Dimensional analysis of models and data sets

James F. Price[a)]
Woods Hole Oceanographic Institution, Woods Hole, Massachusetts 02543

(Received 22 May 2002; accepted 4 November 2002)

Dimensional analysis is a widely applicable and sometimes very powerful technique that is demonstrated here in a study of the simple, viscous pendulum. The first and crucial step of dimensional analysis is to define a suitably idealized representation of a phenomenon by listing the relevant variables, called the physical model. The second step is to learn the consequences of the physical model and the general principle that complete equations are independent of the choice of units. The calculation that follows yields a basis set of nondimensional variables. The final step is to interpret the nondimensional basis set in the light of observations or existing theory, and if necessary to modify the basis set to maximize its utility. One strategy is to nondimensionalize the dependent variable by a scaling estimate. The remaining nondimensional variables can then be formed in ways that define aspect ratios or that correspond to the ratio of terms in a governing equation.
[DOI: 10.1119/1.1533057]

I. ABOUT DIMENSIONAL ANALYSIS

Dimensional analysis is a remarkable tool insofar as it can be applied to virtually all quantitative models and data sets. Topics in the recent literature include donuts, dinosaurs,[1] and the most fundamental theories of physics.[2] In some instances dimensional analysis is very powerful; results include the log-layer profile of a turbulent boundary layer and the spectral slope in the inertial subrange of isotropic turbulence, both landmarks in fluid mechanics.[3] More often the result of dimensional analysis is a hint at the form of a solution or a more effective way to display or correlate a large data set. These kinds of results, though seldom complete if taken alone, are an essential element of many investigations.

This paper is an introduction to dimensional analysis that aims to make the method and the results as accessible as possible.[3,4] The plan is to investigate the motion of a simple pendulum while emphasizing the use of dimensional analysis as an adjunct to experimental, numerical, and theoretical methods.[5] If the simple pendulum seems too familiar, skip ahead to Sec. III. If the use of nondimensional variables is also familiar, skip ahead to Sec. IV, where a general method of computing a basis set of nondimensional variables is presented. The effects of drag are considered in Sec. V, and concluding remarks are in Sec. VI.

II. MODELS OF A SIMPLE PENDULUM

Consider a pendulum that can be made and observed with simple tools; a small lead fishing sinker having a mass of a few tens of grams suspended on a thin monofilament line a few meters in length. The motion of such a pendulum will be only lightly damped by drag with the surrounding air and can be characterized by two distinct time scales—a regular, fast time-scale oscillation having a period, P, and a slow, more-or-less exponential decay with a time-scale, Γ^{-1}. Our goal will be to learn how these time scales and some other variables, for example, tension in the line, vary with line length and the mass of the bob.

A. A physical model

To analyze the motion of the pendulum, we begin by listing the variables that are presumed to be relevant to the aspect of the motion that is of interest. To start, consider the fast time-scale, oscillatory motion. The line will be idealized as rigid, so that the bob must swing along a constant radius. The motion of the bob is then defined by the angle of the line to the vertical, $\phi(t)$, and its time derivatives; the angle ϕ is the dependent variable of this physical model and the time, t, is the only independent variable. Several properties of the pendulum would seem to be important—the mass of the bob, M, the length of the supporting line, L, and the acceleration of gravity, g. To account for why there is motion at all, the initial angle, ϕ_0, or an initial angular velocity (here assumed to be zero) must also be included. This list of relevant variables constitutes a physical model for the oscillatory motion of a simple, inviscid pendulum:

(1) The angle of the line, $\phi \doteq \text{nond}$, the dependent variable.
(2) The time, $t \doteq m^0 l^0 t^1$, the independent variable.
(3) The mass of the bob, $M \doteq m^1 l^0 t^0$, a parameter.
(4) The length of the line, $L \doteq m^0 l^1 t^0$, a parameter.
(5) The acceleration of gravity, $g \doteq m^0 l^1 t^{-2}$, a parameter.
(6) The initial angle, $\phi_0 \doteq \text{nond}$, a parameter.

The notation $X \doteq m^a l^b t^c$ indicates the dimensions mass, length, and time (or nond if the variable is nondimensional). Parameters are variables that are constant during a particular realization—M, L, g, and ϕ_0 in this list. The range of these parameters defines the family of pendula and environments that are of interest.

B. A mathematical model

Dimensional analysis is most useful when a mathematical model is not known. Mathematical models of the simple pendulum are well known, and we will use them to generate numerical data and to show how dimensional analysis can be applied to a mathematical model. For an inviscid pendulum the rate of change of angular momentum of the bob is due solely to the torque associated with the downward force of gravity acting on the bob,

$$L^2 M \frac{d^2\phi}{dt^2} = -LMg \sin\phi. \tag{1}$$

If we divide by $L^2 M$, the equation of motion becomes

$$\frac{d^2\phi}{dt^2} = -\frac{g}{L}\sin\phi. \tag{2}$$

For experimental purposes it is preferable to start from a state of rest and so the initial conditions at $t=0$ are taken to be

$$\phi = \phi_0, \quad \frac{d\phi}{dt} = 0. \tag{3}$$

It may also be of interest to compute the tension in the line, T, from the radial equation of motion, $dr/dt=0$, and thus

$$T = gM\cos\phi + LM\left(\frac{d\phi}{dt}\right)^2. \tag{4}$$

The appropriate solution method to Eqs. (2) and (3) depends upon the initial angle, ϕ_0. If ϕ_0 is restricted to values less than about 0.1 rad, then $\sin\phi$ in Eq. (2) can be well approximated by ϕ and the resulting linear model has the well-known solution

$$\phi = \phi_0 \cos(t/\sqrt{L/g}). \tag{5}$$

In the general case where ϕ_0 may take any value from $-\pi$ to π, Eq. (2) is nonlinear and a solution cannot be given in elementary functions. Numerical integration of Eqs. (2) and (3) is straightforward, however, and yields (numerical) data (see Fig. 1) that we will treat as an intermediary between experiment and theory; we know exactly the physical model, but not the specific parameter dependence.

