We have limited attention until now to boundary layers that are located on boundaries of the fluid.

Boundary layers can occur within a fluid as well.

Two examples:
1) Internal boundary layer in the oceanic thermocline
2) The equatorial undercurrent (EUC).
Consider flow in a pipe. For simplicity assume the flow is one dimensional and has constant velocity $U$.

At the entrance to the pipe, the temperature, considered a tracer is held to a value, $T_I$.

And that initially, and so at large distances from the entrance, the temperature is $T_O$.

\[ x = Ut \]
Model Equation

\[
\frac{\partial T}{\partial t} + U \frac{\partial T}{\partial x} = \kappa \frac{\partial^2 T}{\partial x^2}
\]

For small \(\kappa\) we might ignore the diffusion term leading to the \textit{discontinuous} solution:

\[
T = T_1, \quad x - Ut \leq 0,
\]

\[
T = T_o, \quad x - Ut > 0
\]
The transition zone

New coordinates

\[ \xi = x - Ut \quad -\infty < \xi < \infty \]
\[ \tau = t \]

In frame moving with the temperature front

\[ \frac{\partial T}{\partial \tau} = \kappa \frac{\partial^2 T}{\partial \xi^2} \]

\[ T = \frac{T_0 - T_I}{2} \text{erf} \left( \frac{\xi}{2\sqrt{\kappa \tau}} \right) + \frac{T_0 + T_I}{2} \]
The internal boundary layer

Thickness of layer \( \delta : \sqrt{\kappa t} \)
The ventilated thermocline

We are going to apply these ideas to the oceanic thermocline in the subtropical gyres where the wind stress produces a \textit{downward} Ekman pumping.

In analogy with the pipe, the bowl shaped region is where fluid enters the thermocline and is pumped downward carrying the surface density distribution to depth.

Some of the fluid flows to the equator, a region requiring special dynamics.

The deep water beneath the thermocline is water of polar origin that slowly upwells to establish a temperature contrast with the thermocline.
Two principal questions (at least)

1) Why does the surface density forcing extend *only* to about 1km?

2) Why does the bowl become shallow at *low* latitudes?

pole __________________________________________ equator

warm

cold

cold
Ekman Pumping

\[ \vec{V}_e = \hat{k} \times \frac{\dot{\tau}}{\rho f} \]

\[ f = 2\Omega \sin \theta \]

This drives the entire wind driven circulation. \( W_e \) is \( O(10^{-4} \text{ cm/sec}) \).
We first need to explain the structure of the bowl. (why is it shallow at the equator?). The LPS model:

Planetary geostrophy

\( U/\beta L^2 \ll 1 \) for large scales (greater than about 50-100 km)

\[
\rho_n f u_n = -\frac{\partial p_n}{\partial y},
\]

\[
\rho_n f v_n = \frac{\partial p_n}{\partial x},
\]

\[
\frac{\partial u_n}{\partial x} + \frac{\partial v_n}{\partial y} + \frac{\partial w_n}{\partial z} = 0.
\]

\[
\rho_n g = -\frac{\partial p_n}{\partial z}
\]

\[
f = 2\Omega \sin \theta
\]

\[
\beta v_n = f \frac{\partial w_n}{\partial z}
\]

\[
\beta = \frac{df}{dy}
\]

Integrating over all moving layers:

\[
\beta \sum_n v_n h_n = f w_e
\]

Continuity of \( w_n \)
Layer equation for mass conservation

\[
\begin{align*}
(u_2 h_2)_x + (v_2 h_2)_y &= -w_e \quad y > y_2 \\
&= 0 \quad y < y_2
\end{align*}
\]

\[
(u_1 h_1)_x + (v_1 h_1)_y = -w_e \quad y < y_2
\]

from integrating over each layer

\[
\beta v_n = f \frac{\partial w_n}{\partial z}
\]

\[
\beta v_n = -f (u_{nx} + v_{ny})
\]

\[
y \geq y_2, n = 2 \\
y \leq y_2, n = 1 \\
\text{else} \quad 0
\]

yields the potential vorticity equation for each layer
Potential vorticity equations

\[ u_2 \frac{\partial}{\partial x} \left( \frac{f}{h_2} \right) + v_2 \frac{\partial}{\partial y} \left( \frac{f}{h_2} \right) = \frac{f}{h_2^2} w_e \Theta(y - y_2), \]

\[ u_1 \frac{\partial}{\partial x} \left( \frac{f}{h_1} \right) + v_1 \frac{\partial}{\partial y} \left( \frac{f}{h_1} \right) = \frac{f}{h_1^2} w_e \Theta(y_2 - y) \]

\( \Theta(x) = 1, \ x > 0 \)

\( = 0, \ x < 0 \)

Is Heavyside fnc.

