

# Intermittency in Some Simple Models for Turbulent Transport

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## 1 Introduction

Consider the passive scalar equation

$$T_t + (u \cdot \nabla)T = \varepsilon \Delta T + F, \quad T|_{t=0} = \phi \quad (1)$$

where  $T$  is a quantity which is passively advected by a fluid with velocity  $u$  and  $F$  is an external forcing. The quantity  $T$  can be for instance heat, or dye used in visualizing turbulent effects, or a pollutant. The passive term refers to the fact that the effect of  $T$  on the fluid is negligible so that one can regard  $u$ , the velocity of the fluid, as being an externally given quantity, which does not depend on the evolution of  $T$ .

Although (1) is a linear equation for  $T$ , the relation between the passive scalar field  $T$  and the velocity field  $u$  is nonlinear. The influence of the velocity field on the statistics of  $T$  is very subtle and difficult to analyze in general. For instance, the interplay between  $u$  on the one hand and  $F$  and  $\phi$  on the other hand may lead to rare but large amplitude fluctuations of  $T$  (in space, time or both) which differ considerably from the average and contribute significantly to the statistics.

The matter of interest is then how rare these large fluctuations are. In many situations, based on the Central Limit Theorem as a heuristical principle, one would expect things to organize themselves so that in the average the distribution of the variable of interest is Gaussian. But large fluctuations can be more frequent than what is required for the Central Limit theorem to apply, and then fluctuations can dominate the statistics in a non-Gaussian way. This phenomenon is referred to as intermittency.

One signature of intermittency is the presence of non-Gaussian tails for the probability distribution function (PDF from now on) of  $T$ . It should be mentioned that there are physical experiments where such a behavior has been observed ([1],[2]).

We will be interested in identifying flows as simple as possible in which the large scale intermittency appears. Our goal is thus to identify some of the simplest mechanisms capable of producing intermittency.

While the flows we choose are simplistic, these models can provide intuition about the phenomena that occur in real turbulence, and with these specific choices the calculations are completely rigorous and unambiguous. This is the path followed also in [5] and there

one can find some more discussions on the relevance of this kind of approach. Results in a similar framework can be found in [3], [4],[6].

Our choice of flows will fall in the general class of flows proposed by M. Avellaneda and A. Majda, namely flows of the type

$$u(x, y, t) = \begin{pmatrix} w(t) \\ v(x, t) \end{pmatrix}$$

which can be regarded as nonlinear two dimensional shear velocity fields.

## 2 Heuristics

We will start by offering a heuristical interpretation of the mechanism of intermittency in a general setting and then rigorously prove it for a particular choice of flow. The explanation for the decaying case (no forcing) has already been given in [3] and it is included here for the sake of completeness.

In the decaying case we have the following representation formula for the solution of (1)

$$T(x, t) = \int_{\mathbb{R}^d} \phi(y) g(t, x, y) dy \quad (2)$$

where  $g(t, x, y)$  is the random function (for fixed  $x$  and  $t$ ) giving the probability density function of  $X(t)$  in each realization of  $u$ , where  $X(t)$  is the solution of the characteristic SDE associated to (1))

$$dX(t) = u(X(t), t)dt + \sqrt{2\epsilon}d\beta(t), X(0) = x \quad (3)$$

where  $\beta(t)$  is a Brownian motion accounting for molecular diffusion. In terms of  $X(t)$ , (2) can be written as

$$T(x, t) = \mathbb{E}_x^\beta \phi(X(t))$$

where  $\mathbb{E}_x^\beta$  denotes expectation over  $\beta(t)$  conditional on  $X(0) = x$ .

As time evolves  $g(t, x, y)$  broadens and assuming that  $\phi$  has mean zero, it is clear from the representation formula (2) that the dynamics will smooth out any spatial fluctuations in the initial data, with an average rate depending on the average growth rate of the width of  $g(t, x, y)$ .

On the other hand in any realization where  $g(t, x, y)$  broadens abnormally slowly, one will observe a large fluctuation in the scalar field amplitude at point  $x$ , even if the initial data sampled by  $X(t)$  is very typical (see Figure 1).

