# Lecture 8: The Shallow-Water Equations

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## 1 Introduction

The shallow-water equations describe a thin layer of fluid of constant density in hydrostatic balance, bounded from below by the bottom topography and from above by a free surface. They exhibit a rich variety of features, because they have infinitely many conservation laws. The propagation of a tsunami can be described accurately by the shallow-water equations until the wave approaches the shore. Near shore, a more complicated model is required, as discussed in Lecture 21.

## 2 Derivation of shallow-water equations

To derive the shallow-water equations, we start with Euler's equations without surface tension,

$$\text{free surface condition}: \quad p = 0, \quad \frac{\mathrm{D}\eta}{\mathrm{D}t} = \frac{\partial \eta}{\partial t} + \boldsymbol{v} \cdot \nabla \eta = w, \quad \text{on } z = \eta(x, y, t) \tag{1}$$

$$\text{momentum equation}: \quad \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \frac{1}{\rho}\nabla p + g\hat{z} = 0, \tag{2}$$

$$\text{continuity equation}: \quad \nabla \cdot \boldsymbol{u} = 0, \tag{3}$$

$$\text{bottom boundary condition}: \quad \boldsymbol{u} \cdot \nabla (z + h(x, y)) = 0, \quad \text{on } z = -h(x, y). \tag{4}$$

Here, $p$ is the pressure, $\eta$ the vertical displacement of free surface, $\boldsymbol{u} = (u, v, w)$ the three-dimensional velocity, $\rho$ the density, $g$ the acceleration due to gravity, and $h(x, y)$ the bottom topography (Fig. 1).

For the first step of the derivation of the shallow-water equations, we consider the global

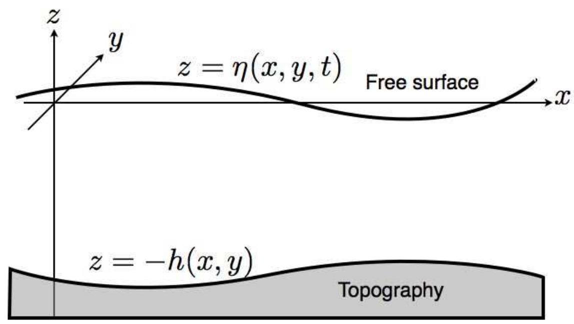


Figure 1: Schematic illustration of the Euler's system.

mass conservation. We integrate the continuity equation (3) vertically as follows,

$$
\begin{aligned}
0 &= \int_{-h}^{\eta} [\nabla \cdot \boldsymbol{u}] \mathrm{d}z, && (5)\\
&= \int_{-h}^{\eta} \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] \mathrm{d}z, && (6)\\
&= \frac{\partial}{\partial x} \int_{-h}^{\eta} u \mathrm{d}z - u|_{z=\eta} \frac{\partial \eta}{\partial x} + u|_{z=-h} \frac{\partial(-h)}{\partial x}, \\
&\quad + \frac{\partial}{\partial y} \int_{-h}^{\eta} v \mathrm{d}z - v|_{z=\eta} \frac{\partial \eta}{\partial y} + v|_{z=-h} \frac{\partial(-h)}{\partial y}, \\
&\quad + w|_{z=\eta} - w|_{z=-h}, && (7)\\
&= \frac{\partial}{\partial x} \int_{-h}^{\eta} u \mathrm{d}z - u|_{z=\eta} \frac{\partial \eta}{\partial x} + \frac{\partial}{\partial y} \int_{-h}^{\eta} v \mathrm{d}z - v|_{z=\eta} \frac{\partial \eta}{\partial y} + w|_{z=\eta}. && (8)
\end{aligned}
$$

where the bottom boundary condition (4) was used in the fourth row. With the surface condition (1), equation (8) becomes

$$
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \int_{-h}^{\eta} u \mathrm{d}z + \frac{\partial}{\partial y} \int_{-h}^{\eta} v \mathrm{d}z = 0. \tag{9}
$$