\( f u_2 = -\gamma_2 \frac{\partial h}{\partial y}, \)

\( f v_2 = \gamma_2 \frac{\partial h}{\partial x}. \)

\( h = h_1 + h_2, \quad \gamma_2 = \frac{\rho_3 - \rho_2}{\rho_o} g \)
Single moving layer region, 
\[ y > y_2 \]

\[ \beta_2 h_2 = f w_e \]
\[ v_2 = \frac{\gamma_2 \partial h}{f \partial x}, \quad h_2 = h, \quad h_1 = 0 \]

Integrate to eastern boundary \( x = x_e \), where \( u_2 = 0 \). We will satisfy that bc by taking \( h_2 = 0 \) there (not necessary, it need only be a constant but it suffices for our purposes)

\[ \frac{\partial h_2^2}{\partial x} = 2 \frac{f^2}{\beta \gamma_2} w_e \]

\[ h_2^2 = - \frac{2f^2}{\beta \gamma_2} \int_x^{x_e} w_e(x',y)dx' \quad y \geq y_2 \]
The process of subduction

Fluid in layer 2 is driven southward by Ekman pumping. At \( y = y_2 \) it *subducts* beneath layer 1.

It is then no longer driven directly by the Ekman pumping which directly forces layer 1 in that region south of \( y_2 \).
Conservation of potential vorticity

Streamlines in layer 2 are coincident with pressure field. (Geostrophy). In layer 2 this means the streamlines are lines of constant $h=h_1 + h_2$. For $y < y_2$

$$
\vec{u}_2 \cdot \nabla \left( \frac{f}{h_2} \right) = 0 \quad \Rightarrow \quad \frac{f}{h_2} = Q_2(h)
$$

Potential vorticity is an arbitrary function of streamline.

$h=const.$ \quad \quad \quad \quad f/h_2 = const.$
The determination of the function $Q_2$

By definition at $y = y_2$, $h_2 = h$ so on that line

$$\frac{f}{h_2} = \frac{f_2}{h}$$
The conserved relation

On streamline $h = \text{const.}$ the relationship established at the outcrop line is maintained and is valid for all points reached by streamlines emanating from the outcrop line.
The layer thicknesses in the 2-layer region

\[ \frac{f}{h_2} = \frac{f_2}{h} \]

From geostrophy

\[ u_1 = -\frac{1}{f} \frac{\partial}{\partial y} (\gamma_2 h + \gamma_1 h_1), \quad f u_2 = -\gamma_2 \frac{\partial h}{\partial y}, \]

\[ v_1 = \frac{1}{f} \frac{\partial}{\partial x} (\gamma_2 h + \gamma_1 h_1), \quad f v_2 = \gamma_2 \frac{\partial h}{\partial x}. \]

\[ \gamma_1 = g \frac{\rho_2 - \rho_1}{\rho_o}, \quad \gamma_2 = \frac{\rho_3 - \rho_2}{\rho_o} g, \quad h = h_1 + h_2, \]

\[ h_2 = \frac{f}{f_2} h, \]

\[ h_1 = \left(1 - \frac{f}{f_2}\right) h. \]
The Sverdrup relation

$$\beta \sum_n \nu_n h_n = f w_e$$

With geostrophy, yields,

$$\frac{\partial}{\partial x} \left( h^2 + \frac{\gamma_1}{\gamma_2} h_1^2 \right) = \frac{2f^2}{\beta \gamma_2} w_e(x, y)$$

$$h^2 + \frac{\gamma_1}{\gamma_2} h_1^2 = -\frac{2f^2}{\beta \gamma_2} \int_x^{x'} w_e(x', y) \, dx'$$

Then with pv conservation

$$h_1 = \left( 1 - \frac{f}{f_2} \right) h$$

$$h = \frac{\left( D_o^2 \right)^{1/2}}{\left( 1 + \frac{\gamma_1}{\gamma_2} \left[ 1 - \frac{f}{f_2} \right]^2 \right)^{1/2}}$$

$$D_o^2 = -2 \frac{f^2}{\gamma_2 \beta} \int_x^{x'} w_e(x', y) \, dx' \geq 0$$
The thermocline bowl

\[ D_0^2 = (x_e - x) \frac{2}{\rho \gamma} \left( \frac{d\tau f}{dy} \right) \]

