Title:

Intrinsic Nonlinearity and Spectral Structure of Internal Tides at an Idealized Mid-Atlantic Bight Shelfbreak

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Abstract – To quantify dynamical aspects of internal tide generation at the Mid-Atlantic Bight shelfbreak, this study employs an idealized ocean model initialized by climatological summertime stratification and forced by monochromatic barotropic tidal currents at the offshore boundary. The Froude number of the scenario is subunity and the bathymetric slope offshore the shelfbreak is supercritical. A barotropic to baroclinic energy conversion rate of 335 W m$^{-1}$ is found, with 14% of the energy locally dissipated through turbulence and bottom friction and 18% radiated onto the shelf. Consistent with prior studies, nonlinear effects result in additional superharmonic and subharmonic internal waves at the shelfbreak. The subharmonic waves are subinertial, evanescent, and mostly trapped within a narrow beam of internal waves at the forcing frequency, and they likely result from non-resonant triad interaction associated with strong nonlinearity. Strong vertical shear associated with the subharmonic waves tends to enhance local energy dissipation and turbulent momentum exchange (TME). A simulation with reduced tidal forcing shows an expected diminished level of harmonic energy. A quasi-linear simulation verifies the role of momentum advection in controlling the relative phases of internal tides and the efficiency of barotropic to baroclinic energy conversion. The local TME is tightly coupled with the internal wave dynamics: for the chosen configuration, neglecting TME causes the internal wave energy to be over-estimated by 12%, and increasing it to high levels damps the waves on the continental shelf. This work implies a necessity to carefully consider nonlinearity and turbulent processes in the calculation of internal tidal waves generated at the shelfbreak.
1. Introduction

Internal waves generated at continental shelf edges can be dissipated within a few tens of kilometers, influencing local stratification (Holloway et al. 2001; Klymak et al. 2010; Levine and Boyd 2006), or transported onshore or offshore. Onshore propagation of the waves into water of decreasing depth raises their energy density, possibly leading to short nonlinear waves with nonhydrostatic pressure (Apel et al. 1997; Sandstrom and Elliott 1984) and strong dissipation over a distance of 50 km or greater (Shroyer et al. 2010a). Here, generation of internal tides (tidal frequency internal waves) at an idealized Mid-Atlantic Bight (MAB) shelfbreak by barotropic tides is examined (Fig. 1). The practical motivation for the study is a desire to define a prediction capability for nonlinear internal waves on the shelf. These waves can have large acoustical effects (Lynch et al. 2010), and the possibility of predicting their locations, directions, sizes, shapes, and other characteristics is being examined. This is a challenging problem, as illustrated in a recent paper suggesting that internal waves incident on the continental slope, potentially unknowable, may at times play a significant role (Nash et al. 2012). Despite this, it is likely that internal tides generated at the shelf edge can be important precursors to nonlinear wave packets on the shelf, and we wish to determine the prediction capability, while recognizing the inherent impossibility of complete prediction of nonlinear phenomena (Eckmann and Ruelle 1985; Krishnamurthy 1993).

A number of geometric factors control the characteristics of the internal waves at a shelf edge, including particle excursion, topography scale length, steepness of slope, and so on (Garrett and Kunze 2007). Because the geometry and the stratification can vary so broadly, computational models are often used to determine the effects (e.g., Gerkema et al. 2006; Green et al. 2008; Holloway and Barnes 1998; Legg and Adcroft 2003). However, questions remain, including which modeling techniques are best, which are adequate, and what errors are
introduced by specific modeling procedures. This study also employs a computational flow
model and tries to answer some of the modeling questions. A primitive-equation hydrostatic
model is used to study only the low-frequency (with periods longer than two hours) internal
waves generated locally by the barotropic M₂ tide at the idealized shelfbreak.

The internal wave fields resulting from our simulations are formed by only one of the
processes responsible for ocean internal waves. This paper addresses two mechanisms at work in
the energy balance of tidally driven internal waves in their generation region: nonlinear boundary
zone effects and nonlinear wave-wave interaction. Both of these affect the level and shape of the
resultant internal wave frequency spectrum. The questions addressed here are: (1) What fraction
of internal wave energy provided by tidal processes resides in the tidal frequencies, and what
fraction in other frequencies? (2) How important is nonlinear momentum advection in the
generation region? (3) How does imposed sub-grid turbulent momentum exchange (TME) affect
the internal tide generation?

Resonant nonlinear wave-wave interaction (McComas and Bretherton 1977; Müller et al.
1986) is a potential mechanism of filling the internal wave frequency band, $f$ (inertial frequency)
to $N$ (buoyancy frequency), with energy moving from waves at a few source wavenumbers ($k$)
and frequencies ($\omega$) (i.e. near-inertial, tidal) to other $k$ and $\omega$, with net flux to waves that
dissipate. Parametric subharmonic instability (PSI), one of the many types of resonant
interactions, generates waves of frequency near one-half the forcing frequency and high vertical
wavenumbers, and can therefore potentially induce strong TME, especially near the critical
latitude (Gerkmann et al. 2006; Lamb 2004; MacKinnon and Winters 2005; Young et al. 2008).
Therefore, better understanding nonlinear wave-wave interactions would be an important step
toward full knowledge of the wavefield itself, from the generation processes to the physical-
biogeochemical impacts of wave dissipation. Nonlinear wave-wave interaction has been
demonstrated a few times in the field (Carter and Gregg 2006; MacKinnon et al. 2013; Sun and
Pinkel 2012; 2013) and numerous times in models (e.g., Hazewinkel and Winters 2011; Korobov
and Lamb 2008; Legg and Huijts 2006; Nikurashin and Legg 2011), and has been shown to
enhance turbulence dissipation in the open ocean environment (Lvov et al. 2012; Nikurashin and
Legg 2011; Polzin 2004a; 2004b; Winters and D'Asaro 1997).

In addition to demonstrating the development of expected fundamental internal waves and
superharmonic waves (via nonlinear resonant wave-wave interactions), this three-dimensional
study examines subinertial evanescent waves appearing trapped in an internal tide beam. A study
by Korobov and Lamb (2008) examined the generation of subinertial, trapped waves using two-
dimensional numerical simulations and attributed the phenomenon to nonlinear non-resonant
interaction of internal waves. Their findings echo observations of standing evanescent waves at
frequencies greater than $N$ in the intersections of two internal wave beams in both laboratory
(Teoh et al. 1997) and numerical (Javam et al. 1999; 2000) settings. Here, the transfer of internal
wave energy from the primary frequency to other (sub and super) harmonics is quantified for the
chosen shelfbreak environment.

Substantial internal wave activities have been observed in the MAB shelfbreak region (Nash
et al. 2004; Shroyer et al. 2011; Tang et al. 2007) and may dominate local TME and tracer
mixing (MacKinnon and Gregg 2005; Shroyer et al. 2010b). $M_2$ is the dominant tidal constituent
in the area, and the $M_2$ tide propagates mainly towards the coast (Fig. 1). The barotropic tidal
current speed increases dramatically across the shelfbreak, from about 1.5 cm s$^{-1}$ at the 1000 m
isobath to about 10 cm s$^{-1}$ at the 100 m isobath. In this study, we neglect influences of factors,
such as other tidal constituents, mesoscale activities, meteorological forcing, bathymetric
irregularity, and internal waves radiating into the area (Nash et al. 2012), on the internal tide generation. Barotropic to baroclinic energy conversion at the shelfbreak region under a climatological summer condition is quantified, and influences of nonlinear momentum advection and TME on the conversion are investigated. The model setup represents a highly idealized single scenario, and the results achieved here provide only an instructive example of the dynamics presumably associated with internal tide generation over a broader range of scenarios than can be examined on one paper.

2. Experiment Setup

2.1. Basic Model Configuration

The hydrostatic Regional Ocean Modeling System, with terrain-following vertical coordinates, is used for the simulations (ROMS; http://www.myroms.org; Shchepetkin and McWilliams 2005; Shchepetkin and McWilliams 2008). Using the hydrostatic pressure approximation in the model is justifiable because internal tides generated at the shelfbreak are of low frequency, and the associated vertical accelerations (a major component of the nonhydrostatic pressure) are small. As these waves propagate in nature onto the shallow shelf they evolve into bores and short nonlinear internal waves ($O(100 \text{ m})$), which are not properly handled by the hydrostatic model. Therefore, details in that regime are not examined here.

