Lecture 8: Asymptotic techniques for SDEs

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Here we discuss techniques by which one can study SDEs evolving on very different time-scales and derive closed equations for the slow variables.

1 The case of stiff ordinary differential equations

We start with an ODE example. Consider

$$\begin{cases} \dot X_t = -Y_t^3 + \sin(\pi t) + \cos(\sqrt{2}\pi t) & X_0 = x \\ \dot Y_t = -\dfrac{1}{\varepsilon}(Y_t - X_t) & Y_0 = y. \end{cases} \tag{1}$$

If ε is very small, Y_t is very fast and one expects that it will adjust rapidly to the current value of X_t, i.e. $Y_t = X_t + O(\varepsilon)$ at all times. Then the equation for X_t reduces to

$$\dot X_t = -X_t^3 + \sin(\pi t) + \cos(\sqrt{2}\pi t). \tag{2}$$

The solutions of (1) and (2) are compared in figure 1.

Here is a formal derivation of the limiting equation (2) which uses the backward Kolmogorov equation. For simplicity we drop the term $\sin(\pi t) + \cos(\sqrt{2}\pi t)$. Generalizing the derivation below with this term included is easy but requires a slightly different backward equation because (2) is non-autonomous. Let f be a smooth function and consider

$$u(x, y, t) = f(X_t).$$

(This function depends on both x and y since X_t depends on both these variable because X_t and Y_t are coupled in (1), and there is no expectation since (1) is deterministic.) The backward equation is

$$\frac{\partial u}{\partial t} = L_x u + \frac{1}{\varepsilon} L_y u,$$

where

$$L_x = -y\frac{\partial}{\partial x}, \qquad L_y = -(y - x)\frac{\partial}{\partial y}.$$

Look for a solution of the form $u = u_0 + \varepsilon u_1 + O(\varepsilon^2)$, so that $u \to u_0$ as $\varepsilon \to 0$. Inserting this expansion into the backward equation, and grouping terms of same order in ε, one obtains

$$\begin{aligned} L_y u_0 &= 0, \\ L_y u_1 &= \frac{\partial u_0}{\partial t} - L_x u_0, \end{aligned} \tag{3}$$

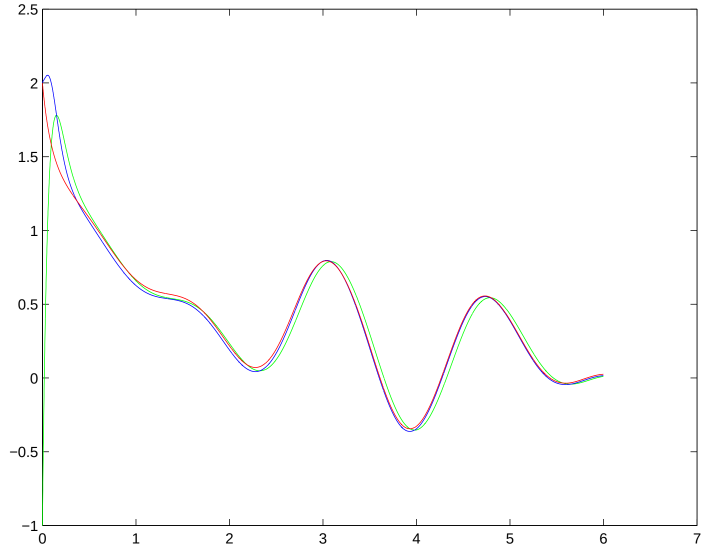

Figure 1: The solution of (1) when $\varepsilon = 0.05$ and we took $X_0 = 2$, $Y_0 = -1$. X_t is shown in blue, and Y_t in green. Also shown in red is the solution of the limiting equation (2).

and so on. The first equation tells that u_0 belong to the null-space of L_y, i.e. $u_0 = u_0(x,t)$. The second equation requires as a solvability condition that the right hand-side belongs to the range of L_y. To see what this condition actually is, multiply the second equation in (3) by a test function $\rho(y)$, and integrate both sides over $\mathbb{R}$. After integration by part at the left hand-side, this gives

$$\int_{\mathbb{R}} L_y^\star \rho(y) u_1 dy = \int_{\mathbb{R}} \rho(y) \Big(\frac{\partial u_0}{\partial t} - L_x u_0 \Big) dy.$$

where $L_y^\star$ is the adjoint of L_y viewed as an operator in y at fixed x, i.e.