Layer thicknesses remain finite as latitude goes to zero.
The horizontal circulation (with shadow zone)

Layer 2 streamlines

Layer 1 streamlines

In shadow zone $u_2 = 0$
Further references


Adding more layers one approaches a high resolution finite difference form of the solution of the continuous model in $z$ as Huang (1989) has done.
The equatorial inertial boundary layer

\[ D_o^2 = (x_e - x) \frac{2}{\rho \gamma_2} \left( \frac{\partial \tau}{\partial y} \frac{f}{\beta} - \tau \right) \]

So the layer thicknesses remain finite as \( y \) and \( f \) go to zero at the equator

\[ h^2 + \frac{\gamma_1}{\gamma_2} h_1^2 = D_o^2 \]

But

\[ \nu_2 = \frac{\gamma_2}{f} \frac{\partial h}{\partial x} \]

\[ \nu_1 h_1 + \nu_2 h_2 = \frac{\tau}{\rho_0 f} \]

“geostrophic” transport equal and opposite to Ekman transport. Need to remove the singularity in \( \nu \)

Dominates as \( f \) goes to zero
Equations of motion in the equatorial region.
Layer model.

\[ u_n \frac{\partial v_n}{\partial x} + v_n \frac{\partial v_n}{\partial y} + \beta y u_n = -\frac{1}{\rho_o} \frac{\partial p_n}{\partial y} \]

\[ f = \beta y \]

\[ u_n \frac{\partial u_n}{\partial x} + v_n \frac{\partial u_n}{\partial y} - \beta y v_n = -\frac{1}{\rho_o} \frac{\partial p_n}{\partial x}, \]

\[ \frac{\partial}{\partial x} (u_n h_n) + \frac{\partial}{\partial y} (v_n h_n) = 0 \quad \text{Scaling} \]

\[ x = Lx' \quad (u, v) = U(u', \frac{1}{L} v') \]

\[ y = l y' \quad p = \rho_o \beta^2 U p' \]

\[ h = Hh' \]
Pierre Welander

Veronis: *J. Marine Res.* 1997, **55**, i-vii
Equations of motion
Non dimensional

dropping primes

\[
\frac{U}{\beta^2} \left( u_n \frac{\partial v_n}{\partial x} + v_n \frac{\partial v_n}{\partial y} \right) + yu_n = -\frac{\partial p_n}{\partial y}
\]

\[
\frac{U}{\beta^2} \left( u_n \frac{\partial u_n}{\partial x} + v_n \frac{\partial u_n}{\partial y} \right) - yv_n = -\frac{1}{\rho_o} \frac{\partial p_n}{\partial x},
\]

As we approach the equator need to keep advective terms in second equation to heal singularity in \(v_n\)

\[
\frac{\partial}{\partial x} (u_n h_n) + \frac{\partial}{\partial y} (v_n h_n) = 0
\]

\[
U = \beta^2
\]

Note zonal velocity remains in geostrophic balance
Pressure-depth scaling

From geostrophy of \( u \) and the hydrostatic balance.

\[
p = \mathcal{O}(\rho_o U \beta^2) = \rho_o \beta^2 | \frac{4}{4} = \mathcal{O}(\gamma_2 H)
\]

so

\[
H = \frac{U \beta^2}{\gamma_2} = \frac{\beta^2 |^4}{\gamma_2}
\]

\[
D_o^2 = (x_e - x) \frac{2}{\rho \gamma_2} \left\{ \frac{\partial f}{\partial y} \beta - \tau \right\}
\]

Matching to the ventilated thermocline solution in the matching region as we leave the equatorial zone.

\[
H^2 = \frac{\tau_o L}{\rho_o \gamma_2}
\]
Further scaling

From the balance of transport between the Ekman layer and the equatorial thermocline:

\[ y = 0 \]

With geostrophic balance for \( U \)

\[ \rho f U | = p = \rho \gamma H \]

\[ \therefore \]

\[ \tau L = \rho \gamma H^2 \]