The situation is different in the forced case. Then, the representation formula for the solution becomes

$$T(x, t) = \int_{\mathbb{R}^d} \phi(y) g(t, x, y) dy + \int_0^t \int_{\mathbb{R}^d} F(y) g(t-s, x, y) dy ds \quad (4)$$

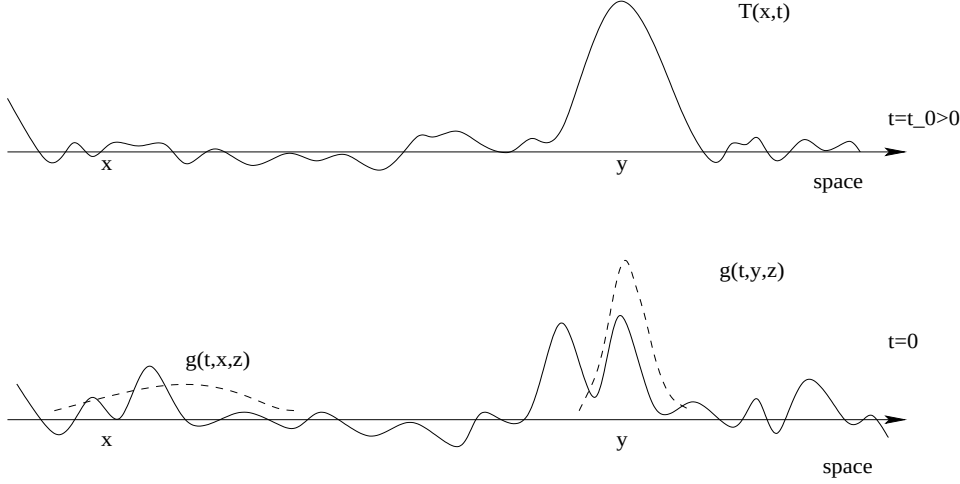


Figure 1: The heuristics in the decaying case

Equivalently, the above formula can be written in terms of averages over  $X(t)$ , giving us a Lagrangian picture of the evolution of  $T$

$$T(x,t) = \mathbb{E}_x^\beta \phi(X(t)) + \int_0^t \mathbb{E}_x^\beta F(X(t-s)) ds \quad (5)$$

In general, as  $t \rightarrow \infty$  one has  $\mathbb{E}_x^\beta \phi(X(t)) = \int_{\mathbb{R}^d} \phi(y) g(t,x,y) dy \rightarrow 0$  and thus one is left with analyzing the effect of the forcing term

$$\int_0^t \mathbb{E}_x^\beta F(X(t-s)) ds \quad (6)$$

Generically the trajectories of the  $X(t-s)$  will tend not to be on the level curves of  $F$ , and given the mixing effect of the flow they will be relatively uniformly spread within a short time, so  $F(X(t-s))$  will average to a zero value. On the other hand there will be (rare!) realizations of  $u$  where the effect of mixing will not be so strong, the diffusion will be the main (slow) mechanism for the spreading of  $X(t)$ , which will happen in a slow time. Thus, those  $X(t-s)$  which start on a level curve of  $F$  will have the possibility of remaining on the level curves for a time long enough so that the average of  $F(X(t-s))$  will be equal to a nonzero constant (close to the value of  $F$  on that level set).

In both cases rare realizations may have very strong effects on large scales. This will prevent the type of averaging which leads, by the Central Limit Theorem to a Gaussian distribution for the PDF of  $T$ , and indeed one will observe "fat" (non-Gaussian) tails for the PDF, consistent with large scale intermittency.

In our approach we will use a Lagrangian picture as this offers a simple understanding of the phenomena which occur. Indeed, we will consider the associated stochastic differential equations associated to the passive scalar equation and we will use them to obtain

representations formula for  $T$  from which we will compute the PDF of  $T$ . Thus, one can see the appearance of intermittency as the result of clustering of close trajectories in the realizations where the effect of turbulent mixing is abnormally weak.

### 3 The Decaying Case

We will take the flow to be

$$u = \begin{pmatrix} 0 \\ g \sin(x) \end{pmatrix} \quad (7)$$

where  $g$  is a Gaussian random variable, with mean zero and variance one. This is a time independent, "periodic shear" analogue of the random shear model of A.Majda (see [5]), model in which the  $\sin(x)$  from our equation is just  $x$  and  $g$  is time dependent.