The positive $x$ direction is offshore and southeastward (to mimic MAB), positive $y$ is northeastward, and positive $z$ is upward with $z = 0$ at the sea surface. With the hydrostatic pressure assumption, the model solves the horizontal momentum equation,

$$\frac{\partial \mathbf{u}_k}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}_k + \mathbf{f} \times \mathbf{u}_k = -\frac{1}{\rho_0} \nabla p + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( A \frac{\partial \mathbf{u}_k}{\partial z} \right), \quad (1)$$

continuity equation,
\( \nabla \cdot \mathbf{u} = 0, \quad (2) \)

and scalar equations for temperature and salinity and nonlinear equation of state. Here, \( \mathbf{u} \) is a three-dimensional velocity vector \((u, v, w)\), \( \mathbf{u}_h \) is horizontal velocity vector \((u, v, 0)\), \( \mathbf{f} \) is the Coriolis vector \((0, 0, f)\), \( \rho_0 \) is reference density, \( p \) is pressure, and \( A_v \) is the TME coefficient (vertical eddy viscosity) acting on vertical shear. Model bathymetry is uniform in the along-shelf direction, and its cross-shelf form (Fig. 2a) is given by

\[
H = \max(0, \alpha(x_1 - x)) + h_0 \tanh\left(\frac{x_0 - x}{L}\right) - h_1, \quad (3)
\]

where, \( \alpha = 0.001, x_1 = 64 \text{ km}, x_0 = 105 \text{ km}, L = 16.5 \text{ km}, h_0 = 465 \text{ m}, \) and \( h_1 = 540 \text{ m}. \) This function of \( x \) has a uniform slope \( \alpha \) on the shelf transitioning to a hyperbolic tangent shape in the slope sea. Values of the parameters are chosen to represent the mean slope of the MAB continental shelf, to fit the along-shelf averaged MAB shelfbreak bathymetry (water depth of 100 – 400 m), and to limit the maximum depth at 1000 m. Limiting the maximum depth maintains model vertical resolution in the deep sea.

The model domain spans an area of 1931 km \((x\) direction) \(\times 27 \text{ km}\) \((y\) direction). The nearshore 150 km in the \(x\) direction is the study area. The horizontal resolution in the study area is 121 and 149 m in the \(x\) and \(y\) directions, respectively. The region outside the study area is to delay the reflection of internal waves at the offshore boundary; its \( y\)-resolution is 149 m and \( x\)-resolution coarsens gradually from 121 m to 6.2 km toward the offshore boundary. There are 100 stretched vertical layers with enhanced resolution near the surface and bottom (about 0.2 m at the shelf break). The Coriolis parameter is \( f = 2\Omega \sin(39^\circ) \), where \( \Omega \) is earth’s rotation rate.

The coastal boundary is a solid wall with depth of 10 m. Periodic boundary conditions are used in the \( y\) direction. The deep-sea boundary is open with the Chapman (1985), Flather (1976) and an Orlanski-type radiation (Orlanski 1976) conditions used for sea-level, two-dimensional
momentum and three-dimensional variables, respectively. In the control simulation, barotropic tidal velocity of only principal lunar M2 frequency is added on the offshore boundary, adjusted to generate a deep-sea current ellipse with major-axis of 0.02 m s\(^{-1}\) and inclination pointing onshore, consistent with the averaged properties at 1000 m isobath in the MAB retrieved from the OTIS regional tidal solution ([http://volkov.oce.orst.edu/tides/region.html](http://volkov.oce.orst.edu/tides/region.html)). The resultant barotropic tidal current at the shelfbreak (100 m isobath) has amplitude of 0.12 m s\(^{-1}\) in the cross-shelf direction, the same value as OTIS (Fig. 1), with this consistency justifying the limited depth of 1000 m. A 600-km-wide sponge layer is applied at the offshore end of the domain (outside of the study area) to further prevent reflection of internal waves at the offshore boundary. No explicit horizontal viscosity or diffusivity is applied in the interior. The numerical advection schemes are third-order upstream bias and fourth-order centered for three-dimensional horizontal and vertical momentum advection, respectively, and fourth-order Akima for both horizontal and vertical tracer advection.

The control simulation employs the Generic Length Scale (GLS) turbulence closure K-KL scheme ([Umlauf and Burchard 2003; Warner et al. 2005](#)) for the vertical TME and tracer mixing. The turbulence closure model solves two dynamical equations (one for turbulence kinetic energy, \(K\), and the other for \(KL\), where \(L\) is the turbulent length scale) derived from transport of the Reynolds stress tensor. Typically, the resulting vertical eddy viscosity, \(A_v\), and diffusivity, \(\kappa_v\), are on the order of \(10^{-3} \text{ m}^2 \text{s}^{-1}\) in the bottom boundary layer (BBL; about 5 m near the bottom) and gradually reduce to \(10^{-4} \text{ m}^2 \text{s}^{-1}\) in the interior and then \(10^{-5} \text{ m}^2 \text{s}^{-1}\) in the thermocline. The Prandtl number (\(A_v/\kappa_v\)) is about 1 in the BBL and 1.5 in the thermocline. Note that \(A_v\) and \(\kappa_v\) are derived separately from \(K\) and \(L\) using different quasi-equilibrium stability functions ([Kantha and Clayson 1994](#)).
All model simulations start from rest with horizontally uniform density structure (Fig. 3) obtained from summertime $T$ and $S$ climatology in the region (Zhang et al. 2011). Quadratic bottom drag is used with a drag coefficient, $C_d = 0.003$. The model has no heat, salt or momentum exchange with the atmosphere. All simulations last for 25 days; within the first 10 days, internal waves with period greater than two hours are well developed (see below). We therefore chose Day 10 as the time window to analyze characteristics of modeled internal tide fields. For the time series analyses, we selected two sites, A and B (Fig. 4a). Site A, 145 m below the surface, is within the M$_2$ beam and slightly above the critical slope, and Site B, 37 m below the surface, is about 10 km onshore of the shelfbreak and slightly below a surface-reflected M$_2$ internal-tidal beam (see below).

2.2 Equations of baroclinic motion

Under the assumption that the flow can be decomposed into barotropic and baroclinic components, we write

$$u = U + u'; \quad p = P + p',$$

where $U = \int_{-H}^{0} u \, dz$ and $P = \int_{-H}^{0} p \, dz$ are barotropic velocity and pressure, and $u'$ and $p'$ are baroclinic velocity and perturbation pressure, respectively. Subtracting the equation of barotropic flow (with no TME) from (1) gives the horizontal baroclinic momentum equation:

$$\frac{\partial u'_h}{\partial t} + U \cdot \nabla u'_h + u' \cdot \nabla U_h + u'_h \cdot \nabla u'_h + f \times u'_h = -\frac{1}{\rho_0} \nabla_h p' + M. \quad (5)$$

Here, $M$ is the eddy viscosity term that contains both barotropic and baroclinic flows. Note that the nonlinear momentum advection term in (1), $u \cdot \nabla u_h$, breaks into four parts, three of which appear in (5). Among them, $U \cdot \nabla u'_h$ and $u' \cdot \nabla u'_h$ represent barotropic and baroclinic advection
of the baroclinic momentum, respectively. We refer to $\mathbf{U} \cdot \nabla \mathbf{u}'_h$ as “baroclinically linear” and $\mathbf{u}' \cdot \nabla \mathbf{u}'_h$ as “baroclinically nonlinear” processes. The term $\mathbf{u}' \cdot \nabla \mathbf{U}_h$ is also “baroclinically linear”, but negligible because $\mathbf{U}$ has a small wavenumber. The equation for baroclinic flow kinetic energy averaged over a tidal period is:

$$\frac{\partial E_{kz}}{\partial t} = C_z + T_z - \nabla \cdot \mathbf{F}_z - \varepsilon_z - D'. \tag{6}$$