Work potential energy balance

\[ V_e = \frac{\tau}{\rho f} \quad f = \beta l \]

\[ V_g H = U \frac{1}{L} H = -V_e \]
Scaling results

\[ \ell = \left( \frac{\gamma_2 \tau_o L}{\rho_o \beta^4} \right)^{1/8} \quad \text{200 km} \ll L \]

\[ H = \left( \frac{\tau_o L}{\gamma_2 \rho_o} \right)^{1/2} \quad \text{100 meters} \]

\[ U = \left( \frac{\gamma_2 \tau_o L}{\rho_o} \right)^{1/4} \quad \text{1 meter/second} \]

Scaled boundary layer equations (1)

\[-(y + \zeta_n) v_n = -\frac{\partial B_n}{\partial x}, \quad \zeta_n = -\frac{\partial u_n}{\partial y}, \quad \text{Relative vorticity dominated by} \quad -\frac{\partial u}{\partial y}\]

\[B_n = p_n + \frac{1}{2}u_n^2, \quad \text{Stream function for layers beneath the surface.}\]

\[h_n u_n = \hat{k} \times \nabla \psi_n,
\]

thus

\[q_n \frac{\partial \psi_n}{\partial x} = \frac{\partial B_n}{\partial x}, \quad \text{where} \quad q_n = \frac{y - \frac{\partial u_n}{\partial y}}{h_n}\]
Scaled boundary layer equations (2)

\[ yu_n = -\frac{\partial p_n}{\partial y} \]

Zonal geostrophy

Equivalent to

\[ q_n \frac{\partial \psi_n}{\partial y} = \frac{\partial B_n}{\partial y} \]

\[ q_n \nabla \psi_n = \nabla B_n \]
Conservation of $B_n$ and $q_n$

from

\[ q_n \nabla \psi_n = \nabla B_n \]

\[ q_n \vec{u}_n \nabla \psi_n = \vec{u}_n \nabla B_n = 0 \]

\[ \nabla q_n \times \nabla \psi_n = 0 \]

\[ \vec{u}_n \nabla q_n = 0 \]

\[ q_n = Q_n(B_n) \]
The equation of motion for layer 2

\[ p_2 = h, \quad p_1 = h + \Gamma_{12} h_1, \]

Hydrostatic relation (n.d.)

\[ \Gamma_{12} = \frac{\gamma_1}{\gamma_2} \]

conservation of pv and \( B_2 \)

\[ \frac{y - \frac{\partial u_2}{\partial y}}{h_2} = Q_2 \left( h + \frac{1}{2} u_2^2 \right) \]

geostrophic balance for \( u_2 \)

\[ \frac{\partial h}{\partial y} = -yu_2 \]
The determination of $Q_2$

The link to mid-latitudes.

For large $y$ (bdy layer coordinate) must merge with mid latitude dynamics

$$q_2 \approx \frac{y}{h^2}, \quad B_2 \approx h$$

From the ventilated thermocline solution

$$\frac{y}{h^2} = Q_2(h) = \frac{y_2}{h}$$

thus

$$Q_2(B_2) = \frac{y_2}{B_2}$$

$$\frac{y - \partial u_2}{\partial y} = \frac{y_2}{h + \frac{1}{2}u_2^2}$$
The boundary layer differential equations

\[ \frac{\partial u_2}{\partial y} = y - \frac{y_2 h_2}{h + \frac{1}{2} u_2^2}, \]

ODEs only in \( y \)

\[ \frac{\partial h}{\partial y} = -y u_2 \]

need a relation between \( h \) and \( h_2 \)

but Sverdrup relation no longer valid.

\[ h_2 = h - h_1 \]
Two closures: both sketchy

let $y_n \gg 1$ be the northern latitude where the solution merges with the mid-latitude solution.

One closure assumes: $h_1(x,y) = h_1(x,y_n)$ for all $y$.

The second assumes

$$h_1(x,y) = h_1(x,y_n) + \frac{(h(x,y_n) - h(x,y))}{\Gamma_2}$$

or upper layer pressure gradient independent of $y$. 
Boundary condition at the equator

Fluid does not cross equator (pv conserved).

