $< \rho g D \delta_t >= 0$,

$$\int_{oc} < W_t > dxdy = \frac{1}{2} Re \left\{ int_0 - \sigma\rho\zeta_0\Gamma^* + i\sigma g\rho\zeta_0\delta^* \right\},$$

where $(\cdot)^*$ is the complex conjugate. Using the Love number decomposition from §3,

$$\begin{aligned}\int_{oc} < W_t > dxdy &= \frac{1}{2} Re \int_0 \sum_n -i\sigma\rho\zeta_{0n} \left((1+k_n)u_n^* + g\alpha_n\zeta_{0n}^* + k_n' g\alpha_n\zeta_{0n}^* \right) \\ &\qquad + i\sigma\rho\zeta_{0n}(h_n u_n^* + \delta h_n' \alpha_n \zeta_{0n}^*) \\ &= \frac{1}{2} Re \sum_n \int -i\sigma\rho(1+k_n-h_n)\zeta_{0n}u_n^*. \end{aligned} \tag{41}$$

This last equation is true provided that the solid earth tide is dissipation-less, that is to say, that the Love numbers are real. Now, since again $\int_{ocean} \nabla\cdot < P > dxdy = 0$,

$$\int_{ocean} < u \cdot F > dxdy = \int_{ocean} < W_t > dxdy = \frac{1}{2} Re \sum_n \int -i\sigma\rho(1+k_n-h_n)\zeta_{0n}u_n^* \tag{42}$$

and we can estimate the dissipation of energy due to the tides if we know the observed tides, ζ_0.

Notes by Vineet Birman and Eleanor Williams Frajka

References

[1] C. L. Provost, M. L. Genco, F. Lyard, P. Vincent, and P. Canceil, "Spectroscopy of the world ocean tides from a finite element hydrodynamic model," J. Geophys. Res. **99**, 24777 (1994).

[2] W. E. Farrell, "The effect of ocean loading on tidal gravity," EME Symposium Marees Terrestres Strasbourg: OBS. ROY. BELG. COMM 116 .

[3] G. D. Egbert and R. D. Ray, "Significant dissipation of tidal energy in the deep ocean inferred from satellite altimeter data," Nature **405**, 775 (2000).

Investigator	Type of Model	Boundary Condition	Dissipation	Earth Tide
Zahel (1970)	M2, K1 tides for globe	Impermeable coast	Bottom stress in shallow water	None
Pekeris et al. (1969)	M2 tide for globe	Impermeable coast	Linearized artificial bottom stress in shallow water	None
Hendershott (1972)	M2 tide for globe	Coastal elevation specified	At coast only	None
Bogdanov et al. (1967)	M2, S2, K1, 01 tides for globe	Coastal elevation specified	At coast only	None
Tiron et al. (1967)	M2, S2, K1, 01 tides for plane model	Coastal elevation specified where data exist, otherwise impermeable coast	At coast only	None
Gohin (1971)	M2 tide for Atlantic, Indian Oceans	Coastal impedance specified	At coast only	Yielding to astronomical force only
Platzman (1972)	Normal modes for major basins, especially Atlantic	Adiabatic boundaries	Zero	-
Munk et al. (1970)	Normal mode representation of California coastal tides	Impermeable coast	Zero, although energy flux parallel to coast may reflect dissipation up coast	Yielding to astronomical force only
Cartwright (1971)	β plane representation for South Atlantic tides	Island values specified	Zero, although energy flux may reflect distant dissipation	Yielding to astronomical force only
Ueno (1964)	M2 tide for globe	Impermeable coast	None	None
Gordeev et al. (1972)	M2 tide for globe	Impermeable coast	Linearized bottom stress in shallow water	None

Table 1: Summary of Large-Scale Tidal Models.