The terms and additional quantities are defined as follows. The period-averaged internal wave kinetic energy is

$$E_{kz} = \frac{1}{2} \rho_0 \langle \mathbf{u}' \cdot \mathbf{u}' \rangle; \tag{7}$$

the barotropic to baroclinic energy conversion rate is

$$C_z(x, z) = \langle \rho' g W_z \rangle; \tag{8}$$

the advection of baroclinic kinetic energy is

$$T_z = -\rho_0 \langle \mathbf{u}'_h \cdot (\mathbf{U} \cdot \nabla \mathbf{u}'_h) \rangle - \rho_0 \langle \mathbf{u}'_h \cdot (\mathbf{u}' \cdot \nabla \mathbf{U}_h) \rangle - \rho_0 \langle \mathbf{u}'_h \cdot (\mathbf{u}' \cdot \nabla \mathbf{u}'_h) \rangle; \tag{9}$$

the baroclinic energy flux vector is

$$\mathbf{F}_z = (F_{xz}, F_{yz}, F_{wz}) = \langle \mathbf{u}' \mathbf{p}' \rangle; \tag{10}$$

the interior dissipation rate is (Kang and Fringer 2012)

$$\varepsilon_z' = \left\langle \rho_0 A_v \frac{\partial \mathbf{u}'}{\partial z} \cdot \frac{\partial \mathbf{u}'}{\partial z} + g \kappa_v \frac{\partial \rho}{\partial z} \frac{\partial \zeta}{\partial z} \right\rangle; \tag{11}$$

the dissipation due to bottom friction is (Kang and Fringer 2012)

$$D' = \rho_0 C_d \left| \mathbf{u}_b \right| (\mathbf{u}_b \cdot \mathbf{u}'_h) \tag{12}$$

and the vertical velocity associated with the barotropic flow is
\[ W_\zeta = -\frac{d[U(z+H)]}{dx}. \] (13)

Here, \( \rho' \) is perturbation density and \( \zeta \) is vertical isopycnal displacement. The angle brackets in (7) – (12) indicate time average over a M2 wave period. Because advection of available potential energy in this study is at least one order of magnitude smaller than the other terms, it is neglected in energy budget analyses.

### 2.3 List of model runs and parameter scalings

To investigate the influences of nonlinear momentum advection and vertical TME on the internal tide generation, five runs were conducted as shown in Table 1. The “control run” is intended to be the most realistic simulation, while the other sensitivity runs with altered physics serve to illustrate the roles of specific effects. The half-forcing and no-advection runs are to examine the role, in the internal wave generation process, of nonlinearity (largely caused by \( \mathbf{u} \cdot \nabla \mathbf{u} \), but also arising from the equation of state and from TME. The reduced and enhanced TME simulations serve to demonstrate the effect of vertical TME.

Several nondimensional parameters are important for the problem of internal wave generation (Legg and Huijts 2006). Their values for the simulations are given here to provide context for interpreting the model results. The first parameter is the relative topographic height,

\[ \delta = \frac{h_0}{H_d}, \] (14)

where \( h_0 \) is the topographic height, defined here as the depth change across the continental slope, and \( H_d \) the bottom depth of the deep sea. In this study, as \( h_0 \approx 900 \text{ m} \) and \( H_d = 1000 \text{ m} \), \( \delta \approx 0.9 \), indicating that the topographic variation is drastic and strong baroclinic responses to barotropic tidal flow is expected. Note that \( \delta \) is even closer to 1 near the real MAB shelfbreak. The next parameter is the tidal excursion parameter

\[ R = \frac{U_0}{\omega L}. \] (15)
This is the ratio of the excursion length at frequency $\omega$, $U_0/\omega$, to the characteristic horizontal length scale $L$ of the local topography (Legg and Klymak 2008; Rayson et al. 2011). Here, $U_0$ is the characteristic tidal current speed in the shelfbreak region. Because $L \approx 20$ km (Fig. 2a) and $U_0 \approx 0.1$ m s$^{-1}$ at the shelfbreak, $R = U_0/(\omega_{M2}L) \approx 0.0053$ in this work, suggesting that low-frequency internal tides, instead of high frequency lee waves, are likely to be generated at the topography.

The third parameter is the normalized seabed slope

$$\gamma = s/\alpha,$$  (16)

which is related to the intensity of the internal wave generation, with the intensity increasing with $\gamma$ for fixed topographic height. Here, $s$ is the seabed slope, while $\alpha$ is the slope of the (depth-dependent) internal wave energy propagation characteristic satisfying

$$\alpha(z) = \tan \theta = \left( \frac{\omega^2 - f^2}{N(z)^2 - \omega^2} \right)^{1/2} = \frac{k_x}{k_z}. \quad (17)$$

In (17), $\theta$ is the angle with respect to horizontal, $N(z)$ is the local buoyancy frequency, and $k_x$ is the wavenumber in $x$ direction. Here, $\alpha$ is evaluated at the seabed. In this study, $\gamma = s/\alpha_{M2}$ is greater than one (supercritical) for the $M_2$ tide in the steepest region, while $\gamma$ is less than one on the shelf and in the deep sea (Fig. 2a). The shallow critical seabed location ($\gamma = 1$) for $M_2$ frequency is at about 13 km off the shelfbreak where bottom depth is 220 m and bottom slope 0.0138, and intensive generation of internal tides is expected there. Stratification and internal tide generation are small at the deep critical location. The final parameter is the topographic Froude number

$$Fr = U_0/(h_0N), \quad (18)$$
measuring the influence of stratification and topography on tidal flow (Garrett and Kunze 2007; Legg and Klymak 2008). Here, $N > 0.004 \, s^{-1}$ (Fig. 3b) gives $Fr < 0.03 << 1$, meaning that the stratification and/or topography of the region exert strong influence on the tidal flow, and also that mode-one internal wave speed far exceeds $U_0$.

3. Results

3.1 Control run

All of the simulations show generation of internal waves at the shelf break, similar to many prior studies of this type (e.g., Gerkema et al. 2006; Lamb 2004; Legg 2004; Nash et al. 2012). Fig. 4 shows cross-shelf transects of control-run $u'$, $v'$, $w$ and $\rho'$ at the time of peak onshore barotropic current at Day 10. All panels show beams of $M_2$ frequency internal waves emitted from the critical slope and reflected at sea surface and bottom. The vertically integrated baroclinic energy fluxes, $F_x = \int_{-H}^{0} F_{xz} \, dz$ and $F_y = \int_{-H}^{0} F_{yz} \, dz$, (Fig. 2b-c) show that baroclinic energy is generated in the region surrounding the critical slope site, and propagates both onshore and offshore, with more flux offshore. Peak offshore $F_x$ of 360 W m$^{-1}$ occurs 45 km offshore of the shelfbreak (Fig. 4b), and peak onshore $F_x$ of about 60 W m$^{-1}$ occurs right at the shelfbreak.

The vertically integrated internal wave kinetic energy, $E_k = \int_{-H}^{0} E_{kz} \, dz$, (Fig. 5a) shows a broad peak around 40 km offshore of the shelfbreak and drops gradually toward the coast on the shelf.

To examine the energy budget and determine the roles of conversion, advection, flux divergence and dissipation, we computed the right-hand-side terms in (6) over Day 10 when $E_{kz}$ has reached its quasi-equilibrium state. 1) Cross-shelf section of $C_z/\rho_0$ (Fig. 6a) shows peak values along the $M_2$ beam and maximum conversion near the critical site where vertically integrated conversion rate, $C = \int_{-H}^{0} C_z \, dz$, reaches a peak of 0.02 W m$^{-2}$ (Fig. 2d). $C$ declines to
zero at the shelfbreak and also 45 km offshore of that, and becomes negative further away from
the shelfbreak (indicating energy transfer from baroclinic flow to barotropic flow). The cross-
shelf integrated $C$ between 30 km onshore and offshore the shelfbreak is about 335 W m$^{-1}$ (Table
2), slightly smaller than the maximum coherent (presumably locally generated) conversion rate
of 400 W m$^{-1}$ estimated from three moorings across the MAB shelfbreak (Fig. 10e in Nash et al.
2012). In Fig. 6a, there is a minor peak of $Cz/\rho_0$ at 30 m below surface near the shelfbreak,
corresponding to relatively large $\rho'$ at the thermocline depth (Fig. 4d).

2) The flux gradient (Fig. 6c) is large both on the shelf and along the M$_2$ beam offshore the
shelfbreak, and it is distributed in organized patches of large positive and negative values.

$\nabla \cdot \mathbf{F}_z$ integrated over the area 60 km across the shelfbreak is -295 W m$^{-1}$ (Table 2), making
flux gradient the largest sink term for kinetic energy density.