C. Models generally

Model equations are a relation between a dependent variable, the angle ϕ or the tension T, and the independent variables and parameters that make up the physical model. Even if we had no idea of the mathematical model, we could still assert that a complete physical model could be used to define a relation

$$\phi = F(t,g,L,M,\phi_0), \tag{6}$$

where F will be used to indicate an unknown function. If our goal were to solve for the period of the oscillation, then we would evaluate the time at some (arbitrary) repeated value of ϕ to find

$$P = F(g,L,M,\phi_0). \tag{7}$$

For the tension, T, and the maximum tension during an oscillation, $T_{\max}$, we could similarly write

$$T = F(t,g,L,M,\phi_0), \tag{8}$$

and

$$T_{\max} = F(g,L,M,\phi_0). \tag{9}$$

It will often happen that the list of variables for the physical model will include one or more parameters that do not appear in the mathematical model. If we compare Eqs. (6) and (2), the physical model includes the mass, M, while in the mathematical model, the mass appeared as a coefficient in the gravitational force (right-hand side of Eq. (1)) and in the inertial force (left-hand side) and cancels. In this regard, the mathematical model, Eq. (2), is a considerable advance over the physical model, Eq. (6). Note too that the angular velocity, $d\phi/dt$, appears in the mathematical model for the tension, Eq. (4), although not in the physical model Eq. (8). Even if we were aware that the mathematical model of tension depends upon $d\phi/dt$, we should still omit this second dependent variable in Eq. (8) because $d\phi/dt$ must itself depend upon t, g, L, M and ϕ_0 and should not be written into the physical model again.

Relations (6)–(9) could be written in one of several forms, for example,

$$\phi/F(t,g,L,M,\phi_0) = 1, \tag{10}$$

or reusing F yet again,

$$F(\phi,t,g,L,M,\phi_0) = 1. \tag{11}$$

What is most important is the assertion that the physical model is complete, meaning that it includes all of the variables required to construct a mathematical model that could in principle yield a unique solution. If we do not know the corresponding mathematical model, then completeness can only be a hypothesis.

Although it is essential that the physical model be complete, it is also highly desirable that the physical model be as concise as possible, that is, it include only those variables that have a significant effect on the dependent variable. The selection of variables for the physical model thus requires considerable judgment.

III. AN INFORMAL DIMENSIONAL ANALYSIS

A. Invariance to a change of units

We take it for granted that every equation must be dimensionally consistent, or homogeneous.[6] But how about the units used to measure length, time, etc.? The premise of dimensional analysis is that the physical relationship expressed by a complete equation does not depend on the choice of units, that is, whether SI, British engineering, or any other.[5] Invariance to the choice of units implies a constraint on the form that the dimensional variables can take in a complete equation, and dimensional analysis is a systematic procedure for learning what that form is.

Angles are an interesting and relevant case. An angle is the ratio of two lengths, an arclength and a radius, and is thus inherently nondimensional. (Angles may be specified in units of radians or degrees, among others.) If we compute an angle ϕ by measurements of arclength and radius in units of meters, we will get a certain number. If we then use feet to measure these same lengths, we will get precisely the same number, that is, the same angle. Thus the left-hand side of Eq. (6) is invariant to a change in the units of length. How about the right-hand side of Eq. (6)? For invariance to the choice of units to hold, the length and the acceleration of gravity must appear in the ratio g/L (or any power of the ratio, for example, $\sqrt{L/g}$), and not as g or L separately, because the latter would imply a change of F with a change in the units of length. Thus, we already know something about the invariant form of Eq. (6). Consider the mass, M. A change in the units of mass should also leave F unchanged, and yet it is impossible to see how that could hold because M is the only variable in the physical model having dimensions of mass. This informal analysis leads to the conclusion that an equation for ϕ that is invariant to a change of units cannot depend upon the mass of the bob alone. This conclusion is an obvious result of the mathematical model, Eqs. (2) and (3), but can be deduced by dimensional analysis in the absence of

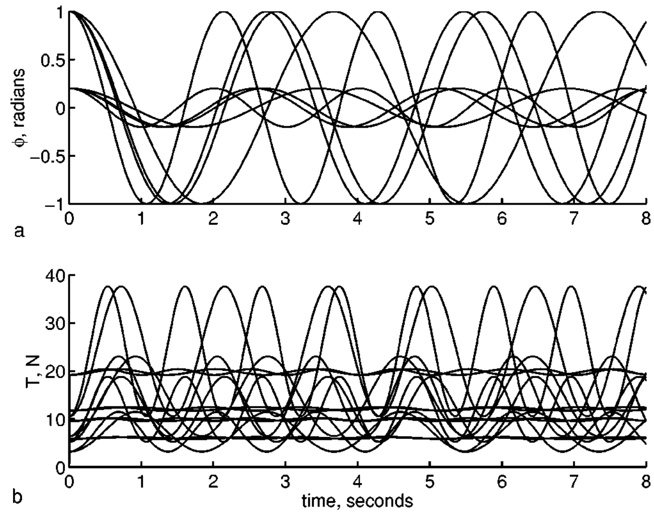

Fig. 1. Numerical solutions of the simple, inviscid pendulum for two values each of L (1, 1.8) m, M (1, 2) kg, g (9.8, 6) m^2/s, and ϕ_0 (0.2, 1.0) rad or 16 solutions in total. (a) The angle, ϕ. The mathematical model, Eqs. (2) and (3), does not depend upon M, and so there are 8 distinct solutions here. (b) Tension (Newtons) for the same set of solutions. Here there are 16 distinct solutions, though some are difficult to distinguish. As these data were acquired it was noticed that the maximum tension did not vary with L.

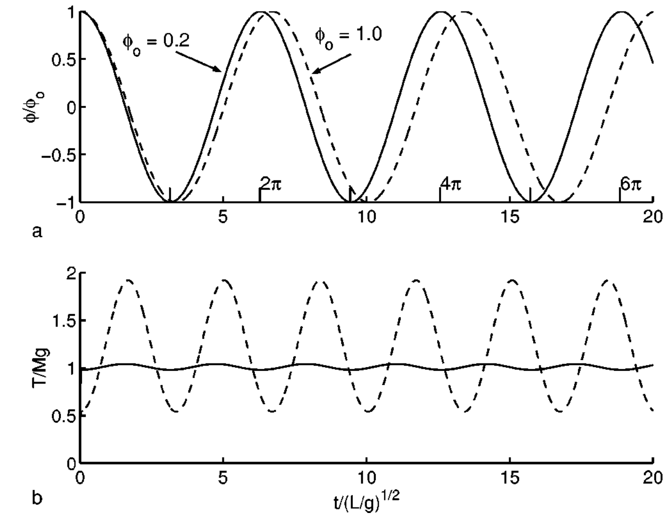

Fig. 2. The numerical solutions of Fig. 1 (two values each of L, M, g, and ϕ_0) plotted in a nondimensional format. The time is nondimensionalized by $\sqrt{L/g}$. In (a) the angle ϕ is normalized by the initial angle, ϕ_0. This helps us to compare the period of the two solutions, but obscures the important difference in amplitude. The eight distinct solutions of Fig. 1(a) collapse to just two curves that correspond to the cases $\phi_0=0.2$ (solid curve) and $\phi_0=1.0$ (dashed curve). In (b) the tension is nondimensionalized by Mg. The 16 separate curves of Fig. 1(b) collapse to just two curves that have the ϕ_0 as in (a).