In this case (1) reduces to

$$\frac{\partial T}{\partial t} + g \sin(x) \frac{\partial T}{\partial y} = \epsilon \Delta T, \quad T|_{t=0} = \phi(y) \quad (8)$$

We also assume that the initial data depends only on  $y$  and it is a mean zero Gaussian random process, statistically independent of the random velocity field, and

$$\phi(y) = \int_{\mathbb{R}} e^{ipy} \sqrt{E(p)} dW(p) \quad (9)$$

with energy spectrum

$$E(p) = C_E |k|^\alpha \psi(k) \quad (10)$$

where  $\psi(k)$  is a cutoff function, rapidly decaying for  $|k| > 1$ ,  $\psi(0) = 1$  and satisfying  $\psi(k) = \psi(-k)$ . The quantity  $C_E$  is a normalizing constant and  $dW$  is a complex white noise process (independent of  $g$ ), with

$$\langle dW(p), d\bar{W}(q) \rangle = \delta(p - q) dp dq$$

The exponent  $\alpha > -1$  in formula (10) measures the decay of the spatial correlation of the initial condition  $\phi(y)$ ; the smaller  $\alpha$ , the longer the spatial correlation.

In the case where there is no forcing, it is well known that  $T$  will decay to zero. Therefore, in order to observe the intermittency we will look at  $T$  rescaled by the energy  $E = \mathbb{E}T^2$  which we shall denote by  $\theta = \frac{T}{\sqrt{E}}$ . We will compute in the following the PDF of  $\theta$  in the long time limit and we will obtain that

$$\mathbb{P}(\bar{X} > \lambda) \approx C_1 \lambda^{-\frac{2}{\alpha+1}} \text{ as } \lambda \rightarrow \infty \quad (11)$$

where  $C_1$  is a constant, independent of time, whose value can be computed explicitly (in general in the following  $C_1, C_2, \dots$  will be used to denote constants which can be explicitly computed, and are independent of time).

Consider the stochastic differential equations associated with (8)

$$\begin{cases} X(t) = \sqrt{2\epsilon}d\beta_x(t) \\ Y(t) = g \sin(X(t))dt + \sqrt{2\epsilon}d\beta_y(t) \end{cases} \quad (12)$$

The equations (12) have the solution

$$\begin{cases} X(t) = x + \sqrt{2\epsilon}\beta_x(t) \\ Y(t) = y + g \int_0^t \sin(x + \sqrt{2\epsilon}\beta_x(s))ds + \sqrt{2\epsilon}\beta_y(t) \end{cases} \quad (13)$$

Since  $g$  and  $\beta$  are independent, we have then that

$$\begin{aligned} \mathbb{E}_\beta Y(t) &= y + g \int_0^t \sin(x) e^{-\epsilon s} ds = y + \frac{g}{\epsilon} \sin(x)(1 - e^{-\epsilon t}) \\ &\sim y + \frac{g}{\epsilon} \sin(x), \text{ as } t \rightarrow \infty \end{aligned} \quad (14)$$

and

$$\begin{aligned} &\mathbb{E}_\beta(Y(t) - \mathbb{E}_\beta(Y(t)))^2 = \\ &= \mathbb{E}_\beta g^2 \int_0^t \int_0^t \sin(x + \sqrt{2\epsilon}\beta_x(s)) \sin(x + \sqrt{2\epsilon}\beta_x(s')) ds ds' + \epsilon t + g^2 \left( \int_0^t \sin(x) e^{-\epsilon s} ds \right)^2 \\ &= \frac{g^2}{2} \int_0^t \int_0^t (e^{-\epsilon|s-s'|} - \cos(2x) e^{-\epsilon(s+s'+2\min(s,s'))}) ds ds' + \epsilon t + \frac{g^2}{\epsilon^2} \sin^2(x)(1 - e^{-\epsilon t})^2 \\ &= \epsilon t + g^2 \left( \frac{t}{\epsilon} + \frac{1-e^{-\epsilon t}}{\epsilon^2} \right) - \frac{g^2 \cos(2x)}{2\epsilon^2} \left( \frac{e^{-4\epsilon t}-1}{3} - \frac{7(e^{-\epsilon t}-1)}{12} \right) + g^2 \sin^2(x) \frac{(e^{-\epsilon t}-1)^2}{\epsilon^2} \end{aligned} \quad (15)$$