$$L_y^\star \rho(y) = \frac{\partial}{\partial y}((y-x)\rho(y)).$$

Choosing $\rho(y)$ such that

$$0 = L_y^\star \rho(y), \tag{4}$$

one concludes that the solvability of (3) requires that

$$0 = \int_{\mathbb{R}} \rho(y) \Big(\frac{\partial u_0}{\partial t} - L_x u_0 \Big) dy. \tag{5}$$

It can be shown that this equation is also sufficient for the solvability of (3) – the calculation above actually tells the range of L_y is the space perpendicular to the null-space of the adjoint of L_y. Now, (4) is simply the forward Kolmogorov equation for the equilibrium density of the process Y_t at fixed $X_t = x$. Here the equilibrium density is a generalized function

$$\rho(y|x) = \delta(y-x).$$

Using this $\rho(y|x)$, the solvability condition (5) becomes

$$0 = \frac{\partial u_0}{\partial t} + x\frac{\partial u_0}{\partial x},$$

which is the backward equation for

$$\dot{X}_t = -X_t^3, \qquad X_0 = x.$$

A similar argument with the term $\sin(\pi t) + \cos(\sqrt{2}\pi t)$ included gives the backward equation for (2).

2 Generalization to stochastic differential equation

The derivation that lead to (2) can be generalized to SDEs. Consider

$$\begin{cases} dX_t = f(X_t, Y_t)dt, & X_0 = x \\ dY_t = \dfrac{1}{\varepsilon}b(X_t, Y_t)dt + \dfrac{1}{\sqrt{\varepsilon}}\sigma(X_t, Y_t)dt, & Y_0 = y, \end{cases} \tag{6}$$

and assume that the equation for Y_t at $X_t = x$ fixed has an equilibrium density $\rho(y|x)$ for every x. Then going through a derivation as above with

$$u(x, y, t) = \mathbb{E}f(X_t),$$

one concludes that the backward equation associated with this SDE also reduces to (5) as $\varepsilon \to 0$, i.e.

$$\frac{\partial u_0}{\partial t} = F(x)\frac{\partial u_0}{\partial x},$$

where

$$F(x) = \int_{\mathbb{R}} f(x, y)\rho(y|x)dy.$$

Thus the limiting equation for X_t is

$$\dot{X}_t = F(X_t), \qquad X_0 = 0.$$

The main difference with the deterministic example treated before is that the fast process Y_t does not rapidly settle to an equilibrium point depending on the current value of X_t – only its density does.

Here is an example generalizing (1). Consider

$$\begin{cases} dX_t = -Y_t^3 dt + \sin(\pi t) + \cos(\sqrt{2}\pi t), & X_0 = x \\ dY_t = -\dfrac{1}{\varepsilon}(Y_t - X_t)dt + \dfrac{\alpha}{\sqrt{\varepsilon}}dW_t, & Y_0 = y. \end{cases} \tag{7}$$

The equation for Y_t at fixed $X_t = x$ defines an Ornstein-Uhlenbeck process whose equilibrium density is

$$\rho(y|x) = \frac{e^{-(y-x)^2/\alpha^2}}{\sqrt{\pi}\alpha}.$$

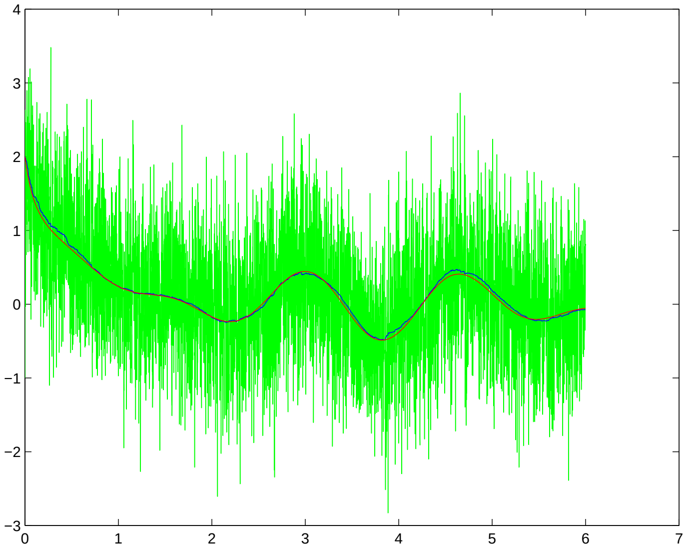

Figure 2: The solution of (7) with $X_0 = 2$, $Y_0 = -1$ when $\varepsilon = 10^{-3}$ and $\alpha = 1$. X_t is shown in blue, and Y_t in green. Also shown in red is the solution of the limiting equation (8). Notice how noisy Y_t is.