the mathematical model. A similar consideration of the units used to measure time indicates that t and g must also appear together in a nondimensional variable, say $t/\sqrt{L/g}$. Again, any power of this variable is possible, but we might as well leave the independent variable t to the first power. The upshot of this reasoning is that the dimensional variables appear in combinations that are nondimensional. The simplest (but not the only) form for Eq. (6) is

$$\phi=F(t/\sqrt{L/g},\phi_0). \tag{12}$$

The essential result is that in place of a dependence on one independent dimensional variable and four parameters as in the original Eq. (6), we now have a dependence on one nondimensional independent variable, $t/\sqrt{L/g}$, and one nondimensional parameter. When the data of Fig. 1 are plotted using this nondimensional format in Fig. 2, there is a very significant reduction in the volume of data required to display and define the data set, an important benefit of dimensional analysis applied to a presentation of data.

The period of the motion can be written in a way analogous to Eq. (7) as

$$P/\sqrt{L/g}=F(\phi_0). \tag{13}$$

If ϕ_0 is small, say less than about 0.1 rad, the dependence on ϕ_0 is found experimentally to be negligible [Fig. 3(a)], and $F(\phi_0\ll 1)=\text{constant}$. The period of a simple pendulum undergoing small amplitude oscillations thus increases in proportion to the square root of the length of the supporting line divided by the local acceleration of gravity, g. The measurement of the period of just one linear pendulum is sufficient to fix the constant, $F(\phi_0\ll 1)=2\pi$, for all such pendula. If ϕ_0 is not small, then from dimensional analysis and Eq. (13) it is evident that the nondimensional period will depend on the single parameter ϕ_0. The function $F(\phi_0)$, often referred to as a similarity law,[6] might be determined by experiment (assuming that viscous effects are negligible), by the analysis of numerical simulations, or from theory [Fig. 3(a)].

B. Natural units

A complementary way to come to the same result is to consider the units used to measure time in the mathematical

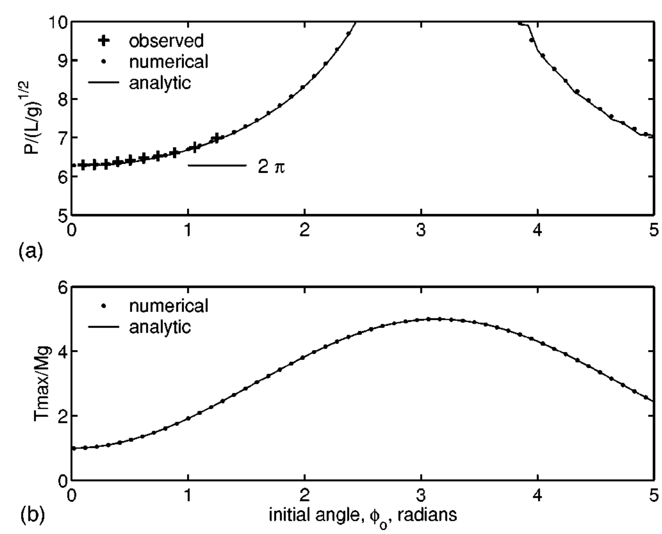

Fig. 3. (a) The period of a simple pendulum as determined from a series of numerical solutions (dots), computed from energy conservation, which leads to an elliptic integral that is evaluated numerically (solid line), and observations (crosses). The observations were acquired by measuring the time required for ten oscillations of a nearly conservative pendulum using an electronic stopwatch (the observed period is accurate to about 0.3%). The flexible line of this pendulum and the initial condition $d\phi/dt=0$ limit the initial angle to about $-\pi/2<\phi_0<\pi/2$. The period goes to infinity as $\phi_0\to\pi$ because the initial condition is $d\phi(t=0)/dt=0$. From dimensional analysis we expect that this result, $F(\phi_0)$ from the right-hand side of Eq. (13), holds for all simple, inviscid pendula. (b) The maximum tension (during an oscillation), $T_{\max}$, determined from a series of numerical solutions (dots) and as computed from energy conservation and the radial equation of motion (solid line); we had no means to observe this variable.

model, Eqs. (2) and (3). There is no compelling reason to use seconds, aside from the practical convenience that clocks are calibrated in this unit. But suppose that our aim was to simplify the mathematical model by choosing a unit of time that is natural to the problem itself. The natural time scale of the pendulum is, of course, the (linear) period, which can be used to define a nondimensional time (omitting the factor 2π),

$$t^* = t/(P/2\pi) = t/\sqrt{L/g}. \tag{14}$$

The variable t^* is a pure number that has the same numerical value regardless of the units used to measure t, g, and L, a hint that there might be something useful here.

Nondimensional time may sound a little esoteric, but amounts to nothing more than counting time in units of the linear period while taking explicit account of the $\sqrt{L/g}$ dependence of the period. If we were to consider only one pendulum, then the whole exercise would amount to dividing the time by a constant. But if we were to consider all possible pendula (all possible L and g), then there is real merit in this. To see why, let's follow through by rewriting the equation of motion, Eq. (2), using the nondimensional time, t^*. Time derivatives transform as $dt = dt^* \sqrt{(L/g)}$ and so the equation of motion becomes

$$\frac{d^2\phi}{dt^{*2}} = -\sin\phi, \tag{15}$$

with the initial conditions as before. The solution will be of the form

$$\phi = F(t^*, \phi_0), \tag{16}$$

which is just like Eq. (12). If the amplitude of the motion is small, then the linear solution of Eq. (15) is just

$$\phi = \phi_0 \cos t^*. \tag{17}$$

The dependence upon L and g has not been omitted, but is rather subsumed into the nondimensional time, t^*, so that Eq. (17) suffices for all L and g.