Thus, for  $t \gg 1$  we have

$$\mathbb{E}_\beta(Y(t) - \mathbb{E}_\beta(Y(t)))^2 \sim \epsilon t + g^2 \frac{t}{\epsilon} \quad (16)$$

Next observe that, using (9), we have an explicit representation of  $T$  as

$$T = \mathbb{E}_\beta \phi(Y_t) = \int_{R^2} e^{ipm(t) - \frac{1}{2}p^2 v(t)} \sqrt{E(p)} dW(p) \quad (17)$$

where we used the fact that  $\phi$  is a function of only one variable;  $m(t)$  and  $v(t)$  are respectively the mean and variance of  $Y(t)$  with respect to the Brownian motion  $\beta$  which, taking into account (14) and (16), for large  $t$  become

$$\begin{aligned} m(t) &\sim y + \frac{g}{\epsilon} \sin(x) \\ v(t) &\sim \epsilon t + g^2 \frac{t}{\epsilon} \end{aligned} \quad (18)$$

Introduce the rescaled variable

$$z = p\sqrt{v(t)} \quad (19)$$

and the rescaled white noise

$$d\hat{W}_t(z) \stackrel{d}{=} v(t)^{\frac{1}{4}} dW\left(\frac{z}{\sqrt{v(t)}}\right) \quad (20)$$

where  $\stackrel{d}{=}$  stands for the equality in the sense of distributions. In terms of these quantities we can rewrite the representation formula of  $T$  as

$$T \stackrel{d}{=} \int_{\mathbb{R}} e^{i\frac{z}{\sqrt{v}}m - \frac{z^2}{2}} v(t)^{-\frac{1+\alpha}{4}} \sqrt{C_E} |z|^{\frac{\alpha}{2}} \psi^{1/2} \left( \frac{z}{\sqrt{v(t)}} \right) d\hat{W}_t(z) \quad (21)$$

Therefore (using (18))

$$\bar{\phi} - \sqrt{C_E} \int_{\mathbb{R}} e^{i\frac{z}{\sqrt{v(t)}}m(t) - \frac{z^2}{2}} |z|^{\frac{\alpha}{2}} \psi^{1/2} \left( \frac{z}{\sqrt{v(t)}} \right) d\hat{W}_t(z) \rightarrow 0 \text{ as } t \rightarrow \infty$$

where the limit here and below is understood in the sense of distributions and

$$\bar{\phi} = \sqrt{C_E} \int_{\mathbb{R}} e^{-\frac{z^2}{2}} |z|^{\frac{\alpha}{2}} d\hat{W}_t(z) \quad (22)$$

This implies that for large times we have

$$T(t, \cdot) \stackrel{d}{\sim} v(t)^{-\frac{(1+\alpha)}{4}} \bar{\phi} \quad (23)$$

Using the explicit formula for  $T$  we can compute  $E(t) = \mathbb{E}_{g,\beta} T^2$  which is

$$\mathbb{E}_{g,\beta} T^2 = C_E \mathbb{E}_g \left[ v(t)^{-\frac{1+\alpha}{2}} \int_{\mathbb{R}} e^{-z^2} |z|^{\alpha} \psi \left( \frac{z}{\sqrt{v(t)}} \right) dz \right] \quad (24)$$

From (18), this is

$$E(t) \stackrel{d}{=} C_2 t^{-\frac{(1+\alpha)}{2}} + o(t^{-\frac{(1+\alpha)}{2}}) \quad (25)$$

By rescaling  $T$  we will obtain a finite limit, namely let us consider the quantity:

$$\theta = \frac{T}{\sqrt{E}} \quad (26)$$

Then the above allow us to conclude that

$$\theta(t, \cdot) \stackrel{d}{\sim} \bar{\theta}(t) \text{ as } t \rightarrow \infty \quad (27)$$

where

$$\bar{\theta} = C_3 \left( \epsilon + \frac{g^2}{\epsilon} \right)^{-\frac{1+\alpha}{4}} \bar{\phi} \quad (28)$$