Therefore

$$F(x) = -\int_{\mathbb{R}} y^3 \frac{e^{-(y-x)^2/\alpha^2}}{\sqrt{\pi}\alpha} dy = -x^3 - \tfrac{3}{2}\alpha^2 x,$$

and the limiting equation is

$$\dot{X}_t = -X_t^3 - \tfrac{3}{2}\alpha^2 X_t + \sin(\pi t) + \cos(\sqrt{2}\pi t), \qquad X_0 = x. \tag{8}$$

Note the new term $-\frac{3}{2}\alpha^2 X_t$, due to the noise in (7). The solution of (7) and (8) are shown in figure 2.

3 Strong convergence and the property of self-averaging

The derivation in section 2 only give weak convergence, or convergence in distribution. But stronger results can be obtained. Consider a system of the form

$$\dot{X}_t^\varepsilon = f(X_t^\varepsilon, Y_{t/\varepsilon}), \tag{9}$$

where Y_t is a given stochastic process. Assume that Y_t is ergodic, in the sense that for any fixed x,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(x, Y_s) ds = \bar{f}(x). \tag{10}$$

Then we can show that, as $\varepsilon \to 0$, X_t^ε converges strongly to the solution of

$$\dot{\bar{X}}_t = \bar{f}(\bar{X}_t) \tag{11}$$

To see this, consider the integral form of (9):

$$X^{\varepsilon}_{t+\Delta t} - X^{\varepsilon}_t = \int_t^{t+\Delta t} f(X_s, Y_{s/\varepsilon}) ds. \tag{12}$$

We rewrite this equation in a way that allows us to exploit the self-averaging property (10).

$$X^{\varepsilon}_{t+\Delta t} - X^{\varepsilon}_t = \int_t^{t+\Delta t} f(X_t, Y_{s/\varepsilon}) ds + \int_t^{t+\Delta t} \big(f(X_s, Y_{s/\varepsilon}) - f(X_t, Y_{s/\varepsilon})\big)\, ds. \tag{13}$$

We will consider the behavior of these two integrals as $\varepsilon \to 0$ separately.

Using (10), the first integral

$$\int_t^{t+\Delta t} f(X_t, Y_{s/\varepsilon}) ds = \varepsilon \int_{t/\varepsilon}^{(t+\Delta t)/\varepsilon} f(X_t, Y_s) ds \to \Delta t \bar{f}(X_t), \tag{14}$$

as $\varepsilon \to 0$. To investigate the contribution of the second integral, let

$$A(t, \Delta t, \varepsilon) = \int_t^{t+\Delta t} \big(f(X_s, Y_{s/\varepsilon}) - f(X_t, Y_{s/\varepsilon})\big)\, ds. \tag{15}$$

We then have

$$|A(t, \Delta t, \varepsilon)| \le \int_t^{t+\Delta t} \big|f(X_s, Y_{s/\varepsilon}) - f(X_t, Y_{s/\varepsilon})\big|\, ds. \tag{16}$$

Assuming f is uniformly Lipschitz in Y_t with constant K, we then write

$$\begin{aligned} |A(t, \Delta t, \varepsilon)| &\le \int_t^{t+\Delta t} K\, |X_s - X_t|\, ds \\ &\le \int_t^{t+\Delta t} K \Big| X_s - X_t - \int_t^s f(X_t, Y_{s'/\varepsilon}) ds' \Big| \\ &\quad + \int_t^{t+\Delta t} K \Big| \int_t^s f(X_t, Y_{s'/\varepsilon}) ds' \Big| ds \end{aligned}$$

It is straightforward to show using (14) that, for sufficiently small ε,

$$\int_t^{t+\Delta t} K \Big| \int_t^s f(X_t, Y_{s'/\varepsilon}) ds' \Big| ds < C \Delta t^2 \tag{17}$$

for some constant $C < \infty$. Gronwall's lemma then implies that

$$\begin{aligned} \Big| X_{t+\Delta t} - X_t - \int_t^{t+\Delta t} f(X_t, Y_{s/\varepsilon}) ds \Big| &= |A(t, \Delta t, \varepsilon)| \\ &\le C \Delta t^2 \exp K \Delta t = o(\Delta t). \end{aligned} \tag{18}$$

This shown that

$$\lim_{\varepsilon \to 0} \big(X^{\varepsilon}_{t+\Delta t} - X^{\varepsilon}_t \big) = \Delta t \bar{f}(X^{\varepsilon}_t) + o(\Delta t). \tag{19}$$

which is sufficient to demonstrate that X^{ε}_t converges strongly to $\bar{X}_t$.