Recall that the linear pendulum has the solution $\phi = \phi_0 \cos(t/\sqrt{L/g})$, and note that the argument of that cosine function is the nondimensional time—it was there all along! (because the arguments of trigonometric and exponential functions are nondimensional). The difference between Eqs. (5) and (17) is in how you look at them; do you see the dimensional time, t, as the independent variable, or do you see instead the nondimensional time, $t^* = t/\sqrt{L/g}$? The answer will probably depend on the stage of an investigation (and no doubt on our familiarity with dimensional analysis); experimental data are almost always recorded in dimensional units, and it may be helpful to carry out a numerical integration using dimensional units. But when it comes time to report a mass of data from many experiments or integrations, there is often a great advantage to the use of nondimensional variables defined by dimensional analysis.

C. Extra and omitted variables

Dimensional analysis revealed that the period of a simple, inviscid pendulum did not depend on the mass of the bob, M. This result might suggest that the inclusion of extra or superfluous variables in a physical model will not spoil the result. However, in most cases an extra variable will not be detected in this way and will lead to an extra nondimensional variable. For example, if we had included the bob diameter, D_b, in the physical model of the inviscid pendulum, it would have been carried through to a nondimensional variable, D_b/L. If we had access to an experiment, we would soon find that D_b/L was of no evident importance in determining the period of a nearly conservative pendulum, and would drop it from the final result.

We may ask whether the omission of a relevant variable would be detected. The answer is yes, rarely, if the omission makes it impossible to nondimensionalize the dependent variable. For example, if we analyzed tension under the assumption that the mass would be irrelevant as it was for the period, then it would not be possible to find a nondimensional tension. That would be a clear signal that something important had been omitted from the physical model. However, if the dependent variable can be nondimensionalized with the variables that are included, then the purely formal procedure of dimensional analysis is not able to identify an incomplete model.[7]

IV. A BASIS SET OF NONDIMENSIONAL VARIABLES

Once a preliminary physical model has been defined, the second and mathematical step of a dimensional analysis is to find a complete set of nondimensional variables for that model. With a little experience and for small problems such as the simple, inviscid pendulum, this can be done by inspection. For larger problems it may be helpful to use the following technique[4] that relies on the matrix methods of linear algebra. Elements of linear algebra are commonly used in dimensional analysis,[3,5] and an exhaustive exposition of matrix methods can be found in Ref. 8. Brückner and colleagues[9] have shown how matrix methods can be applied to very large problems. The following development differs from most others in that it does not rely on the Buckingham Pi theorem, although it comes to the same result, and utilizes the null space basis to find a basis set of nondimensional variables.[10]

A. The mathematical problem

What can we infer about a function given only that it is invariant to a change of units? An arbitrary change of units for the dimensional variable X_i can be written as

$$X_i' = \alpha_1^{D_{1i}} \alpha_2^{D_{2i}} \cdots \alpha_J^{D_{Ji}} X_i, \tag{18}$$

where α_1 is the scale change associated with mass, α_2 the scale change associated with length, α_3 is for time and so on up to J fundamental units.[11] For pendulum problems and for mechanics generally, $J=3$ (mass, length, and time), which is assumed to simplify later expressions. The doubly indexed object, D_{ji}, is the dimensionality of the ith dimensional variable with respect to the jth fundamental unit, and when written as a matrix is called the dimensional matrix, $\mathbf{D}$. We have already listed the elements of $\mathbf{D}$ as part of the physical model. If we assume a physical model with I dimensional variables, invariance for $J=3$ may be written as

$$F(X_1, X_2, \ldots, X_I) = F(\alpha_1^{D_{11}} \alpha_2^{D_{21}} \alpha_3^{D_{31}} X_1, \\ \alpha_1^{D_{12}} \alpha_2^{D_{22}} \alpha_3^{D_{32}} X_2 \cdots \alpha_1^{D_{1I}} \alpha_2^{D_{2I}} \alpha_3^{D_{3I}} X_I) \tag{19}$$

for all α (all possible changes of units). Thus for α_j, for example, we can write that

$$\frac{\partial F}{\partial \alpha_j}=\frac{\partial F}{\partial X_1}\frac{\partial X_1}{\partial \alpha_j}+\frac{\partial F}{\partial X_2}\frac{\partial X_2}{\partial \alpha_j}+\cdots\frac{\partial F}{\partial X_I}\frac{\partial X_I}{\partial \alpha_j}=0. \tag{20}$$

If we multiply Eq. (20) by α_j/F [assuming F to be nonzero as in Eq. (10)], and use

$$D_{ji}=\frac{\alpha_j}{X_i}\frac{\partial X_i}{\partial \alpha_j}, \tag{21}$$

which follows from Eq. (18), we obtain $J=3$ equations, one for each α:

$$D_{11}\frac{X_1}{F}\frac{\partial F}{\partial X_1}+D_{12}\frac{X_2}{F}\frac{\partial F}{\partial X_2}+\cdots D_{1I}\frac{X_I}{F}\frac{\partial F}{\partial X_I}=0, \tag{22a}$$

$$D_{21}\frac{X_1}{F}\frac{\partial F}{\partial X_1}+D_{22}\frac{X_2}{F}\frac{\partial F}{\partial X_2}+\cdots D_{2I}\frac{X_I}{F}\frac{\partial F}{\partial X_I}=0, \tag{22b}$$

$$D_{31}\frac{X_1}{F}\frac{\partial F}{\partial X_1}+D_{32}\frac{X_2}{F}\frac{\partial F}{\partial X_2}+\cdots D_{3I}\frac{X_I}{F}\frac{\partial F}{\partial X_I}=0. \tag{22c}$$

This set of equations is best written and solved in matrix form

$$D_{ji}S_i=0, \tag{23}$$

where $\mathbf{D}$ is a known $J\times I$ matrix, and $\mathbf{S}$ is an unknown $I\times 1$ vector of the (logarithmic) derivatives of F with respect to the dimensional variables that we seek to find:

$$S_i=\frac{\partial \log F}{\partial \log X_i}. \tag{24}$$

We will discuss a solution method in the following, but we anticipate here that there will usually be several solution vectors denoted by $\mathbf{S_k}$, with $k=1\cdots K$ (a bold subscript denotes a particular vector, not an element of the vector as in Eq. (26)). For example, let's say that there are $I=4$ dimensional variables and $K=2$ solution vectors (written in row form) that happened to be $\mathbf{S_1}=[\beta_1 0 \beta_2 0]$ and $\mathbf{S_2}=[0\beta_3 00]$, where the β are usually small rational numbers. The first solution vector indicates that

$$\begin{aligned}\frac{X_1}{F}\frac{\partial F}{\partial X_1}&=\beta_1, \quad \frac{X_2}{F}\frac{\partial F}{\partial X_2}=0,\\ \frac{X_3}{F}\frac{\partial F}{\partial X_3}&=\beta_2, \quad \frac{X_4}{F}\frac{\partial F}{\partial X_4}=0.\end{aligned} \tag{25}$$