We can now compute the tails of the probability distribution of  $\bar{\theta}$ . As  $\bar{\phi}$  is normally distributed with mean zero and variance  $\bar{\sigma}$  (which can be explicitly computed, see (22)), we have (assuming without loss of generality that  $C_3 = 1, \bar{\sigma} = 1$  for the sake of simplifying the computation):

$$\mathbb{P}(\bar{\theta} > \lambda) = P\left(\left(\epsilon + \frac{2g^2}{\epsilon}\right)^{-\frac{1+\alpha}{4}} \bar{\phi} > \lambda\right) = \int_0^\infty \int_{|a|\delta^{-\frac{1+\alpha}{4}} \geq \lambda} \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} da dP_\delta \quad (29)$$

where

$$P_\delta = \mathbb{P}\left(\epsilon + \frac{2g^2}{\epsilon} \leq \delta\right) = \int_{-\sqrt{\frac{|\epsilon\delta - \epsilon^2|}{2}}}^{\sqrt{\frac{|\epsilon\delta - \epsilon^2|}{2}}} e^{-\frac{z^2}{2}} dz \quad (30)$$

Integrating by parts in (29) we obtain:

$$\mathbb{P}(\bar{\theta} > \lambda) = \frac{(1 + \alpha)\lambda}{2\sqrt{2\pi}} \int_0^\infty \delta^{\frac{\alpha-3}{4}} e^{-\frac{1}{2}\lambda^2\delta^{\frac{1+\alpha}{2}}} P_\delta d\delta \quad (31)$$

In order to compute the integral, for  $\lambda \gg 1$  thanks to the exponential factor and to the Laplace method, we only need to know  $P_\delta$  for small  $\delta$ . Using the change of variables  $\delta = s\lambda^{-\frac{4}{1+\alpha}}$  by standard, though tedious, computations we get (11) to the leading order in  $\lambda$  (as  $\lambda \rightarrow \infty$ ).

## 4 The Forced Case

### 4.1 The one mode, time independent, stirring

We will consider the flow to be given by:

$$u = \begin{pmatrix} g \\ \sin(x + \varphi) \end{pmatrix} \quad (32)$$

where  $g$  is a Gaussian random variable which has mean zero and variance one and  $\varphi$  is a random variable uniformly distributed on  $[0, 2\pi]$ . The two random variables are independent.

The passive scalar equation becomes

$$T_t + gT_x + \sin(x + \varphi)T_y = \epsilon\Delta T \quad (33)$$

Assume also that the mean gradient of  $T$  is imposed

$$T = \frac{y}{L} + \tilde{T} \quad (34)$$

Then  $\tilde{T}$  will satisfy the equation

$$\tilde{T}_t + g\tilde{T}_x + \sin(x + \varphi)\tilde{T}_y + \frac{\sin(x + \varphi)}{L} = \epsilon\Delta\tilde{T} \quad (35)$$

In this specific case, the general heuristics from the second section can be made more precise and we have a simpler mechanism which is responsible for intermittency and can be understood as follows.

Let us assume that  $T$  represents the temperature and we have a region made of two parts, one hot ( $(x, y) \in \mathbb{R}^2$  with  $x > 0$ ) and one cold ( $(x, y) \in \mathbb{R}^2$  with  $x \leq 0$ ). In the generic case, when  $g \neq 0$  we will have transport in both  $x$  and  $y$  directions, and thus mixing of the cold and hot which will lead to a decrease in the average temperature. In the realizations

when  $g \approx 0$ , however, the flow  $u$  points only in the  $y$  direction, so (neglecting the effect of diffusion) there is transport only in the  $y$  direction. The hot region remains hot, and the cold one cold. The extreme values of temperature will not be significantly changed. Therefore one expects that the rare realizations where  $g \approx 0$  will strongly influence the average over all the realizations leading to a non-Gaussian distribution of  $T$ . Indeed, we will obtain that the tails of the PDF of  $T$  decay like  $\lambda^{-2}$ .