4 Diffusive time-scale

An interesting generalization of the situation presented in section 2 arises when

$$\int_{\mathbb{R}} f(x,y)\rho(y|x)dy = 0. \tag{20}$$

In this case the limiting equation reduces to the trivial ODE, $\dot{X}_t = 0$, i.e. no evolution at all. In fact, the interesting evolution then occurs on a longer time-scale of order ε^{-1}, and the right scaling to study (6) is

$$\begin{cases} dX_t = \frac{1}{\varepsilon} f(X_t, Y_t)dt, & X_0 = x \\ dY_t = \frac{1}{\varepsilon^2} b(X_t, Y_t)dt + \frac{1}{\varepsilon}\sigma(X_t, Y_t)dt, & Y_0 = y, \end{cases} \tag{21}$$

To obtain the limiting equation for X_t as $\varepsilon \to 0$, we proceed as above and consider the backward equation for $u(x,y,t) = \mathbb{E}f(X_t)$, which is now rescaled as

$$\frac{\partial u}{\partial t} = \frac{1}{\varepsilon} L_x u + \frac{1}{\varepsilon^2} L_y u.$$

Inserting the expansion $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^2)$ (we will have to go one order in ε higher than before) in this equation now gives

$$\begin{aligned} L_y u_0 &= 0, \\ L_y u_1 &= -L_x u_0, \\ L_y u_2 &= \frac{\partial u_0}{\partial t} - L_x u_1, \end{aligned} \tag{22}$$

and so on. The first equation tells that $u_0(x,y,t) = u_0(x,t)$. The solvability condition for the second equation is satisfied by assumption because of (20) and therefore this equation can be formally solved as

$$u_1 = -L_y^{-1} L_x u_0.$$

Inserting this expression in the third equation in (22) and considering the solvability condition for this equation, we obtain the limiting equation for u_0:

$$\frac{\partial u_0}{\partial t} = \bar{L}_x u_0,$$

where

$$\bar{L}_x = \int_{\mathbb{R}} dy \rho(y|x) L_x L_y^{-1} L_x.$$

To see what this equation is explicitly, notice that $-L_y^{-1} g(y)$ is the steady state solution of

$$\frac{\partial v}{\partial t} = L_y v + g(y).$$

The solution of this equation with the initial condition $v(y,0) = 0$ can be represented by Feynman-Kac formula as

$$v(y,t) = \mathbb{E}\int_0^t g(Y_s^x)ds,$$

where Y_t^x denotes the solution of the second SDE in (21) at $X_t = x$ fixed and $\varepsilon = 1$, i.e.

$$dY_t^x = b(x, Y_t^x)dt + \sigma(x, Y_t^x)dW_t, \qquad Y_0^x = y.$$

Therefore

$$-L_y^{-1} g(y) = \mathbb{E} \int_0^\infty g(Y_t^x)dt,$$

and the limiting backward equation above can be written as

$$\frac{\partial u_0}{\partial t} = \mathbb{E} \int_0^\infty dt \int_{\mathbb{R}} dy \rho(y|x) f(x,y) \frac{\partial}{\partial x} \Big(f(x, Y_t^x) \frac{\partial u_0}{\partial x} \Big),$$

This is the backward equation of the SDE

$$dX_t = \bar{b}(X_t)dt + \bar{\sigma}(X_t)dW_t, \qquad X_0 = x,$$

where

$$\begin{aligned} \bar{b}(x) &= \mathbb{E} \int_0^\infty \int_{\mathbb{R}} \rho(y|x) f(x,y) \frac{\partial}{\partial x} f(x, Y_t^x) dy dt, \\ \bar{\sigma}^2(x) &= 2\mathbb{E} \int_0^\infty \int_{\mathbb{R}} \rho(y|x) f(x,y) f(x, Y_t^x) dy dt. \end{aligned}$$

The interesting new phenomena is that the limiting equation for X_t has become an SDE. This means that fluctuations are important on the long-time scale and give rise to stochastic effects in the evolution of X_t that were absent on the shorter time-scale.