For the next step, we make the long-wave approximation, by assuming that the wave length is much longer than the depth of the fluid. However, we do not assume that perturbations have a small amplitudes, so that nonlinear terms are not neglected. Through the long-wave approximation, we can neglect the vertical acceleration term in (2), and deduce the hydrostatic pressure by integrating the vertical component of the momentum equation,

$$
\begin{aligned}
\int_{z}^{\eta} \frac{\partial p}{\partial z} \mathrm{d}z &= -\int_{z}^{\eta} \rho g \mathrm{d}z \\
p(x, y, \eta, t) - p(x, y, z, t) &= -\rho g(\eta(x, y, t)) - z) \\
p(x, y, z, t) &= \rho g(\eta(x, y, t) - z). && (10)
\end{aligned}
$$

where we used the surface condition $p(x, y, \eta, t) = 0$. Using this expression for the hydrostatic pressure (10) and further assuming that there are no vertical variations in $(u, v)$, the

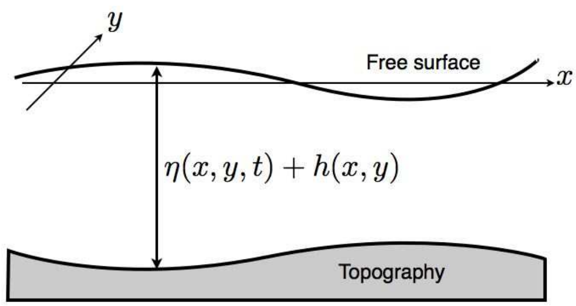


Figure 2: Schematic illustration of the shallow-water system.

horizontal momentum equations of the shallow-water system are obtained as follows,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial \eta}{\partial x} = 0, \tag{11}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial \eta}{\partial y} = 0. \tag{12}$$

The conservation of mass given by (9) becomes

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[(\eta + h)u] + \frac{\partial}{\partial y}[(\eta + h)v] = 0. \tag{13}$$

Then, equations (11), (12), and (13) are the shallow-water equations (Fig. 2).

These equations are similar to the equations for gas dynamics in 2-D. Indeed, the equations describing the dynamics of an inviscid, non-heat-conducting, isentropic (i.e. entropy is constant and $p \propto \rho^\gamma$) gas are (Kevorkian, 1990, chapter 3.3.4, [2])

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \tag{14}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \rho^{\gamma-2}\frac{\partial \rho}{\partial x} = 0, \tag{15}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \rho^{\gamma-2}\frac{\partial \rho}{\partial y} = 0, \tag{16}$$

where $\gamma \equiv C_p/C_v$ is the ratio of specific heats, $C_p$ is the specific heat at constant pressure, and $C_v$ is the specific heat at constant volume. These equations, (14), (15), and (16), are the exact analog of the shallow-water equations (13), (11), and (12) if we identify $u, v$ in both cases, set $g(\eta + h) = \rho$, assume flat bottom topography ($h$ = constant), and take $\gamma = 2$.

If we consider the shallow-water equations in a rotating frame (the rotation axis is perpendicular to $x$-$y$ plane), the Coriolis term should be added to the momentum equation.

In that case (Vallis, 2006, chapter 3, [3]),

$$\begin{aligned}
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[(\eta + h)u] + \frac{\partial}{\partial y}[(\eta + h)v] &= 0, \quad (17)\\
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + g\frac{\partial \eta}{\partial x} &= 0, \quad (18)\\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + g\frac{\partial \eta}{\partial y} &= 0, \quad (19)
\end{aligned}$$

where $f$ is the Coriolis parameter.

# 3 Mathematical structure

In this section, we explore the mathematical structure of the shallow-water equations.

## 3.1 Hyperbolic partial differential equations

The shallow-water equations can be written in the matrix form

$$\frac{\partial}{\partial t}\begin{pmatrix}\eta\\ u\\ v\end{pmatrix} + \begin{bmatrix} u & \eta + h & 0\\ g & u & 0\\ 0 & 0 & u\end{bmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\eta\\ u\\ v\end{pmatrix} + \begin{bmatrix} v & 0 & \eta + h\\ 0 & v & 0\\ g & 0 & v\end{bmatrix}\frac{\partial}{\partial y}\begin{pmatrix}\eta\\ u\\ v\end{pmatrix} = -\begin{pmatrix} u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\\ 0\\ 0\end{pmatrix}. \quad (20)$$

The eigenvalues of the first coefficient matrix are

$$u, \quad u \pm \sqrt{g(\eta + h)}, \quad (21)$$

and those of the second coefficient matrix are

$$v, \quad v \pm \sqrt{g(\eta + h)}. \quad (22)$$

Since the eigenvalues (21) and (22) are real and distinct, the shallow-water equations are hyperbolic partial differential equations (PDEs).