3) Advection of energy (Fig. 6b) is small but present at the shelfbreak, primarily along the
M$_2$ beam, and slightly stronger on the shelf with a patchy distribution. Vertical integrated total
advection, $T = \int_{-H}^{0} T_z \, dz$, is small and fluctuates, especially on the shelf (Fig. 2e), and $T$
integrated over the area of 60 km across the shelfbreak is negligible comparing to the other terms
(Table 2). Fig. 7 shows that all terms in (9) are small and, therefore, small $T$ is not caused by
cancellation of the terms. Among the terms in (9), $-\langle \mathbf{u}' \cdot (\mathbf{u}' \cdot \nabla \mathbf{u}') \rangle$ is the largest, meaning that
the baroclinic self-advection is the dominant advection process.

4) Energy dissipation (Fig. 6d) occurs mostly at the two locations where M$_2$ beams interact
with the seabed: near the critical site and 20 km onshore of the shelfbreak, and vertically
integrated dissipation, $\varepsilon' = \int_{-H}^{0} \varepsilon' \, dz$, reaches peak values at those locations (Fig. 2f). The terms
$\varepsilon'$ and $D'$ integrated over the area 60 km across the shelfbreak are 41.8 and 4.6 W m$^{-1}$,
respectively, together occupying about 14% of the internal tide energy in the region (Table 2). Note that the along-beam patchy patterns in Fig. 6b-d around 20 km offshore the shelfbreak resembles that of the subharmonic waves (see below).

3.2 Sensitivity tests

When the offshore tidal forcing is reduced by 50% (Run 2), $F_x$, $F_y$, $C$, $T$, $\varepsilon'$ (Fig. 2b – 2f) and $E_k$ (Fig. 5a) are all greatly diminished (also in Table 2). Since $F_x$, $F_y$, $\varepsilon'$ and $E_k$ are all quadratic quantities of baroclinic variables, they are expected to reduce to a quarter of the prospective values in Run 1 if the system is baroclinically linear. Fig. 5c and Table 2 show that the ratio of $E_k$ in Runs 2 and 1 is around 1/4. Similar ratios are obtained for $F_x$ and $F_y$. These suggest that the baroclinic $M_2$ generation process behaves in linear fashion, consistent with $C$ and $-\nabla \cdot \mathbf{F}$ being the predominant terms in the energy budget. The increased nonlinearity in Run 1 versus Run 2 does affect development of low-energy but potentially high-shear harmonics, however, discussed in another section.

However, completely neglecting $\mathbf{u} \cdot \nabla \mathbf{u}$ (Run 3) substantially modifies the modeled internal wave field energy. In particular, $C$ and $E_k$ integrated over the area spanning 60 km across the shelfbreak are reduced by 33% and 27%, respectively (Table 2). The reduction of $F_x$ on the shelf is even greater, more than 50% (Fig. 2b). There is an increase of turbulence dissipation at the critical site (Fig. 2f), but the overall dissipation around the shelfbreak remains similar to that in Run 1 (Table 2). Note that $\mathbf{U}$ is nearly identical in Runs 1 and 3, so that all the described changes are caused by variations in $\mathbf{u}'$. Examination of the Run 3 fields shows that the reduction of $E_k$ is accompanied by large phase changes of the main-beam $M_2$ internal tides, relative to the Run 1 phases (Fig. 8b). Fig. 9 shows that, in Run 1, $W_z$ and $\rho'$ at Site A in the main $m_2$ beam are in phase, while they are not in Run 3. As $C_z$ is the greatest in the $M_2$ beam (Fig. 6a), the phase
change greatly reduces $C$ (Fig. 2d) defined in (8), and thus also $E_k$. This demonstrates the first of
our highlighted results, that phases of internal tide at the shelfbreak are crucial information (be
they entirely locally generated, or the sum of local and incident waves), and they affect the
arrival time and the energy level of internal tides on the shelf. This agrees with the findings of
Kelly and Nash (2010), obtained with a simpler idealized model (see below).

Differences between Runs 1, 4 and 5 quantify the noticeable influence of vertical TME on
the modeled internal tides. Explicitly decreasing $A_v$ (Run 4) slightly modifies $F_x$ (Fig. 2b) and $C$
(Fig. 2d), greatly reduces dissipation everywhere (Fig. 2f), and magnifies the fluctuation of $T$
(Fig. 2e). This also increases $E_k$ everywhere (Fig. 5a), and cross-shelf integrated $E_k$ by 12%
(Table 2), consistent with the relative size of $\varepsilon'$ in the energy budget of Run 1. The M$_2$ internal
tide phase does not change much with the reduced $A_v$ (Fig. 8c). The effects of a high $A_v$ (Run 5)
are more pronounced: This lowers $E_k$ by about 53% (Table 2), with greater reduction (about
80%) on the shelf (Fig. 5a); this also reduces $F_x$, $C$ and $T$ (Figs. 2b – 2e) and essentially inhibits
the onshore propagation of the baroclinic energy. The reduction of $C$ is apparently caused by
diminished $\partial u'/\partial z$, reduced M$_2$ beam strength, and reduced $\rho'$. Increasing $A_v$ also modifies the
phases of the internal M$_2$ tide in the beam by about 45° near the shelfbreak and much more on
the shelf (Fig. 8d). Note that, in Run 5, while $A_v$ is increased to $10^{-2}$ m$^2$ s$^{-1}$, $\kappa_v$ is reduced to $10^{-6}$
m$^2$ s$^{-1}$, a value typically used in non-hydrostatic simulations of internal waves. Increasing $A_v$
alone has greatly reduced $E_k$, we speculate that increasing both $A_v$ and $\kappa_v$ would reduce it even
further.

3.3 Internal Wave-Wave Interaction

Sub-harmonic and super-harmonic internal waves are generated in our control simulation, as
shown by the power spectral densities (PSDs) of the modeled $u'$ at Sites A and B (Figs. 10a and
10b), and as found in related previous studies (e.g., Korobov and Lamb 2008; Nikurashin and Legg 2011), The PSDs are computed from velocity time series over the period of Days 5 – 25. The expected maximum PSD at both sites is at $M_2$ frequency. The Site A PSD contains peaks, significant at 95% confidence level, at frequencies of $M_2/2$, $M_3$, $M_4$, $M_5$ and $M_6$, whereas the Site B PSD has significant peaks only at frequencies that are harmonics of $M_2$. Due to the absence of external disturbances, there is no peak at inertial frequency at either site.

To verify the modeled sub- and super-harmonic waves, PSDs were computed for velocities observed with Shallow Water ’06 Experiment moorings (Newhall et al. 2007; Tang et al. 2007). Figs. 10c and 10d show spectra of observed $u'$ at depths of 458 and 126 m, respectively, during the period of 27-07-2006 to 21-09-2006 by Mooring SW43 at a site with water depth 480 m (see Fig. 1 for its location). Average of the spectra at all depths at the site is also given for reference. A PSD peak significant at 95% confidence level occurs near $M_2/2$ at 458 m (Fig. 10c). Note that $M_2/2$ frequency is very close to $O_1$ frequency, and they are inseparable in the given data. Although it cannot be proven to be $M_2/2$ waves, the observed peak near $M_2/2$ demonstrates the existence of forced and/or spatially trapped waves in the MAB shelfbreak region, as the mooring site is north of the critical latitude of both $M_2/2$ and $O_1$ internal waves ($\sim 29.9^\circ$). Similar forced subinertial baroclinic waves of diurnal frequency have been reported at Yermak Plateau (Fer et al. 2010; Padman et al. 1992). An additional significant peak at $M_4$ is found in the Mooring SW43 $u'$ spectrum for 126 m depth (Fig. 10d). These confirm natural occurrence of the modeled super-harmonic internal waves near the shelfbreak. Note that the observed spectral peaks are much less evident than modeled, a possible consequence of remotely generated waves and local disturbances from various factors (e.g., winds and mesoscale activities) that are not represented in the model. The absence of these processes and of non-hydrostatic effects, along with
numerical errors, might also cause the modeled spectrum to drop more rapidly versus frequency than the observed.

To examine the inhomogeneous structures of modeled internal wave fields, $u'$ signals at $M_2/2$, $M_2$, $M_3$, and $M_4$ frequencies were extracted at every grid point using harmonic demodulation. A snapshot of the $M_2$ velocity component at Day 10 (Fig. 11b) shows a smooth $M_2$ beam tangent to the critical slope. The $M_3/2$ velocity (Fig. 11a) depicts an oscillatory pattern in vertical direction, strongest in the $M_2$ beam, with typical vertical wavelengths of 20 to 30 m. Some weak $M_2/2$ signal is also visible along an $M_3$ beam. A hotspot of strong $M_2/2$ velocity exists approximately 4 km inshore of the critical isobath. Site A sits in the middle of the hotspot. Although $M_2/2$ motions are most intense at the hotspot, features of the $M_2/2$ waves (e.g., $k_z$) are similar throughout the $M_2$ beam. Patterns of the $M_3$ velocity resemble those of $M_2/2$, except that $M_3$ energy is less concentrated in the $M_2$ beam, and weak $M_3$ beams are visible both onshore and offshore of the hotspot (Fig. 11c). The $M_4$ velocity shows a beam originating at the critical slope and reflecting multiple times off the surface and bottom (Fig. 11d).