In order to make the above reasoning rigorous let us consider the stochastic differential equations associated to (33)

$$\begin{cases} dX(t) = gdt + \sqrt{2\epsilon}d\beta_x(t), & X_0 = x \\ dY(t) = \sin(X(t) + \phi)dt + \sqrt{2\epsilon}d\beta_y(t), & Y_0 = y \end{cases}$$

which have the solution

$$\begin{cases} X(t) = x + gt + \sqrt{2\epsilon}\beta_x(t) \\ Y(t) = y + \int_0^t \sin(x + \phi + gs + \sqrt{2\epsilon}\beta_x(s))ds + \sqrt{2\epsilon}\beta_y(t) \end{cases}$$

Assuming that the initial data is zero (if not it can be shown it decays) by Feynman-Kac formula we get the following representation of the solution

$$\tilde{T} = -\mathbb{E}_\beta \int_0^t \frac{1}{L} \sin(x + \phi + gs + \sqrt{2\epsilon}\beta_x(s))ds \quad (36)$$

Thus in each realization we have that

$$\lim_{t \rightarrow \infty} \tilde{T}(x, y, t) - \bar{T}(x, t) = 0 \quad (37)$$

where  $\bar{T}$  is

$$\bar{T}(x) = \frac{1}{L} \frac{[g \cos(x + \phi) - \epsilon \sin(x + \phi)]}{\epsilon^2 + g^2} \quad (38)$$

We are interested now in computing

$$f(\lambda) = \mathbb{P}(\bar{T}(x) \geq \lambda) \quad (39)$$

To this extent, taking into account the independence of  $\phi$  and  $g$ , we will compute first the moments only with respect to the uniformly distributed random variable  $\phi$ . Indeed, we have

$$\mathbb{E}_\phi T^{2n} = [L(\epsilon^2 + g^2)]^{-2n} \frac{1}{2\pi} \int_0^{2\pi} (g \cos(\phi) - \epsilon \sin(\phi))^{2n} d\phi \quad (40)$$



One can compute the last integral, namely

$$\begin{aligned}
\int_0^{2\pi} (g \cos(\phi) - \epsilon \sin(\phi))^{2n} d\phi &= \frac{1}{2^{2n}} \int_0^{2\pi} [(g + \epsilon i)e^{ix} + (g - \epsilon i)e^{-ix}]^{2n} dx \\
&= \frac{1}{2^{2n}} \int_0^{2\pi} \sum_{m=0}^{2n} \binom{2n}{m} (g + \epsilon i)^{2n-m} (g - \epsilon i)^m e^{ix(2n-m-m)} dx \\
&= \frac{1}{2^{2n}} \binom{2n}{n} (g^2 + \epsilon^2)^n
\end{aligned} \tag{41}$$

So

$$\begin{aligned}
\mathbb{E}_\phi e^{ikT} &= \sum_{n=0}^{\infty} \frac{(-1)^n k^{2n}}{(2n)!} \mathbb{E}_\phi T^{2n} \\
&= \sum_{n=0}^{\infty} (-1)^n \left(\frac{k}{2L}\right)^{2n} \frac{1}{(n!)^2} (\epsilon^2 + g^2)^{-n} \\
&= J_0 \left( \sqrt{\frac{-k^2}{L^2(\epsilon^2 + g^2)}} \right)
\end{aligned} \tag{42}$$

(where  $J_0$  is the Bessel function of the first kind) and thus

$$\hat{f}(k) = \int_{\mathbb{R}} \frac{e^{-\frac{g^2}{2}}}{\sqrt{2\pi}} J_0 \left( \sqrt{\frac{-k^2}{L^2(\epsilon^2 + g^2)}} \right) dg \tag{43}$$

Expressing  $f(\lambda)$  in terms of its inverse Fourier transform and using the fact that  $f(\lambda)$  is real valued we have