On \( y = 0 \) \( B_2 = \text{const.} = B_o \)

and \( B_o = h(0,y_n) \)

\[
(u_2h_2) = -\frac{1}{q_2} \frac{\partial B_2}{\partial y} = -\frac{\partial}{\partial y} \left( \frac{B_2^2}{2y_2} \right)
\]

\[
\left[ \int_0^{y_n} u_2 h_2 dy \right]_{x=0} = \frac{B_o^2 - h^2(0,y_n)}{2y_2}
\]
Solutions

Solutions obtained by a shooting method. Starting with $h$ from the mid-latitude solution, guess a starting $u_2$ at $y = y_n$ and integrate to the equation and try to “hit” $B_0$. Then adjust guess for $u_2$.

With the first closure

![Graphs showing solutions](image-url)
The equatorial thermocline

Fig. 6.4.3. Base of the moving layer representing the core of the under-current shown as a solid line at the equator and as a dashed line in the matching region at \( y = y_e \). (From Pedlosky 1987)
Results with second closure

Fig. 6.4.4. Profiles of $u_2$, $\partial u_2 / \partial y$, $h$, and $h_1$ for the case in which (6.4.19) is used. The parameters are otherwise as in Fig. 6.4.2. The calculation is at $x = 0.5$. The maximum velocity of the eastward velocity is now 0.910.
The link to mid-latitudes

Fig. 6.4.5. Several lines of constant $B_2$, i.e., streamlines for the flow calculated in Fig. 6.4.2. (From Pedlosky 1987)
A multi-layer model

Fig. 6.4.6. Results of a four-layer model showing the monotonic decrease of the velocity with depth in the undercurrent solution. (Courtesy of R. Samelson, pers. comm.)
EUC observations

Fig. 6.1.2. a Contours of zonal velocity in the EUC measured dire current meter in the Pacific at the same longitude as Fig. 6.1.1. b The density field. Note that the meridional density gradient vanishes at the equator. (From Johnson and Luther 1994)

Fig. 6.1.3. Temperature and zonal velocity profiles from the Atlantic and Pacific oceans. In each case the measurements represent 2-year means. (From Halpern and Weisberg 1989)
An Internal boundary layer in the thermocline

The coldest water in the subtropical thermocline comes from water downwelled from the *southern* boundary of the subpolar gyre.

Water that has sunk at the pole and, rising slowly, fills the abyss.
The diffusive internal layer
refs:

To smooth out the temperature difference between the abyss and the thermocline a diffusive layer might be expected as anticipated by Welander

Welander, P. 1971 The thermocline problem, Phil. Trans. R. Soc. Lond. A. 270, 415-421


Boundary layer equations

Following Samelson and Vallis (1997)

\[- f v = - \frac{\partial p}{\partial x} - \varepsilon u , \]

\[f u = - \frac{\partial p}{\partial y} - \varepsilon v , \]

\[\frac{\partial p}{\partial z} = b , \]

\[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]

\[b_t + u b_x + v b_y + w b_z = \kappa_v b_{zz} + \kappa_H \nabla^2 b - \lambda \nabla^4 b \]

\[\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]
The Result of the S&V calculation

Double structure to thermocline and an interior maximum in $T_z$

Upper maximum is the ventilated thermocline contribution. Note deep *positive* $w$ in the abyss with a zero at the base of the thermocline.

Figure 4. Vertical profiles of $T$ (left panel), $T_z$ (center), and $w$ (right) at the center of the domain, $(x, y) = (0.5, 0.5)$, for the solution in Figure 2.
Scaling the internal thermocline (1)

from \( \frac{\partial w}{\partial z} = \frac{\beta}{f} v \) since \( w \) is 0 at the base of the adiabatic, ventilated thermocline.

\( W = \frac{\beta}{f} U \delta \)

From thermal wind \( u_z = -\frac{b_y}{f} \)

The horizontal gradient of buoyancy is determined by the slope of the isopycnals in the ventilated thermocline solution

\( \frac{\Delta b}{L} = \frac{f U}{D_a} \)

\( D_a \) is the vertical scale of the adiabatic thermocline

\( D_a^2 : \frac{f^2 W_e L}{\beta \Delta b} \)
Scaling the internal thermocline (2)

In the diffusive region of the internal thermocline, vertical diffusion balances vertical advection.

\[ wT_z \approx \kappa T_{zz} \]

\[ \frac{W}{\delta}, \frac{\kappa_v}{\delta} \]
The internal thermocline scale

\[ W = \frac{\beta}{f} U \delta \]

\[ \frac{\Delta b}{L} = f U / D_a \]

\[ W = \frac{\beta \Delta b}{f L} D_a \delta = \frac{\kappa_v}{\delta} \]

\[ \delta = \left( \frac{\kappa_v f L}{\beta \Delta b D_a} \right)^{1/2} \]

\[ D_a^2 = \frac{f^2 W e L}{\beta \Delta b} \]

Finally

\[ \delta = \kappa_v^{1/2} \left( \frac{f^2 L}{\beta \Delta b W_e} \right)^{1/4} \]

Distinction is the 1/2 power law and not 1/3 as would obtain if \( \delta \) and not \( D_a \) were used in thermal wind eqn.
Scaling law from calculations

Satisfies the $\kappa^{1/2}$ law.