The development time scales of the sub- and super-harmonic waves indicate the rates of energy transfer. To quantify development, we conducted wavelet analysis on $u'$ at Sites A and B using the Morlet basis function (Torrence and Compo 1998). The results depict the gradual appearance of significant peaks at $M_2$, $M_4$ and $M_6$ frequencies at both sites (Fig. 12). The appearance time increases with frequency. It takes 1 and 2 days for the peaks at $M_4$ and $M_6$ frequencies to become significant, respectively. Site A results show additional significant peaks at $M_2/2$, $M_3$ and $M_5$ frequencies. The Site A $M_2/2$ peak emerges at the very beginning of the simulation, while the $M_3$ and $M_5$ peaks at emerge at days 4 and 7, respectively.
To confirm that the sub- and super-harmonic waves are caused by nonlinear internal wave-wave interaction, bispectral methods (Kim and Powers 1979) were applied to \( u' \) at the two sites (Fig. 13). Bicoherence \( \Phi(\omega_1, \omega_2) \) measures the coherency between waves at three frequencies: the primary frequency pair, \( \omega_1 \) and \( \omega_2 \), and the sum frequency, \( \omega_3 = \omega_1 + \omega_2 \). The bispectrum \( \Psi(\omega_1, \omega_2) \) measures the amount of energy involved in the three-wave coupling. Each point in the bicoherence plot denotes the degree of coupling between waves at \( \omega_1, \omega_2 \), which concur with the plot axes, and \( \omega_3 \), with symmetric appearance about the \( \omega_1 = \omega_2 \) diagonal line. If a wave at \( \omega_3 \) is excited by the interaction of waves at \( \omega_1 \) and \( \omega_2 \), or, in the case of \( \omega_1 = \omega_2 \), waves at \( \omega_1 \) are excited by a wave at \( \omega_3 \), \( \Phi(\omega_1, \omega_2) \) will have a value close to one.

The Site A bicoherence (Fig. 13a) is near one at most of the primary frequency pairs with members of multiples of \( M_2/2 \) and which sum to less than \( M_7 \), e.g., \( (M_2/2, M_2/2), (M_2/2, M_2), (M_2, M_2) \), and \( (M_3, M_3) \). This suggests interaction between waves in frequency groups \( [M_2/2, M_2/2 \text{ and } M_2]; [M_2/2, M_2 \text{ and } M_3]; [M_2, M_2 \text{ and } M_4]; [M_3, M_3 \text{ and } M_6], \) etc. The bispectrum for Site A (Fig. 13c) indicates that the three most energetic coupling frequency groups are: \([M_2, M_2 \text{ and } M_4]; [M_2/2, M_2/2 \text{ and } M_2]; [M_2/2, M_2 \text{ and } M_3]\). Coupling in other frequency groups, although measureable with statistical significance, involves very little energy. The situation at Site B is simpler (Fig. 13b); the only significant bicoherence peaks are for primary pair frequencies at the harmonics of \( M_2 \) (\( M_2, M_4 \) and \( M_6, \) etc.). The corresponding bispectrum distribution (Fig. 13d) shows a consistent pattern that indicates energetic coupling takes place only within two groups: \([M_2, M_2 \text{ and } M_4] \) and \([M_2, M_4 \text{ and } M_6]\). Minor and insignificant peaks at \( (M_2/2, M_2/2) \) and \( (M_2/2, M_2) \) are present, indicating that \( M_2/2 \) and \( M_3 \) waves of very weak intensity are generated at sites like “B” off the \( M_2 \) beam.
The wavelet and bispectral analyses confirm that nonlinear internal wave-wave interactions take place in the model near the shelfbreak internal tide generation area, and that effects are not uniform in space, with relatively strong (initially) M2/2 and (subsequently) M3 waves appearing at Site A, but not B. To further examine the spatial inhomogeneity, bispectrum cross-sections at four discrete primary frequency pairs are shown in Fig. 14. Energetic wave-wave interactions take place mainly along the M2 beam. The frequency group [M2, M2 and M4] shows additional features (Fig. 14c, note that the color scaling in Fig. 14c differs from the others): relatively energetic off-beam interactions occur on the shelf, as also demonstrated by the bispectrum pattern at Site B (Fig. 13).

The energy involved in the wave-wave interactions, and its spatial distribution, can be quantified. Cross-sections of the demodulated \( E_{kz} \) in the control simulation (Fig. 15, the first column) show that M2/2 energy is concentrated mostly in the M2 beam, consistent with the trapping of forced subharmonic waves in a tidal beam found by Korobov and Lamb (2008) (see their Fig. 9a). The M3 wave (Fig. 15i) is relatively intense along the M2 beam, and its propagation along a M3 characteristic is also visible. The M4 wave is distributed much more widely, and its intensity is rather high everywhere on the shelf. The figure shows that the M2/2 and M3 wave energy levels are more than two orders of magnitude smaller than that of the M2 wave in most places, while that of M4 wave is much higher than those of M2/2 and M3 waves, particularly on the shelf. Fig. 5b shows that \( E_k \) at M2/2 and M3 are both 5% of that of M2 at the hotspot (in the \( x \) direction) where Site A lies, respectively, and there is less M2/2 or M3 energy elsewhere. \( E_k \) at M4 is weaker than M2 by about one order of magnitude onshore of the shelfbreak and by more than two orders of magnitude offshore. These energy levels suggest that nonlinear internal wave-wave interaction drains 5-10% of the kinetic energy from the primary
M2 wave on the shelf and near the shelfbreak, with a much lower fraction of M2 energy converted in the deep sea.

Nonlinear momentum advection is the process responsible for most of the wave-wave interaction. Without it there is no internal wave kinetic energy at frequencies M2/2 and M3, and very little energy at frequency M4 (Fig. 15, the second column). The other nonlinear processes in the system, expressed in the vertical mixing parameterization (for both momentum and tracer) and nonlinear equation of state, are responsible for the residual Run 3 M4 waves. This was confirmed with a test simulation having minimum vertical TME and mixing and a linear equation of state (results not shown). Reducing the viscosity and diffusivity to near molecular values (Run 4) increases the internal wave kinetic energy slightly above Run 1 levels in the sub- and super-harmonic frequencies (Fig. 15, the third column). Increasing vertical viscosity to $10^{-2}$ m² s⁻¹ (Run 5) suppresses almost all the internal wave kinetic energy at M2/2 and M3 frequencies and reduces M4 energy substantially. The Run 5 findings have important implications for internal wave modeling and are discussed in Section 4.

4. Discussion

4.1 Effects of Nonlinear Advection

The energy budget calculation (Section 3) suggests that nonlinear momentum advection is not a major source of internal tide energy in the shelfbreak region (Table 2 and Fig. 7). Nevertheless, $u \cdot \nabla u_h$ exerts strong influences on the internal tide generation, evident from its effect on the internal tide phase (Figs. 8b and 9) and energy level (thus also the generation efficiency, if one compares Runs 1 and 3 sharing common forcing). Comparing Runs 1 and 3 indicates that deleting $u \cdot \nabla u_h$ reduces $E_k$ (Fig. 5a), $F_x$, and $C$ (Fig. 2) by roughly 30% each.
Advection of the internal tide by the background flow, $U$, is the main mechanism at work. The
wavelength of $U$ is much larger than that of $u'$, and the relative phase of the two is a strong
function of $x$, giving a strong spatial dependence to this term and the total tidal flow field. The
effect of phase is illustrated by the study of Kelly and Nash (2010), whereby an internal tide
incident on a slope can strongly affect baroclinic tidal conversion at the slope. The total currents
at the seabed are linked to the efficacy of the conversion process; cancellation of barotropic and
baroclinic currents there diminishes the potential energy oscillations created at the sloping
seabed, and thus the baroclinic wave generation. Also, $W_z$ and $\rho'$ being in quadrature would give
no net generation. Note that the energy budget calculation does not divulge the effect of phase.
Also note that disregarding advection does not necessarily reduce internal tide energy as in this
example.