The equations admit discontinuous (weak) solutions (see Kevorkian, 1990, chapter 5.3 for details). Such a discontinuity is called a "bore" and approximates a breaking wave. However, how do waves really break? This question is addressed in more detail in Lecture 21.

## 3.2 Method of characteristics

Because the shallow-water equations are hyperbolic PDEs, the method of characteristics can be applied to reduce them to a family of ordinary differential equations.

If we assume $v \equiv 0$ and $\frac{\partial}{\partial y} \equiv 0$ (i.e. 2-D in $x$-$z$ plane), the shallow-water equations become,

$$\begin{aligned}
\frac{\partial}{\partial t}(\eta + h) + u\frac{\partial}{\partial x}(\eta + h) + (\eta + h)\frac{\partial u}{\partial x} &= 0, \quad (23)\\
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial}{\partial x}(\eta + h) &= g\frac{\partial h}{\partial x}. \quad (24)
\end{aligned}$$

If we define $c^2(x,y,t) = g(\eta + h)$, (23)$\times g$ can be written as

$$\frac{\partial c^2}{\partial t} + u\frac{\partial c^2}{\partial x} + c^2\frac{\partial u}{\partial x} = 0,$$

so

$$c\left[\frac{\partial(2c)}{\partial t} + u\frac{\partial(2c)}{\partial x} + c\frac{\partial u}{\partial x}\right] = 0. \tag{25}$$

Since $c(x,y,t) \neq 0$, (25) becomes

$$\frac{\partial(2c)}{\partial t} + u\frac{\partial(2c)}{\partial x} + c\frac{\partial u}{\partial x} = 0. \tag{26}$$

Likewise, (24) can be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + c\frac{\partial(2c)}{\partial x} = g\frac{\partial h}{\partial x}. \tag{27}$$

Then (27) + (26) and (27) − (26) respectively give

$$\begin{aligned}
\frac{\partial}{\partial t}(u+2c) + u\frac{\partial}{\partial x}(u+2c) + c\frac{\partial}{\partial x}(u+2c) &= g\frac{\partial h}{\partial x}, & (28)\\
\frac{\partial}{\partial t}(u-2c) + u\frac{\partial}{\partial x}(u-2c) - c\frac{\partial}{\partial x}(u-2c) &= g\frac{\partial h}{\partial x}. & (29)
\end{aligned}$$

Equation (28) states that along the curves in the $(x,t)$ plane defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u + c, \tag{30}$$

the quantity $u + 2c$ evolves according to

$$\frac{\partial}{\partial t}(u+2c) + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial}{\partial x}(u+2c) = \frac{\mathrm{d}}{\mathrm{d}t}(u+2c) = g\frac{\partial h}{\partial x}. \tag{31}$$

And also along curves defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u - c, \tag{32}$$

equation (29) can be written as

$$\frac{\partial}{\partial t}(u-2c) + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial}{\partial x}(u-2c) = \frac{\mathrm{d}}{\mathrm{d}t}(u-2c) = g\frac{\partial h}{\partial x}. \tag{33}$$

Therefore, if $h$ is constant, $(u+2c)$ and $(u-2c)$ are Riemann invariants (i.e. functions remain constant along the curves). If $h$ is $h = mx + b$, then $\partial h/\partial x$ is constant and the characteristic equations can again be easily integrated, with this time $(u + 2c - gmt)$ and $(u-2c-gmt)$ being the new Riemann invariants. For more complicated bottom topography more care must be taken. Finally, note that this method does not generalize to 3-D ($x$-$y$-$t$) problems.