The calculation above is easy to generalize if there is a slow term in the original equation for X_t, i.e. if instead of (21) one considers

$$\begin{cases} dX_t = g(X_t, Y_t)dt + \dfrac{1}{\varepsilon} f(X_t, Y_t)dt, & X_0 = x \\ dY_t = \dfrac{1}{\varepsilon^2} b(X_t, Y_t)dt + \dfrac{1}{\varepsilon}\sigma(X_t, Y_t)dt, & Y_0 = y, \end{cases}$$

The limiting equation for X_t is then

$$dX_t = G(X_t)dt + \bar{b}(X_t)dt + \bar{\sigma}(X_t)dW_t, \qquad X_0 = x,$$

with $\bar{b}(x)$ and $\bar{\sigma}(x)$ as above, and

$$G(x) = \int_{\mathbb{R}} \rho(y|x) g(x,y) dy.$$

It is also straightforward to generalize to higher dimensions.

Here is an example.

$$\begin{cases} dX_t = \dfrac{2\alpha}{\varepsilon} Y_t Z_t dt - (X_t + X_t^3)dt, \\ dY_t = \dfrac{3\alpha}{\varepsilon} Z_t X_t dt - \dfrac{1}{\varepsilon^2} Y_t dt + \dfrac{1}{\varepsilon} dW_t^y, \\ dZ_t = -\dfrac{\alpha}{\varepsilon} b_3 Y_t X_t dt - \dfrac{1}{\varepsilon^2} Z_t dt + \dfrac{1}{\varepsilon} dW_t^z. \end{cases}$$

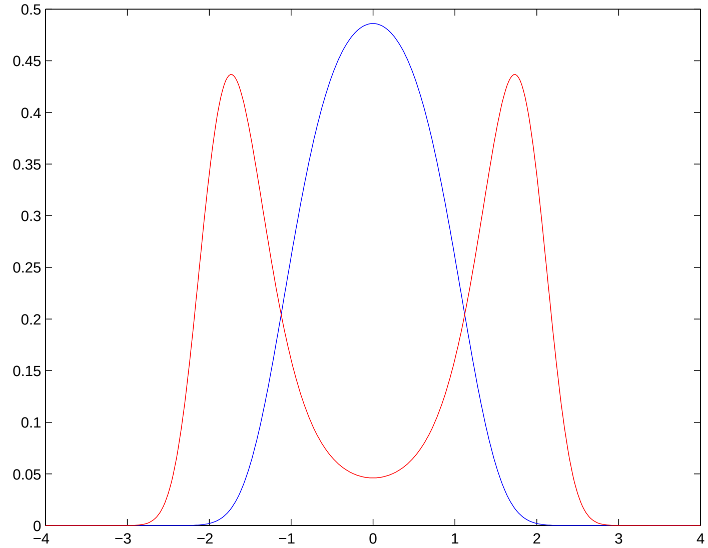

Figure 3: The equilibrium density $\rho(x) = Z^{-1}e^{\frac{1}{2}(\alpha^2-1)x^2-\frac{1}{4}x^4}$ for $\alpha = \frac{1}{2}$ (blue) and $\alpha = 2$ (red).

where W_t^y, W_t^z are independent Wiener processes and α is a parameter. There are two fast variables, Y_t and Z_t, in this example. There is also a slow term, $-(X_t + X_t^3)dt$, in the equation for X_t which, in the absence of coupling with Y_t and Z_t, would drive X_t to the position $x = 0$. We ask to what extend this equilibrium of the uncoupled dynamics is relevant with coupling with Y_t and Z_t.

The limiting equation for X_t is

$$dX_t = ((\alpha^2 - 1)X_t - X_t^3)dt + \alpha dW_t.$$

The equilibrium density for this equation is

$$\rho(x) = Z^{-1}e^{\frac{1}{2}(\alpha^2-1)x^2-\frac{1}{4}x^4}.$$

This density is shown in figure 3. For $|\alpha| \leq 1$, $\rho(x)$ is mono-modal and centered around $x = 0$, the stable equilibrium of the uncoupled dynamics. However, for $|\alpha| > 1$, $\rho(x)$ becomes bi-modal, with two maxima at $x = \pm\sqrt{\alpha^2 - 1}$ and a minimum at $x = 0$. Thus coupling with the fast modes may destroy the structures apparent in the uncoupled dynamics and induce bifurcations.

Notes by Inga Koszalka and Alex Hasha.