The findings also discourage the use of linear internal tide generation theory at supercritical
slopes such as at the MAB shelfbreak. The linear internal wave theory (Bell 1975; Pétrélis et al.
2006) is based on a weak topography approximation, and is valid only when $\gamma<<1$, $\delta<<1$ and
$\gamma R<<1$ (Balmforth et al. 2002; Garrett and Kunze 2007). Because the steep slope offshore of the
shelfbreak makes $\gamma>1$ and $\delta\approx 0.9$, it is not surprising for nonlinearity to play a role there. Garrett
and Kunze (2007) presumed that the breakdown of the linear assumption in the case of steep
topography in deep sea does not affect the overall baroclinic energy flux because the energy flux
is mostly carried out by large-scale internal wave motions and the nonlinearity is mainly
associated with small-scale waves. The order-twofold increase of baroclinic energy flux on the
shelf from quasi-linear to nonlinear models obtained here argues against making this
presumption regarding the onshore radiation of shelfbreak-generated internal tides.
Another difference between Runs 1 and 3 resides in the sub- and super-harmonic waves (Fig. 15). The lack of harmonics with no advection implies that the waves are generated by \( \mathbf{u} \cdot \nabla \mathbf{u}_h \), which is confirmed by the diminished relative harmonic formation in Run 2 (Fig. 5c). \( \nabla \mathbf{U}_h \) is very small, so \( \mathbf{u} \cdot \nabla \mathbf{u}_h' \) is the main term. Overall, nonlinear advection converts 5-10% of the total Run 1 internal wave kinetic energy to internal waves at \( \text{M}_2/2 \), \( \text{M}_3 \) and \( \text{M}_4 \) frequencies, a small fraction, but possibly important for local TME, particularly within the \( \text{M}_2 \) beam (Fig. 16). We will discuss the non-\( \text{M}_2 \) waves in more detail in the sections to follow.

4.2 Effects of Vertical TME

Internal waves are believed to be major sources of mixing in the ocean (Munk and Wunsch 1998), and quantifying ocean mixing triggered by internal waves has been a major objective of many studies (e.g., Green et al. 2008; Lien and Gregg 2001; Nash et al. 2007). The comparison between Run 1 and 4 suggests that the GLS turbulence closure in the model dissipates about 14% of the internal wave kinetic energy (Fig. 5a). This suggests that internal wave kinetic energy near the shelfbreak, and radiating away, may be overestimated if vertical TME is neglected, a simplification that is often used along with the assumption of linearity (e.g., Pétrélis et al. 2006). On the other hand, with elevated \( A_v \), internal wave energy may be underestimated greatly, especially on the shelf. Comparison between Runs 1 and 5 indicates that substantial portion of the internal wave kinetic energy is damped by the high viscosity, from 50% at the shelfbreak (Table 2) to even more on the shelf (Fig. 5a). Since internal wave simulations with primitive equation models sometimes utilize high vertical momentum viscosity values of \( 10^{-3} \) to \( 10^{-1} \) m\(^2\) s\(^{-1}\) (e.g., Green et al. 2008; Nash et al. 2012), caution is needed when interpreting the results on continental shelves. Another consequence of the elevated \( A_v \) is the complete disappearance of the \( \text{M}_2/2 \) and \( \text{M}_3 \) waves and substantial suppression of the \( \text{M}_4 \) waves (Fig. 15). Because the vertical
scale of the subharmonic waves is much smaller than that of the primary M$_2$ waves, they have high shear and are more vulnerable to damping by eddy viscosity. This may explain why subharmonic internal waves are not prominent in some of the previous simulations.

The hydrostatic model cannot resolve small-scale processes of TME and mixing and substitutes a sub-grid mixing parameterization. The quantifications made here about effects of vertical TME may depend on the turbulence closure scheme, and therefore might not be definitive measures of the interactions between internal waves and TME in the modeled scenario. However, we postulate that the qualitative results are valid. That is, internal waves and turbulence processes are very much coupled, and any attempt to separate them may cause errors in quantitative estimates for each processes. Unfortunately, the coupling between internal waves and turbulence involves processes over a broad range of scales, from tens of kilometers (the scale of internal tides) to decimeters (the scale of Kelvin-Helmholtz instability); this multiscale coupling makes simulation of tidally controlled internal waves challenging.

4.3 Effects of Wave-Wave Interaction

The creation of internal waves at harmonic overtones ($\omega = n\omega_0$, where $\omega_0$ is the forcing frequency and $n = 2, 3, \ldots$) through nonlinear wave-wave interaction near an internal tide generation site has been documented in different slope and excursion parameter regimes: 1) $\gamma<1$ and $R>1$ (Bell 1975), 2) $\gamma<1$ and $R<1$ (Legg and Huijts 2006), 3) $\gamma>1$ and $R<1$ (Gayen and Sarkar 2011; King et al. 2010; Korobov and Lamb 2008; Lamb 2004; Legg and Huijts 2006; Sun and Pinkel 2012), and 4) $\gamma>1$ and $R\geq1$ (King et al. 2010; Legg and Huijts 2006). This MAB shelfbreak M$_2$ study falls into the 3rd category. Our simulations suggest that wave-wave interaction acts in this process with a time scale of one to two M$_2$ forcing periods (Fig. 12).

The modeled M$_2$/2 waves in the main M$_2$ beam are intriguing. Generation of subharmonic
features of this type is usually studied at latitudes near or below the critical latitude where $\omega = 2f$ (~28.8° for M$_2$ frequency), and where PSI is considered to be the primary mechanism (Carter and Gregg 2006; Gerkema et al. 2006; Lamb 2004; MacKinnon and Winters 2005). The specified rotation corresponds to the latitude of 39°, and $\omega_{M_2/2} \approx 0.765f$. It is widely believed that PSI does not occur when $\omega_0/2 < f$. However, Korobov and Lamb (2008) demonstrated the generation of forced subinertial M$_2$/2 waves by M$_2$ motions. By squaring (17) with $N > f$ it can be deduced that at least one of the subinertial wavenumber components $k_x$ or $k_z$ must be complex, and that the waves are not free and must decay exponentially in some direction. Consistent with Korobov and Lamb’s finding, the MAB simulation M$_2$/2 oscillations are concentrated in the main M$_2$ beam, which suggests that the beam is a source region around which the nonlinear advection is the strongest (Fig. 6b) and the generated M$_2$/2 waves are trapped.

Departing from the perception that PSI only occurs via weakly nonlinear resonant interactions, Korobov and Lamb attributed the M$_2$/2 waves to PSI associated with non-resonant triad interactions in a regime of strong nonlinearity. That is, PSI passes energy from the primary frequency to lower frequencies (here, forced M$_2$/2) with weak or nonexistent return (resonant) transfer. Although details of the physical process are unclear, the interactions between M$_2$/2 and M$_2$ waves are confirmed by our bispectrum analysis. In the field, Carter and Gregg (2006) observed strong M$_2$/2 motions in a M$_2$ internal wave beam near Hawai’i and suggested non-resonant strong nonlinear interaction as a possible explanation. Non-resonant nonlinear interaction is also believed to be the cause of super-harmonic beams generated in numerical (Lamb 2004) and laboratory (Teoh et al. 1997) internal wave studies. Interestingly, the nonlinear interaction in Teoh et al. (1997) caused evanescent waves at frequencies $\omega > N$, also outside the
internal wave band, and trapped in the intersection of two internal wave beams, where they were
generated, similar to the M$_2$/2 waves shown here.

M$_2$/2 waves are generated not only from M$_2$ waves alone through PSI but also from
nonlinear interactions between M$_2$ and M$_3$ waves. The weak M$_3$/2 signal along M$_3$ beams is
evidence of this (Fig. 11a). The M$_3$ waves generated in the M$_2$ beam through nonlinear
interactions propagate away from the source region. These waves then interact with M$_2$ waves,
generating M$_2$/2 off the M$_2$ beam. These off-beam interactions between M$_2$ and M$_3$ waves
involve little energy, and it is unlikely that they would be observable in nature.