$$\begin{aligned}
f(\lambda) &= \Re \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(k) e^{ik\lambda} dk = \Re \frac{1}{\pi} \int_0^{\infty} \hat{f}(k) \cos(k\lambda) dk \\
&= \Re \frac{1}{\sqrt{2\pi}^{3/2}} \int_0^{\infty} \left( \int_{\mathbb{R}} e^{-g^2/2} J_0 \left( \sqrt{\frac{-k^2}{L^2(\epsilon^2 + g^2)}} \right) dg \right) \cos(k\lambda) dk \\
&= \Re \frac{1}{\sqrt{2\pi}^{3/2}} \int_{-\sqrt{\frac{1}{L^2\lambda^2} - \epsilon^2}}^{\sqrt{\frac{1}{L^2\lambda^2} - \epsilon^2}} e^{-g^2/2} \int_0^{\infty} J_0 \left( \sqrt{\frac{-k^2}{L^2(\epsilon^2 + g^2)}} \right) \cos(k\lambda) dk dg \\
&+ s \Re \frac{1}{\sqrt{2\pi}^{3/2}} \int_{\mathbb{R} \setminus [-\sqrt{\frac{1}{L^2\lambda^2} - \epsilon^2}, \sqrt{\frac{1}{L^2\lambda^2} - \epsilon^2}]} e^{-g^2/2} \int_0^{\infty} J_0 \left( \sqrt{\frac{-k^2}{L^2(\epsilon^2 + g^2)}} \right) \cos(k\lambda) dk dg
\end{aligned}$$

where for the second we used the fact that the Bessel function of the first kind is an even function; also for the fourth equality we used Fubini to interchange the order of integration.

Recall that

$$\int_0^{\infty} J_0 \left( \sqrt{\frac{-k^2}{L^2(\epsilon^2 + g^2)}} \right) \cos(k\lambda) dk = \sqrt{\frac{\epsilon^2 + g^2}{1 - L^2\epsilon^2\lambda^2 - L^2\lambda^2g^2}} \tag{44}$$

which is a real number if

$$g \in [-\sqrt{\frac{1}{L^2\lambda^2} - \epsilon^2}, \sqrt{\frac{1}{L^2\lambda^2} - \epsilon^2}] \quad (45)$$

and purely imaginary (i.e. with zero real part) otherwise. Using this observation and combining the last two relations with get:

$$f(\lambda) = \frac{1}{\sqrt{2}\pi^{3/2}} \int_{-\sqrt{\frac{1}{L^2\lambda^2} - \epsilon^2}}^{\sqrt{\frac{1}{L^2\lambda^2} - \epsilon^2}} e^{-g^2/2} \sqrt{\frac{\epsilon^2 + g^2}{1 - L^2\epsilon^2\lambda^2 - L^2\lambda^2g^2}} dg \quad (46)$$

which clearly holds if and only if  $\lambda < \frac{1}{\epsilon L}$ . On the other hand, taking into account the definition of  $\bar{T}(x)$  and of  $f(\lambda)$  it is easy to see that for  $\lambda > \frac{1}{\epsilon L}$  we will have  $f(\lambda) = 0$  and thus  $f(\lambda)$  is a function with bounded support.

It follows that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} f(\lambda) &= \frac{\sqrt{2}}{\pi^{\frac{3}{2}}} \int_0^\infty \frac{e^{-\frac{g^2}{2}} g}{\sqrt{1 - L^2\lambda^2g^2}} \\ &= \frac{e^{-\frac{1}{2L^2\lambda^2}} \operatorname{erf}(\frac{1}{\sqrt{L|\lambda|}})}{\pi L|\lambda|} \end{aligned} \quad (47)$$

which asymptotically, in the limit  $\lambda \rightarrow \infty$  behaves like

$$\lim_{\epsilon \rightarrow 0} f(\lambda) \approx \frac{\sqrt{2}}{\pi^{\frac{3}{2}}} \lambda^{-2} \quad (48)$$

## 4.2 The Gaussian multimode forcing

In this section we consider the flow

$$u = \begin{pmatrix} g \\ v(x) \end{pmatrix} \quad (49)$$

where  $g$  is a Gaussian random variable which has mean 0 and variance 1, and  $v(x)$  is a Gaussian process specified by

$$v(x) = \int_{\mathbb{R}} dW(k) \sqrt{E(k)} e^{ikx} \quad (50)$$

(we will need to assume that the function  $E(k)$  is compactly supported away from 0 and also that  $\int_{\mathbb{R}} \frac{E(k)dk}{k^2} < \infty$ ).

In this case we obtain a similar behavior as before, though the ingredients are quite different. Namely we will obtain that the tails of the PDF of  $T$  will decay like  $\lambda^{-2}$ .