A solution for F is thus

$$F=X_1^{\beta_1}X_3^{\beta_2}, \tag{26}$$

where it is useful to term the right-hand side a "Pi-variable," that is,

$$\Pi_1=X_1^{\beta_1}X_3^{\beta_2}. \tag{27}$$

The subscript on $\Pi_{(\,)}$ refers to the subscript on the solution vector $\mathbf{S}_{(\,)}$. Any multiple of Π_1 is a solution to Eq. (25), as is any power of Π_1, as is any sum of any power; evidently any function having the argument Π_1 is a solution to Eq. (25). Another solution can be found from the second solution vector $\mathbf{S_2}$ and is some function of $\Pi_2=X_2^{\beta_3}$. In effect, we have integrated a partial differential equation but without supplying boundary or initial data; thus we learn something about the argument of an otherwise arbitrary function.

We find that the dimensional variables can appear in F only in certain combinations that correspond one-to-one with the solution vectors $\mathbf{S_k}$,

$$\Pi_k=X_1^{S_{1k}}X_2^{S_{2k}}\cdots X_I^{S_{1k}}=\mathbf{X}^{\mathbf{S}_k}, \tag{28}$$

where $\mathbf{X}=[X_1X_2\cdots X_I]$ is a vector of the dimensional variables in the order they were entered into the dimensional matrix, $\mathbf{D}$. As anticipated in Sec. II, these Pi-variables are nondimensional. The relationship among the Pi-variables can be written as

$$\Pi_1=F(\Pi_2,\Pi_3,\ldots,\Pi_K) \tag{29}$$

with no loss of generality. In the uncommon case that $K=1$ and there is only one nondimensional variable, the function F must be a constant whose value cannot be determined from dimensional analysis alone. The period of the linear pendulum is an example, and in that case $F=2\pi$ would be determined by experiment or theory [Fig. 3(a)]. Neither can dimensional analysis determine anything further about the form of F in the much more common case that $K>1$.

B. The null space

Equation (23) is underdetermined in the usual case that there are more unknown exponents than there are equations. There will thus be many possible solution vectors that collectively make up the null space of the matrix $\mathbf{D}$. To represent the null space we seek a basis set from which any solution vector can be constructed. The computation of a null space basis is readily automated[4] and so we will not delve into the solution method (see Ref. 12). It is essential, however, to know the following two properties of null space bases.

(P1) *The number of solution vectors, K, is the same for all basis sets and is given by the number of dimensional variables in the physical model minus the rank of the dimensional matrix*, $K=I-R$. K is also the number of nondimensional variables and in that respect all basis sets are equally efficient. One particular basis set may be more useful than others, and so it is often necessary to transform from one basis to another. A transformation is easily accomplished because of the following.

(P2) *The basis set vectors are orthogonal and span the null space*. Any vector that is a solution of the homogeneous system (23) can be found as a linear combination of the vectors in any basis set. For example, if $\mathbf{S_1}$ and $\mathbf{S_2}$ are a null space basis, then their linear combination, say $\mathbf{S_3}=a_1\mathbf{S_1}+a_2\mathbf{S_2}$, with a_1 and a_2 any real number, is in the null space and is thus a solution. The corresponding nondimensional variable is $\Pi_3=\Pi_1^{a_1}\Pi_2^{a_2}$. If $\mathbf{S_3}$, $\mathbf{S_1}$ and Π_3, Π_1 are preferred over, say, $\mathbf{S_2}$, Π_2, then a revised basis set can be taken as $\mathbf{S_1}$, Π_1, and $\mathbf{S}_3$, Π_3 while omitting $\mathbf{S_2}$, Π_2. The revised basis set has the same number of vectors as the initial basis set and it too spans the null space. An initial basis set of nondimensional variables can thus be transformed to another basis set simply by multiplying or dividing the Π's is any order (an example is in Sec. V).

C. A basis set for the simple, inviscid pendulum

An application to the fast time-scale oscillation of the simple, inviscid pendulum may help clarify the use of the null space basis. The dimensional matrix **D** can be read directly from the physical model:

$$\mathbf{D}=\begin{array}{c} \\ m \\ l \\ t \end{array}\begin{array}{c} \begin{array}{cccccc} \phi & t & M & L & g & \phi_0 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 \end{bmatrix} \end{array} \tag{30}$$

where the first row is the dimensionality for mass, the second row is the dimensionality of length, and the third row is for time. The dependent variable ϕ is represented by the first column, 0 0 0, all zeros because angles are nondimensional; the time t by the second column, 0 0 1; the mass M by the third column, 1 0 0, etc. The order of listing the dimensional variables is important only insofar as the algorithm seeks to make the first few dimensional variables appear in the nondimensional variables with exponents of 1. The calculation of a null space basis yields three vectors that are concatenated into a matrix whose columns are the solution vectors, $\mathbf{S}=[\mathbf{S}_1;\mathbf{S}_2;\mathbf{S}_3]$,

$$\mathbf{S}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{31}$$

and the corresponding basis set of nondimensional variables has three elements:

$$\Pi_1=\mathbf{X}^{\mathbf{S}_1}=\phi^1\ \ t^0\ \ M^0\ \ L^0\ \ g^0\ \ \phi_0^0=\phi, \tag{32a}$$

$$\Pi_2=\mathbf{X}^{\mathbf{S}_2}=\phi^0\ \ t^1\ \ M^0\ \ L^{-1/2}\ \ g^{1/2}\ \ \phi_0^0=t/\sqrt{L/g}, \tag{32b}$$

$$\Pi_3=\mathbf{X}^{\mathbf{S}_3}=\phi^0\ \ t^0\ \ M^0\ \ L^0\ \ g^0\ \ \phi_0^1=\phi_0. \tag{32c}$$

The functional relationship among these may be written as $\Pi_1=F(\Pi_2,\Pi_3)$, or

$$\phi=F(t/\sqrt{L/g},\phi_0), \tag{33a}$$

and in analogy with Eqs. (6) and (7)

$$P/\sqrt{L/g}=F(\phi_0), \tag{33b}$$

which is beginning to look familiar. Notice that the mass M has an exponent of zero in all of the solution vectors, consistent with the informal analysis of Sec. II that showed that there was no way to construct a nondimensional variable from a single parameter having dimensions of mass. Also note that the angles ϕ and ϕ_0 sailed into the null space untouched because they were already nondimensional.