The strong vertical shear associated with the energetic large-$k_z$ beam-trapped M$_2$/2 motions
contributes significant vertical TME, as suggested by the resemblance between the patchy
pattern of energy dissipation (Fig. 6d) and M$_2$/2 wave field (Fig. 11a). Run 3, with no M$_3$/2
waves, has less tracer mixing (Fig. 16). The significant Run 1 TME is consistent with enhanced
TME from simulated subharmonic free waves (Hazewinkel and Winters 2011). Enhanced shear
of trapped M$_2$/2 waves might (speculatively) partially explain observed strong turbulence along
an M$_2$ internal tidal beam near the shelfbreak east of Monterey Bay (Lien and Gregg 2001).

5. Summary

This study investigates the generation of internal tides and the associated internal wave-
wave interactions in a continental shelfbreak region using a hydrostatic model. An idealized
configuration that mimics the Mid-Atlantic Bight shelfbreak is used. The model simulations are
forced with oscillating tidal current of M$_2$ frequency at the offshore boundary, and the system
falls into the category of supercritical slope, low excursion and Froude numbers. The control
simulation (Run 1) produces internal tides with integrated barotropic to baroclinic conversion
rate of about 335 W m$^{-1}$ in an area of 60 km across the shelfbreak. About 14% of internal tide
energy (~46.5 W m\(^{-1}\)) is dissipated locally through turbulence and bottom dissipation, 18% (~60 W m\(^{-1}\)) radiates onshore and the remaining radiates offshore. Additional simulations are performed to examine processes in greater detail.

Spectra of internal waves generated within the nonlinear control simulation show discrete and significant peaks at frequencies of M\(_2/2\), M\(_2\), M\(_3\), M\(_4\), and so on. Spectra of produced internal waves vary strongly in space, with two sites examined in detail, one within a narrow beam of strong M\(_2\) internal-wave energy and one outside of the beam. Bispectrum analysis confirms that the sub- and super-harmonic waves are generated by nonlinear internal wave-wave interactions mostly through the nonlinear baroclinic advection in the momentum equations, as verified by reducing tidal forcing and by “turning off” advection in two separate simulations. The nonlinear wave-wave interactions are the most intensive within a beam where M\(_2\) internal waves are the strongest. Consistent with Korobov and Lamb’s (2008) study on the topic in a similar environment, the unusual M\(_2/2\) waves appear predominantly in the beam as trapped waves. It is believed that the subinertial M\(_2/2\) waves are generated through non-resonant triad interaction in the M\(_2\) beam where nonlinearity is strong.

The role of nonlinear effects was quantified by looking at other features of the modeled internal waves. Comparisons between simulations with and without momentum advection indicate that nonlinear effects enhance cross-shelf baroclinic flux, barotropic to baroclinic energy conversion rate, and internal wave kinetic energy, all by about 30% near the shelfbreak and even more on the shelf. However, energy analyses of the control simulation indicate that nonlinear momentum advection is not a direct source of internal tides at the shelfbreak. Rather, it modifies phases of the M\(_2\) internal tides and thus the efficiency of barotropic to baroclinic conversion. This implies that linear internal wave generation theory has limited utility at supercritical-slope
shelfbreak conditions, with phases and amplitudes of continental shelf internal tides linked to nonlinearity. It also exposes the limitation of energy budget calculation in diagnosing the contributions of specific internal tide generation processes.

The model also suggests that vertical turbulent momentum exchange (TME) is important in controlling the internal waves at the shelfbreak. With general length scale turbulence closure, the model converts about 14% of the internal wave kinetic energy to local TME. Although the optimality of this turbulence closure scheme is unproven, the conclusion here that neglecting TME causes nontrivial overestimate of the internal wave energy at the shelfbreak is very likely to be true. A consistent finding is that significant enhancement of vertical eddy viscosity, as sometimes used in internal wave modeling for computational reasons, substantially suppresses the internal wave activity. The results here imply that internal waves and local TME in the boundary zone are very much coupled, and they need to be considered simultaneously for complete understanding of internal wave energy budgets and generation processes. This agrees with the philosophy behind recent works involving unified internal wave generation and dissipation, including those forced by tides (Klymak and Legg 2010; Klymak et al. 2010).

The subject of shelf-edge internal wave generation and transformation is a small part of the large problem of internal wave energy balance in the ocean. Because the shelfbreak is a conduit in the transport of internal wave energy between coastal and open oceans, and because, as demonstrated in this and numerous other studies, it itself is a source of internal waves, processes in shelfbreak regions can affect the internal wave fields in both coastal and deep seas. Finally, the results achieved here concerning internal tides are for a single idealized scenario in the vast shelfbreak parameter space that includes spatially and temporally varying stratification, many types of topography, and many external (non-tidal) influences. Further studies of all these factors
would be needed to obtain a more thorough understanding of internal wave dynamics at the shelf edge and beyond.

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Figure Captions:

Fig. 1. Bathymetry and properties of barotropic M$_2$ tide in the Middle Atlantic Bight area. Grey lines are bathymetric contours (in meters); black solid lines are contours of M$_2$ tidal elevation (in meters); blue dashed lines are contours of M$_2$ tidal phases (in degrees); red ellipses are M$_2$ tidal ellipses at selected locations; the scale on land is for both major and minor axes of the tidal ellipses; black triangle indicates the location of Shallow Water ’06 Mooring 43.

Fig. 2. For the five simulations: (a) central part of the model bathymetry and M$_2$ internal wave characteristics (magenta lines); vertically integrated (b) cross-shelf and (c) along-shelf baroclinic energy fluxes, (d) barotropic-to-baroclinic energy conversion rate, (e) total advection of baroclinic energy, and (f) turbulence dissipation rate. The grey dashed line indicates the site of critical slope.

Fig. 3. Summertime climatological vertical profiles of (left) density and (right) buoyancy frequency in the Middle Atlantic Bight that are used to initialize the model.

Fig. 4. Snapshots of (a) cross-shelf and (b) along-shelf baroclinic velocity, (c) vertical velocity, and (d) density anomaly in the control simulation at Day 10. Black triangle and yellow circular in (a) indicate the locations of Sites A and B, respectively.

Fig. 5. Cross-shelf distributions of vertically integrated internal wave kinetic energy (a) from simulations of different physics and (b) of waves at different frequencies in the control simulation; (c) ratios of kinetic energy of waves at different frequencies in the half-forcing simulation to those in the control simulation. The energy of waves at M$_2$/2, M$_3$ and M$_4$ frequencies in (b) has been enlarged by 10 times, and ratios in (c) have been multiplied by 4.
Fig. 6. Cross-shelf section of the right-hand-side terms in the internal wave kinetic energy equation (6) over Day 10 from the control simulation. The black dashed lines are $M_2$ internal wave characteristics.

Fig. 7. Vertically integrated advection terms in the internal wave kinetic energy equation (6) over Day 10 from the control simulation.

Fig. 8. Differences of the phases of internal $M_2$ waves between the control simulation and the simulations of (a) half-forcing, (b) no advection, (c) reduced vertical TME, and (d) enhanced vertical TME. The white dashed lines represent $M_2$ internal wave characteristics.

Fig. 9. Time series of normalized barotropic and baroclinic variables at Site A (See Fig. 4 for its location) over Days 10 – 11 in the (a) control (Run 1) and (b) no advection (Run 3) simulations. Wave phases in the half-forcing simulation (Run 2) are very similar to those in the control simulation.

Fig. 10. (Black lines) power spectral density of modeled (a-b; see Fig. 4 for the locations of Sites A and B) and observed (c-d; see Fig. 1 for the mooring location) cross-shelf baroclinic velocity. The grey lines indicate 95% confidence intervals at selected frequencies; the dashed lines indicate the inertial frequency; the black solid and dash-dotted straight lines indicate the slopes of -2 and -4, respectively; the red lines are the depth-averaged power spectral density at the same locations offset downward by a factor of 100.

Fig. 11. Snapshots of demodulated cross-shelf baroclinic velocity at different frequencies at Day 10. The triangles and circles indicate the locations of Sites A and B, respectively, and the dashed black, green and magenta lines represent internal wave characteristics at frequencies of $M_2$, $M_3$ and $M_4$, respectively.
Fig. 12. The wavelet power spectra of cross-shelf baroclinic velocity at Sites A and B (see Fig. 4 for their locations) in the control simulation. The white contours outline the 95% confidence level; the dashed lines indicate the inertial frequency; the black solid line indicate the “cone of influence”, where edge effects become important. Note that only the results in the first 15 days are shown.

Fig. 13. (Top) bicoherences and (bottom) bispectra computed from cross-shelf baroclinic velocity at Sites A and B (see Fig. 4 for their locations) in the control simulation. The white contours outline the 95% confidence level of the bicoherence.