### 3.3 Linearization of the shallow-water equations

To linearize the shallow-water equations, we consider small disturbances about a fluid at rest. That is,

$$\eta = 0 + \eta', \quad u = 0 + u', \quad v = 0 + v'. \tag{34}$$

By substituting (34) in the shallow-water equations (11), (12), (13) and neglecting the second-order terms, we obtain the linearized shallow-water equations as follows, (primes are omitted)

$$\frac{\partial \eta}{\partial t} + \frac{\partial (uh)}{\partial x} + \frac{\partial (vh)}{\partial y} = 0, \tag{35}$$

$$\frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x} = 0, \tag{36}$$

$$\frac{\partial v}{\partial t} + g\frac{\partial \eta}{\partial y} = 0. \tag{37}$$

Multiplying (35) by $\sqrt{g}$, and both (36) and (37) by $\sqrt{h}$, we obtain,

$$\frac{\partial}{\partial t}(\eta\sqrt{g}) + \frac{\partial}{\partial x}(u\sqrt{h}\cdot\sqrt{gh}) + \frac{\partial}{\partial y}(v\sqrt{h}\cdot\sqrt{gh}) = 0, \tag{38}$$

$$\frac{\partial}{\partial t}(u\sqrt{h}) + \sqrt{gh}\frac{\partial}{\partial x}(\eta\sqrt{g}) = 0, \tag{39}$$

$$\frac{\partial}{\partial t}(v\sqrt{h}) + \sqrt{gh}\frac{\partial}{\partial y}(\eta\sqrt{g}) = 0. \tag{40}$$

If we eliminate $u\sqrt{h}$ and $v\sqrt{h}$ from the above equations, we obtain the linear 2-D wave equations,

$$\frac{\partial^2}{\partial t^2}(\eta\sqrt{g}) = \nabla\cdot[gh\cdot\nabla(\eta\sqrt{g})]. \tag{41}$$

The phase speed is, as expected from the long-wavelength limit discussed in Lecture 2, $c(x, y) = \sqrt{gh}$, although this time is explicitly derived for varying bottom topography (i.e. $h$ is a function of both $x$ and $y$).

The vorticity equation is derived by defining the vorticity as $\omega(x, y, t) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ and calculating $\frac{\partial}{\partial x}(40) - \frac{\partial}{\partial y}(39)$,

$$\frac{\partial \omega}{\partial t} = 0. \tag{42}$$

This means that the vorticity remains constant in time. In general, the velocity field ($u$, $v$) can be decomposed into rotational and irrotational parts, so $u$ and $v$ can be written as

$$u = -\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x}, \tag{43}$$

$$v = \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}, \tag{44}$$

where $\psi$ and $\phi$ are called streamfunction and velocity potential, respectively. Using (43) and (44), equation (42) can be written as

$$\frac{\partial \omega}{\partial t} = \frac{\partial}{\partial t}(\nabla^2\psi) = 0. \tag{45}$$

Calculating $\frac{\partial}{\partial x}(39) + \frac{\partial}{\partial y}(40)$, we obtain

$$\frac{\partial}{\partial t}(\nabla^2 \phi) + \sqrt{g}\nabla^2(\eta\sqrt{g}) = 0. \tag{46}$$

The rotational part of the velocity field is therefore obtained by integrating the time-independent vorticity field (45) (i.e. that of the initial conditions), while the irrotational part is obtained from the solution of (41) and (46). Nevertheless, $\psi$ and $\phi$ are not entirely independent: they "interact" when applying the lateral boundary conditions on the total velocity field (43,44).

### 3.4 Tracking Vorticity in 2D

From the nonlinear shallow-water equations (11), (12), and (13), the nonlinear vorticity equation is derived as

$$\frac{\partial \omega}{\partial t} + \frac{\partial}{\partial x}(u\omega) + \frac{\partial}{\partial y}(v\omega) = 0. \tag{47}$$

This time, although $\partial\omega/\partial t \neq 0$, the total (integrated) vorticity

$$\iint \omega(x, y)\mathrm{d}x\mathrm{d}y$$

is conserved. Note that the vorticity equation can be written as

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial v} = -\omega\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \tag{48}$$

and the equation (13) can be written as

$$\frac{\omega}{\eta + h}\left[\frac{\partial}{\partial t}(\eta + h) + u\frac{\partial}{\partial x}(\eta + h) + v\frac{\partial}{\partial y}(\eta + h)\right] = -\omega\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \tag{49}$$