Tension can be analyzed in the same manner; the dimensional matrix is

$$\mathbf{D}=\begin{array}{c} \\ m \\ l \\ t \end{array}\begin{array}{c} \begin{array}{cccccc} T & t & M & L & g & \phi_0 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ -2 & 1 & 0 & 0 & -2 & 0 \end{bmatrix} \end{array}, \tag{34}$$

and the null space basis vectors (again in matrix form) are

$$\mathbf{S}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1/2 & 0 \\ -1 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{35}$$

A basis set of nondimensional variables is thus

$$\Pi_1=T/Mg, \tag{36a}$$

$$\Pi_2=t/\sqrt{L/g}, \tag{36b}$$

$$\Pi_3=\phi_0. \tag{36c}$$

The functional relationship for the tension and the maximum tension can be written as

$$T/Mg=F(t/\sqrt{L/g},\phi_0), \tag{37a}$$

and

$$T_{\max}/Mg=F(\phi_0). \tag{37b}$$

Notice that the mass, M, has been retained in the nondimensional tension. That the mass must appear is evident when one considers that the tension in the line will equal the weight of the bob, $T=Mg$, in the absence of motion (the tension of a moving pendulum will exceed this value due to centrifugal acceleration). Note too that the length, L, has been eliminated from the maximum tension. A little thought will reveal that a length cannot be made nondimensional with T, g, and M in any combination, and thus dimensional analysis reveals that the maximum tension of a simple, inviscid pendulum started from rest must be independent of L. This conclusion was suggested by inspection of a few numerical solutions [see Fig. 1(b)] and dimensional analysis assures us that it holds rigorously.[13]

V. THE VISCOUS PENDULUM

Now consider the decay rate defined by

$$\Gamma=\frac{1}{\Phi}\frac{d\Phi}{dt}, \tag{38}$$

where Φ is the amplitude of the motion. We begin with observations of the amplitude Φ made by measuring the cord length at intervals of 30 s to 2 min [the crosses of Fig. 4(a)]. For this purpose it was advantageous to use a longer pendulum, $L=3.70$ m, to minimize the consequence of the random error of the visually measured cord, about 10^{-3} m. This pendulum was supported on a needle bearing (a fishhook on a hard metal surface) to minimize interactions with the pivot, and the line was a smooth monofilament having a diameter $D_l=0.40\times10^{-3}$ m. The bob was a nearly spherical, more or less smooth lead fishing sinker with a diameter $D_b=0.0211$ m and mass $M=0.055$ kg. The observed amplitude history, $\Phi(t)$, was quite repeatable and can be roughly characterized as an exponential decay with a time scale of about 10 min.

A. A physical model of the viscous pendulum

We presume that hydrodynamic drag with the surrounding air is the primary damping process,[14] and that the diameter of

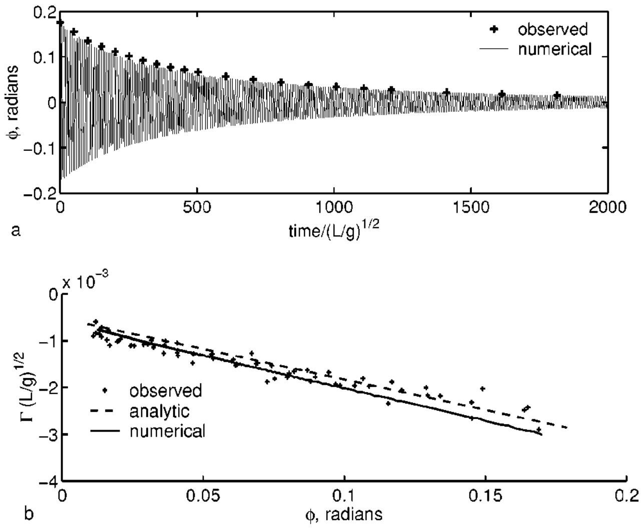

Fig. 4. (a) Observations (crosses) and a numerical solution (the thin solid line) of the motion of a simple, viscous pendulum. The crosses are observations of the amplitude at intervals of 30 s to 2 min. (b) The decay rate computed directly from the observations (crosses, from three repetitions of the experiment), as estimated from an approximate analytical solution Eq. (53) (dashed line) and as determined from the numerical model solution (solid line). Drag that is linear in the angular velocity produces a constant decay rate (simple exponential decay in time), and drag that is quadratic in the angular velocity produces a decay rate that increases linearly with the amplitude, ϕ.

the bob, D_b, and of the line, D_l, are now relevant, as are the density and kinematic viscosity of air, ρ, and ν. When we amend the inviscid model of Sec. II to include these variables, we have a physical model for the decay rate of a simple, viscous pendulum:

(1) the decay rate, $\Gamma \doteq m^0 l^0 t^{-1}$, the dependent variable;
(2) mass of the bob, $M \doteq m^1 l^0 t^0$, a parameter;
(3) length of the line, $L \doteq m^0 l^1 t^0$, a parameter;
(4) acceleration of gravity, $g \doteq m^0 l^1 t^{-2}$, a parameter;
(5) the amplitude of the motion, $\Phi \doteq$nond, a parameter;
(6) diameter of the line, $D_l \doteq m^0 l^1 t^0$, a parameter;
(7) diameter of the sphere, $D_b \doteq m^0 l^1 t^0$, a parameter;
(8) density of air, $\rho \doteq m^1 l^{-3} t^0$, a parameter (1.2 $\mathrm{kg\,m^{-3}}$, nominal);
(9) kinematic viscosity of air, $\nu \doteq m^0 l^2 t^{-1}$, a parameter ($1.5\times 10^{-5}$ $\mathrm{m^2\,s^{-1}}$, nominal).

(For the purpose of defining the amplitude of the motion we might have used ϕ_0 in place of Φ.) Dimensional analysis, from here on omitting all of the intermediate steps, indicates six nondimensional variables,

$$\Gamma \sqrt{L/g} = F(\Phi, D_b/L, D_l/L, \rho D_b^3/M, g^{1/2} L^{3/2}/\nu). \tag{39}$$

The first five nondimensional variables have an obvious interpretation, but the last one involving the viscosity, ν, does not. In any event, we are not ready to make use of such a comprehensive model. We may still be thinking of the nearly conservative pendulum of Sec. II, but the nine-variable physical model includes all possible pendula and fluid mediums. Before we can expect a useful result from dimensional analysis, we will have to identify the most relevant parameters for the kind of nearly conservative pendulum that we have in mind.