Fig. 14. Cross-shelf sections of bi-spectra of different frequency pairs computed from cross-shelf baroclinic velocity in the control simulation. The triangles and circles indicate the locations of Sites A and B, respectively. The white dashed lines represent M2 internal wave characteristics.

Fig. 15. Cross-shelf sections of the internal wave kinetic energy at different frequencies in simulations of different physics. The triangle and circle in (a) indicate the locations of Sites A and B, and the dashed black, white and magenta lines represent internal wave characteristics at frequencies of M2, M3 and M4, respectively. Each colorbar is for panels in the same row.

Fig. 16. Cross-shelf sections of the mean diffusivity in the (a) control and (b) no advection simulations and (c) the differences between them. The white contours in (c) outline the difference of $10^6$ m$^2$ s$^{-1}$. The black dashed lines represent M2 internal wave characteristics.
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<td>Half-forcing</td>
<td>On</td>
<td>0.01 m s(^{-1})</td>
<td>Computed with GLS K-KL scheme</td>
<td>Computed with GLS K-KL scheme</td>
</tr>
<tr>
<td>3</td>
<td>Quasi-linear run (no advection)</td>
<td>Off</td>
<td>0.02 m s(^{-1})</td>
<td>Computed with GLS K-KL scheme</td>
<td>Computed with GLS K-KL scheme</td>
</tr>
<tr>
<td>4</td>
<td>Reduced turbulent momentum exchange</td>
<td>On</td>
<td>0.02 m s(^{-1})</td>
<td>(10^{-6}) m(^2) s(^{-1})</td>
<td>(10^{-6}) m(^2) s(^{-1})</td>
</tr>
<tr>
<td>5</td>
<td>Enhanced turbulent momentum exchange</td>
<td>On</td>
<td>0.02 m s(^{-1})</td>
<td>(10^{-2}) m(^2) s(^{-1})</td>
<td>(10^{-6}) m(^2) s(^{-1})</td>
</tr>
</tbody>
</table>
Table 2. Budget of internal tide kinetic energy averaged over an area between 30 km onshore and offshore of the shelfbreak

<table>
<thead>
<tr>
<th>Simulation index</th>
<th>Simulation description</th>
<th>$E_k$ ($10^5$ J m$^{-1}$)</th>
<th>$C$ (W m$^{-1}$)</th>
<th>$T$ (W m$^{-1}$)</th>
<th>$-\nabla \cdot \mathbf{F}_c$ (W m$^{-1}$)</th>
<th>$-\varepsilon'$ (W m$^{-1}$)</th>
<th>$-D'$ (W m$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>335</td>
<td>0.342</td>
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<td>-41.8</td>
<td>-4.60</td>
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<td>2</td>
<td>Half-forcing</td>
<td>41.2</td>
<td>84.6</td>
<td>0.0632</td>
<td>-76</td>
<td>-6.71</td>
<td>-1.18</td>
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<tr>
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<td>107</td>
<td>223</td>
<td>0</td>
<td>-166</td>
<td>-51.7</td>
<td>-3.43</td>
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<tr>
<td></td>
<td>Reduced turbulent momentum exchange</td>
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<tr>
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<td>Reduced turbulent momentum exchange</td>
<td>174</td>
<td>341</td>
<td>-5.87</td>
<td>-320</td>
<td>-2.60</td>
<td>-0.440</td>
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<tr>
<td>5</td>
<td>Enhanced turbulent momentum exchange</td>
<td>68.6</td>
<td>317</td>
<td>-0.287</td>
<td>-251</td>
<td>-62.1</td>
<td>-0.916</td>
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Fig. 1. Bathymetry and properties of barotropic M$_2$ tide in the Middle Atlantic Bight area. Grey lines are bathymetric contours (in meters); black solid lines are contours of M$_2$ tidal elevation (in meters); blue dashed lines are contours of M$_2$ tidal phases (in degrees); red ellipses are M$_2$ tidal ellipses at selected locations; the scale on land is for both major and minor axes of the tidal ellipses; black triangle indicates the location of Shallow Water ’06 Mooring 43.
Fig. 2. For the five simulations: (a) central part of the model bathymetry and $M_2$ internal wave characteristics (magenta lines); vertically integrated (b) cross-shelf and (c) along-shelf baroclinic energy fluxes, (d) barotropic-to-baroclinic energy conversion rate, (e) total advection of baroclinic energy, and (f) turbulence dissipation rate. The grey dashed line indicates the site of critical slope.
Fig. 3. Summertime climatological vertical profiles of (left) density and (right) buoyancy frequency in the Middle Atlantic Bight that are used to initialize the model.
Fig. 4. Snapshots of (a) cross-shelf and (b) along-shelf baroclinic velocity, (c) vertical velocity, and (d) density anomaly in the control simulation at Day 10. Black triangle and yellow circular in (a) indicate the locations of Sites A and B, respectively.
Fig. 5. Cross-shelf distributions of vertically integrated internal wave kinetic energy (a) from simulations of different physics and (b) of waves at different frequencies in the control simulation; (c) ratios of kinetic energy of waves at different frequencies in the half-forcing simulation to those in the control simulation. The energy of waves at $M_2/2$, $M_3$ and $M_4$ frequencies in (b) has been enlarged by 10 times, and ratios in (c) have been multiplied by 4.
Fig. 6. Cross-shelf section of the right-hand-side terms in the internal wave kinetic energy equation (6) over Day 10 from the control simulation. The black dashed lines are M$_2$ internal wave characteristics.
Fig. 7. Vertically integrated advection terms in the internal wave kinetic energy equation (6) over Day 10 from the control simulation.
Fig. 8. Differences of the phases of internal M$_2$ waves between the control simulation and the simulations of (a) half-forcing, (b) no advection, (c) reduced vertical turbulent momentum exchange, and (d) enhanced vertical turbulent momentum exchange. The white dashed lines represent M$_2$ internal wave characteristics.
Fig. 9. Time series of normalized barotropic and baroclinic variables at Site A (See Fig. 4 for its location) over Days 10 – 11 in the (a) control (Run 1) and (b) no advection (Run 3) simulations. Wave phases in the half-forcing simulation (Run 2) are very similar to those in the control simulation.
Fig. 10. (Black lines) power spectral density of modeled (a-b; see Fig. 4 for the locations of Sites A and B) and observed (c-d; see Fig. 1 for the mooring location) cross-shelf baroclinic velocity. The grey lines indicate 95% confidence intervals at selected frequencies; the dashed lines indicate the inertial frequency; the black solid and dash-dotted straight lines indicate the slopes of -2 and -4, respectively; the red lines are the depth-averaged power spectral density at the same locations offset downward by a factor of 100.
Fig. 11. Snapshots of demodulated cross-shelf baroclinic velocity at different frequencies at Day 10. The triangles and circles indicate the locations of Sites A and B, respectively, and the dashed black, green and magenta lines represent internal wave characteristics at frequencies of $M_2$, $M_3$ and $M_4$, respectively.
Fig. 12. The wavelet power spectra of cross-shelf baroclinic velocity at Sites A and B (see Fig. 4 for their locations) in the control simulation. The white contours outline the 95% confidence level; the dashed lines indicate the inertial frequency; the black solid line indicate the “cone of influence”, where edge effects become important. Note that only the results in the first 15 days are shown.
Fig. 13. (Top) bicoherences and (bottom) bispectra computed from cross-shelf baroclinic velocity at Sites A and B (see Fig. 4 for their locations) in the control simulation. The white contours outline the 95% confidence level of the bicoherence.
Fig. 14. Cross-shelf sections of bi-spectra of different frequency pairs computed from cross-shelf baroclinic velocity in the control simulation. The triangles and circles indicate the locations of Sites A and B, respectively. The white dashed lines represent $M_2$ internal wave characteristics.
Fig. 15. Cross-shelf sections of the internal wave kinetic energy at different frequencies in simulations having different physics. The triangle and circle in (a) indicate the locations of Sites A and B, and the dashed black, white and magenta lines represent internal wave characteristics at frequencies of $M_2$, $M_3$ and $M_4$, respectively. Each colorbar is for panels in the same row.
Fig. 16. Cross-shelf sections of the mean diffusivity in the (a) control Run 1 and (b) no advection Run 3 simulations and (c) the differences between them. The white contours in (c) outline the difference of $10^{-6}$ m$^2$ s$^{-1}$. The white dashed lines represent M$_2$ internal wave characteristics.