Indeed, arguing analogously as before we will obtain the solution will evolve as  $t \rightarrow \infty$  to the solution of stationary equation

$$0 = -gT_x + \epsilon T_{xx} - \frac{v(x)}{L} \quad (51)$$

which will give us the representation formula for the solution:

$$\begin{aligned}
T(x) &= -\frac{1}{L} \int_0^\infty dt \int_{\mathbb{R}} dW(k) \sqrt{E(k)} e^{ik(x-gt)-\epsilon k^2 t} \\
&= -\frac{1}{L} \int_{\mathbb{R}} dW(k) \sqrt{E(k)} e^{ikx} \frac{1}{\epsilon k^2 + ikg}
\end{aligned} \tag{52}$$

This is a Gaussian random variable (as a superposition of Gaussians ) whose moments with respect to  $W$  are:

$$\mathbb{E}_W(T(x))^{2n} = \frac{(2n)!}{2^n n!} (\mathbb{E}_W T^2(x))^n = \frac{2n}{2^n n!} \left( \underbrace{\int_{\mathbb{R}} \frac{E(k) dk}{k^2(\epsilon k^2 + g^2)}}_{F(\epsilon, g)} \right)^n \tag{53}$$

Hence

$$\mathbb{E}_W e^{ikT} = \sum_{n=0}^{\infty} \frac{-1^n k^{2n}}{(2n)!} \cdot \frac{(2n)!}{2^n n!} F(\epsilon, g)^n = e^{-\frac{1}{2} k^2 F(\epsilon, g)} \tag{54}$$

which implies

$$f_\epsilon(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-g^2/2} e^{-\frac{1}{2} F(\epsilon, g)} e^{ik\lambda} dk dg \tag{55}$$

Since  $F(\epsilon, g) \rightarrow \frac{M}{g^2}$  (when  $\epsilon \rightarrow 0$ , with  $M = \int_{\mathbb{R}} \frac{E(k) dk}{k^2} < \infty$ ) and by Lebesgue's dominated convergence theorem it follows that  $f_\epsilon(\lambda) \rightarrow f(\lambda)$  where (assuming without loss of generality for the sake of computational simplicity that  $M = 1$ )

$$\begin{aligned}
f_\epsilon(\lambda) &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\frac{g^2}{2}} e^{-\frac{k^2}{2g^2} + ik\lambda} dk dg = \sqrt{2\pi} \int_{\mathbb{R}} e^{-\frac{g^2(1+\lambda^2)}{2}} g dg \\
&= \frac{1}{1+\lambda^2} \int_{\mathbb{R}} e^{-\frac{\delta^2}{2}} \delta d\delta = \frac{1}{\pi(1+\lambda^2)}
\end{aligned} \tag{56}$$

## 5 Acknowledgments

I want to thank Eric Vanden-Eijnden for suggesting these problems to me and for his help all over the summer. I would also like to thank Charles Doering for many useful suggestions and for his patience in practicing softball with me. Finally I want to thank Joe Keller for useful discussions, as well as all the fellows for a great summer.

## References

- [1] F. Heslot, B.Castaing, A.Libchaber *Transition to turbulence in helium gas*, Phys. Rev. A. 36(1987) 5870-5873
- [2] B.Castaing, G. Gunaratne, F. Heslot, L. Kadanoff, A.Libchaber, S.Thomae, X.Z.Wu, S.Zaleski, G. Zanetti *Scaling of hard thermal turbulence in Rayleigh-Benard convection* J. Fluid Mech. 204(1989) 1-30

- [3] E.Vanden Eijnden *Non-Gaussian invariant measures for the Majda model of decaying turbulent transport* Comm. Pure Appl. Math. 54 (2001), no. 9, 1146–1167
- [4] A.Bourlioux and A.J. Majda *Elementary models with probability distribution function intermittency for passive scalars with a mean gradient* Phys. Fluids 14 (2002), no. 2, 881–897
- [5] A.J. Majda and P.R. Kramer *Simplified models for turbulent diffusion: theory, numerical modelling, and physical phenomena* Phys. Rep. 314 (1999), no. 4-5, 237–574
- [6] R. McLaughlin and A.J. Majda *An explicit example with non-Gaussian probability distribution for nontrivial scalar mean and fluctuation* Phys, Fluids 8 (1996) no.2, 536-547