From (48) and (49), we obtain

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial v} = \frac{\omega}{\eta + h}\left[\frac{\partial}{\partial t}(\eta + h) + u\frac{\partial}{\partial x}(\eta + h) + v\frac{\partial}{\partial y}(\eta + h)\right], \tag{50}$$

which simplifies to

$$\frac{\partial}{\partial t}\left(\frac{\omega}{\eta + h}\right) + u\frac{\partial}{\partial x}\left(\frac{\omega}{\eta + h}\right) + v\frac{\partial}{\partial y}\left(\frac{\omega}{\eta + h}\right) = 0. \tag{51}$$

This illustrate that the quantity

$$q \equiv \frac{\omega}{\eta + h}, \tag{52}$$

is conserved by each fluid particle (water columns). It is called "potential vorticity" and plays a fundamental role in fluid dynamics. The equation (51) is called Ertel's theorem (1942) [1].

Moreover, let $G(\zeta)$ be any differentiable function. Given that,

$$\frac{\partial G(\zeta)}{\partial t} = \frac{\mathrm{d}G(\zeta)}{\mathrm{d}\zeta}\frac{\partial \zeta}{\partial t}, \tag{53}$$

we obtain

$$\begin{aligned}
\frac{\partial G(q)}{\partial t} + u\frac{\partial G(q)}{\partial x} + v\frac{\partial G(q)}{\partial y} &= \frac{\mathrm{d}G(q)}{\mathrm{d}q}\frac{\partial q}{\partial t} + u\frac{\mathrm{d}G(q)}{\mathrm{d}q}\frac{\partial q}{\partial x} + v\frac{\mathrm{d}G(q)}{\mathrm{d}q}\frac{\partial q}{\partial y}, \\
&= \frac{\mathrm{d}G(q)}{\mathrm{d}q}\left[\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y}\right], \\
&= 0.
\end{aligned} \tag{54}$$

Therefore any smooth function of $q$ is also conserved by each water column.

From (47) and (54), we obtain

$$\frac{\partial}{\partial t}[\omega \cdot G(q)] + \frac{\partial}{\partial x}[u\omega \cdot G(q)] + \frac{\partial}{\partial y}[v\omega \cdot G(q)] = 0, \tag{55}$$

which means that there are infinitely many conservation laws, because $G$ can be chosen from arbitrary differential functions.

## 4 Applications: Tsunami

The linearized shallow-water equation derived in the previous section

$$\frac{\partial^2}{\partial t^2}(\eta\sqrt{g}) = \nabla[gh \cdot \nabla(\eta\sqrt{g})], \tag{56}$$

is a good model for the propagation of tsunami across the open ocean, away from shore. Indeed, as seen in Lecture 7, the typical depth of the ocean is about 4 km while the wave length of tsunami is about 100 km. This equation shows that the local speed of propagation of the Tsunami, in any direction, is

$$c_p = c_g = \sqrt{gh(x,y)}, \tag{57}$$

with no dispersion.

Closer to the shore, the wave compresses horizontally and grows vertically, so the linear approximation is no longer valid. The nonlinear shallow-water equations being hyperbolic, they allow for wave breaking. However, the breaking mechanism is presumably much more complex than that captured by shallow-water equations. This topic is addressed in more detail in Lecture 21.

## 5 Summary so far

Let us complete this lecture by summarizing the various equations and respective assumptions made so far. In Lecture 1, we presented the full nonlinear surface water-wave equations. In Lecture 2, we considered the linear approximation to these equations in the case of flat bottom topography, but allowed for waves of any horizontal wavelength. In the following lectures (4-7), we dropped the linear approximation, and considered the weakly non-linear case. However, in order to do this we had to focus on long-wavelength dynamics only, with flat-bottom topography. In this last lecture, we changed approach to consider the effect of varying bottom topography, and derived the fully nonlinear shallow-water equations (and its linearized counterpart). These equations also assume long-wavelength perturbations. However, an additional approximation had to be made in this case, in assuming that the vertical structure of the flow is entirely uniform ($\partial u/\partial z = 0 = \partial v/\partial z$).

# References


[1] H. Ertel, *Ein neuer hydrodynamischer wirbelsatz (a new hydrodynamic eddy theorem)*, Meteorolol. Z., 59 (1942), pp. 277–281.

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[3] G. K. Vallis, *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-Scale Circulation*, Cambridge University Press, Cambridge, U.K., 2006.