Figure 13. (a) Thickness of the internal peak of $T_i$ (□) versus $\kappa_i$, from the profiles in Figure 11. The internal boundary layer scale $\delta_i$ and the advective-diffusive scale $\delta$ are also shown (dashed lines), along with the corresponding thicknesses from solutions of the similarity equations (6.1) (×) and (6.2) (+). (b) Maximum upward vertical velocity at $(x, y) = (0.5, 0.5)$ versus $\kappa_i$, from the solutions in Figure 11. The internal boundary layer scale $W_i$, the asymptotic estimate $W_{as} = W_0$ from Young and Ierley (1986) for solutions of (6.1), and the advective-diffusive scale $W_d$ are also shown (dashed lines).
An alternative picture

There have been calculations in which the entire thermocline is a dissipative boundary layer.


\[
f v = \frac{1}{\rho_o} \frac{\partial p}{\partial x}
\]
\[
\beta v = f \frac{\partial w}{\partial z}
\]
\[
u \rho_x + v \rho_y + w \rho_z = \kappa_v \rho_{zz}
\]

\[
\frac{\partial p}{\rho_o} = \frac{f^2}{\beta} \frac{\partial w}{\partial z}
\]

Implies the existence of a function \( M \) such that:

\[
\frac{p}{\rho_o} = M_z, \quad \frac{f^2 w}{\beta} = M_x
\]

\[
\therefore \quad u = -\frac{M_y}{f}, \quad v = \frac{M_x}{f}, \quad g \rho / \rho_o = -M_z
\]
The M equation and scales

\[ 1 \left[ M_{x}M_{z} - M_{z}M_{x} \right] + \frac{\beta}{f^2} M_{x}M_{zz} = \kappa_{v} M_{zz} \]

Simple case when \( M = M(x,z) \) then horizontal advection terms vanish. Equivalent to system:

Scales \( L, U, d, g', W \)

\[ \beta = \frac{g' \rho_{x}}{f \rho_{o}}, \quad \nu_{z} = \frac{g' \rho_{x}}{f \rho_{o}}, \quad \nu_{z} = \kappa_{v} \rho_{z} \]

\[ U = \frac{f}{\beta} W / d \]

\[ \frac{U}{d} = \frac{g'}{fL} \quad W = \frac{\kappa_{v}}{d} \quad \rightarrow d = \left( \frac{\kappa_{v} f^2 L}{g'} \right)^{1/3} \quad \rightarrow W = \kappa_{v}^{2/3} \left( \frac{\beta g'}{f^2 L} \right) \]

This gives a thicker bl and a weaker w
Fig. 16.4 Solution of the one-dimensional thermocline equation, (16.27), with boundary conditions (16.28), for two different values of the diffusivity: $\tilde{R} = 3.2 \times 10^{-3}$ (solid line) and $\tilde{R} = 0.4 \times 10^{-3}$ (dashed line), in the domain $0 \leq \tilde{z} \leq -1$. ‘Vertical velocity’ is $W$, ‘temperature’ is $\tilde{T}$, and all units are the non-dimensional ones of the equation itself. A negative vertical velocity, $\tilde{W}_E = -1$, is imposed at the surface (representing Ekman pumping) and $B_0 = 10$. The internal boundary layer thickness increases as $\tilde{R}^{1/3}$, so doubling in thickness for an eightfold increase in $\tilde{R}$. The upwelling velocity also increases with $\tilde{R}$ (as $\tilde{R}^{2/3}$), but this is barely noticeable on the graph because the downwelling velocity, above the internal boundary layer, is much larger and almost independent of $\tilde{R}$. The depth of the boundary layer increases as $\tilde{W}_E^{1/2}$, so if $\tilde{W}_E = 0$ the boundary layer is at the surface, as in Fig. 16.5.