B. Drag on a moving sphere

A piecewise approach is tried next. Consider in isolation the hydrodynamic drag on a smooth sphere (the bob) due to a steady motion through an infinite viscous fluid (air) that is otherwise at rest. The physical model of drag is specified by

(1) drag (a force), $H \doteq m^1 l^1 t^{-2}$, the dependent variable;
(2) speed of the sphere, $U \doteq m^0 l^1 t^{-1}$, a parameter;
(3) diameter of the sphere, $D_b \doteq m^0 l^1 t^0$, a parameter;
(4) density of air, $\rho \doteq m^1 l^{-3} t^0$, a parameter;
(5) kinematic viscosity of air, $\nu \doteq m^0 l^2 t^{-1}$, a parameter.

Despite the highly idealized configuration of this problem, it is very difficult to compute the drag in the common case that the flow around the sphere is turbulent. However, dimensional analysis combined with laboratory measurement leads to a useful result. The initial basis set of nondimensional variables for this physical model comes out to be

$$\Pi_1 = \frac{H}{\rho D_b^2 U^2}, \quad \Pi_2 = \frac{\nu}{U D_b}, \tag{40}$$

where we recognize that Π_2 is the inverse of an important nondimensional variable called the Reynolds number:

$$\mathrm{Re} = \frac{U D_b}{\nu}. \tag{41}$$

We know from P2 of the null space (Sec. III) that Π_1 in Eq. (40) is not uniquely determined by dimensional analysis, and that a general basis set can be written as

$${}_n\Pi_1 = \frac{H}{\rho D_b^2 U^2}\left(\frac{U D_b}{\nu}\right)^n, \quad \Pi_2 = \frac{U D_b}{\nu}, \tag{42}$$

where n is any real constant. In writing this equation we are assuming that H and $\Pi_2 = \mathrm{Re}$ should remain to the first power. The functional relation between these nondimensional variables could be written as ${}_n\Pi_1 = F(\mathrm{Re})$, where F depends on n.

We will next consider how to choose the value of n that gives the best or most useful form.[15] Regardless of the form finally chosen, an essential result is that the nondimensional drag, ${}_n\Pi_1$, is expected to be a function of the Reynolds number alone (more on Re in the following). Laboratory measurements can thus be used to define $F(\mathrm{Re})$ which should hold for all steadily moving spheres, just the way that the function $F(\phi_0)$ [see Fig. 3(a)] sufficed to define the period for all inviscid, simple pendula.

1. Scaling analysis

The crucial (and in this case the only) choice is that of the dependent nondimensional variable, Π_1. One strategy is to form Π_1 so that it reflects a physically meaningful, even if highly idealized, solution for the dependent variable. This procedure is often termed a scaling analysis (see Ref. 3, Lin and Segel, Chap. 6).

A scaling analysis requires some sense of the physics of the problem. Visual observations of the flow around a sphere provide hints that drag can arise from two distinct processes. If the sphere is moving very slowly so that the wake behind the sphere is nearly undisturbed, then the drag will be mainly viscous, that is, due to the shear of the flow around the sphere and directly proportional to the viscosity of the fluid. The shear can be estimated by U/D_b, and the viscous stress by $\rho\nu U/D_b$. If this viscous stress acts over an area proportional to D_b^2, then the viscous drag on the sphere would be $H\propto\rho\nu D_b U$. This is the basis set $n=1$ of Eq. (42). If we expected that this was the dominant drag-producing process, then it would be appropriate to nondimensionalize the drag as

$$\frac{H}{\rho\nu D_b U}=F(\mathrm{Re})=C_v(\mathrm{Re}), \tag{43}$$

because the Re-dependence of C_v, the so-called viscous drag coefficient, would then be minimized.

Even if the fluid were nearly inviscid, there would still be drag because fluid must be accelerated as it is displaced by the moving sphere. If the displaced fluid is carried along in a highly disturbed wake, as is more or less observed behind a rapidly moving sphere (we will clarify what is meant by rapidly), then the drag would be roughly proportional to the density of the fluid times the speed squared (a momentum flux) multiplied by the frontal area, $A=\pi D_b^2/4$. Thus the inertial drag would be estimated as $H\propto\rho A U^2$. If we expected that this inertial drag process was dominant, then the initial basis set corresponding to $n=0$ would be appropriate:

$$\frac{H}{\rho A U^2}=F(\mathrm{Re})=C_i(\mathrm{Re}), \tag{44}$$

and the Re-dependence of the inertial drag coefficient C_i would show the departures from inertial drag due to viscous effects. Either form of the drag coefficient effectively conveys the laboratory data and in that regard there is nothing to choose between them.

2. The other nondimensional variables: The Reynolds number

Once the dependent nondimensional variable, Π_1, has been selected, the remaining nondimensional variables can be formed in ways that most clearly define the geometry of the problem, that reflect a balance of terms in a governing equation, or that follow conventions in the field. This prescription is necessarily vague because the possibilities are limitless, however, the task is often easier than might be expected. For the example of drag on a moving sphere, there is only one remaining nondimensional variable, the Reynolds number or its inverse. There are many other such ratios, often termed nondimensional "numbers," that succinctly characterize the balances among terms in mathematical models and thus are the natural terminology of theoretical mechanics.

Recall that for the purpose of modeling drag, a slowly moving sphere is one that has a nearly undisturbed wake. Observational evidence shows that this kind of flow occurs when $\mathrm{Re}\leqslant 1$, regardless of speed per se; dimensional analysis tells us as much in that the drag coefficient depends only on Re. The small Re range is that of a very small bug swimming slowly through water, for example. Note that in this very small Re range the viscous drag coefficient C_v is $O(1)$ and roughly constant [Fig. 5(a)]. For creatures and objects anywhere near our size, Reynolds numbers of $O(10^5)$ and greater are the norm, and inertial drag (often termed "form drag") is generally more important for runners and bicyclers than is viscous drag. Notice that for moderately large values of Re, $10^3\leqslant\mathrm{Re}\leqslant 10^5$, the inertial drag coefficient C_i is $O(1)$ and

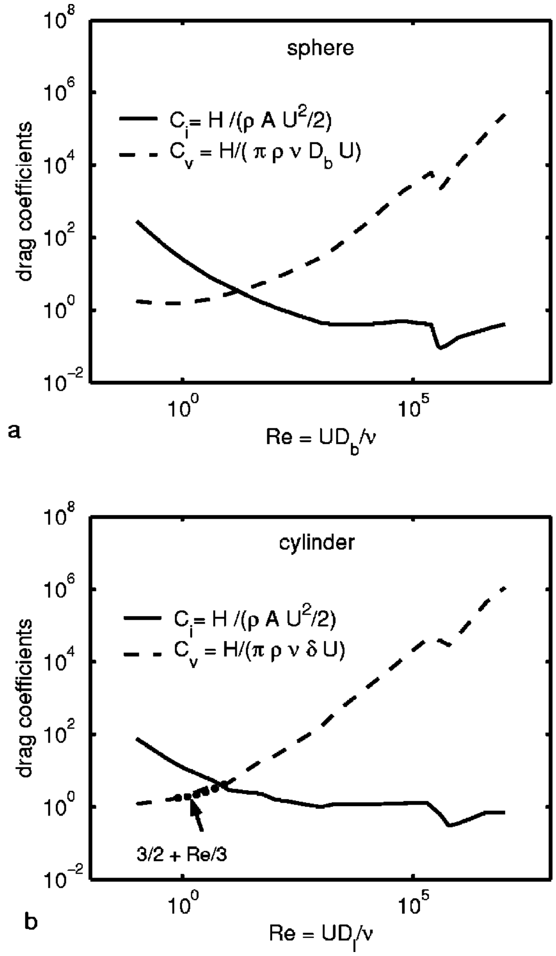

Fig. 5. Drag coefficients of a sphere (a) and a cylinder (b) moving at a steady speed U through viscous fluid. Two forms of drag coefficient are shown here, the viscous drag coefficient is denoted by C_v (the dashed line), and the inertial drag coefficient denoted by C_i (the solid line, usually denoted C_d, and by far the most commonly encountered form). Note that C_v is $O(1)$ if Re is very small, and that C_i is $O(1)$ if the Reynolds number is very large. The inertial drag coefficients were read from Munson *et al.* (Ref. 3, Fig. 7.7), and Rouse (Ref. 16, Figs. 125 and 126).

very roughly constant within subranges.[17] We can anticipate that the motion of our pendulum is in an intermediate Re range in which both viscous and inertial drag will be important.

C. A numerical simulation

To model the decay process we will include hydrodynamic drag on the line and the bob in the angular momentum balance. Drag will be estimated by means of the steady drag laws discussed above, and so it is implicitly assumed that the instantaneous speed of the bob or line gives the same drag as would a steady motion of the same speed. Whether this assumption is appropriate remains to be seen.

The main task is to account for the Re-dependence of the drag coefficients. Because the line is quite thin, the Reynolds numbers of the line are rather small, $\mathrm{Re}_l=UD_l/\nu\leqslant 20$, where $U=r\,d\phi/dt$, r is the distance from the pivot, and an *a priori* estimate of $d\phi/dt$ is $\phi_0/\sqrt{L/g}$. In that small Re range the viscous drag coefficient on a cylinder can be approximated well by $C_v=3/2+\mathrm{Re}/3$ (the heavy dotted line of Fig. 4(b)). The drag per unit length of the line, $\delta=dr$, can then be computed by the drag law corresponding to Eq. (43) as $H = \pi\rho\nu C_v U\,dr$, and the (dimensional) torque due to drag over the length of the line is then

$$\tau_l = \int_0^L rH\,dr = \rho\left(\frac{\pi}{2}\nu L^3 + \frac{1}{12}D_l L^4\left|\frac{d\phi}{dt}\right|\right)\frac{d\phi}{dt}. \tag{45}$$

The absolute value operator ensures that the drag force always opposes the motion. The bob has a much larger diameter and a Reynolds number $\mathrm{Re}_b = L(d\phi/dt)D_b/\nu$ in the range $\mathrm{Re}_b \leqslant 1000$ where no simple formula for a drag coefficient is highly accurate. Thus we will allow an arbitrary $C_i(\mathrm{Re}_b)$ and compute the drag-induced torque on the bob as

$$\tau_b = \frac{\pi\rho}{8}C_i(\mathrm{Re}_b)D_b^2L^3\left|\frac{d\phi}{dt}\right|\frac{d\phi}{dt}, \tag{46}$$

where Re_b and C_i are evaluated at each time step of the numerical integration using the data of Fig. 5(a). The amended angular momentum balance (in dimensional variables),

$$\frac{d^2\phi}{dt^2} = -\frac{g}{L}\sin(\phi) - \frac{\tau_l + \tau_b}{L^2M}, \tag{47}$$

together with Eqs. (53) and (54) and the data of Fig. 4(a) plus the initial condition (3) make a complete if rather cumbersome model that can be integrated numerically.

With drag terms included, the period of the oscillation is nearly unchanged, while the amplitude slowly decays [Fig. 4(a)]. The decay simulated by the numerical solution looks plausible when compared with the observations, suggesting that the steady drag laws have the gist of it (a more critical appraisal is given below).

D. An approximate model of the decay rate

Numerical solutions are not revealing of parameter dependence, but given two modest approximations we can deduce a model of the viscous pendulum that has transparent solutions. First, the angle ϕ is small enough in the case shown in Fig. 4(a) that $\sin\phi$ of Eq. (55) can be well approximated by ϕ. Second, the drag overall is due mostly, $\approx$85%, to the line, and so it should be acceptable to make the approximation that the inertial drag coefficient for the bob is a constant, $C_i = 0.7$, an average for the Re_b range of the bob in the present case. With these approximations we obtain a solvable model for the simple, viscous pendulum (now in nondimensional variables)

$$\frac{d^2\phi}{dt^{*2}} = -\phi - a\frac{d\phi}{dt^*} - b\left|\frac{d\phi}{dt^*}\right|\frac{d\phi}{dt^*}, \tag{48}$$

where the coefficient in the linear drag term is

$$a = \frac{\pi}{2}\frac{\rho\nu L^{3/2}}{Mg^{1/2}}, \tag{49}$$

and the coefficient in the quadratic term is

$$b = \frac{\rho}{8M}\left(0.7D_b^2L + \frac{2\pi}{3}D_lL^2\right). \tag{50}$$

Approximate solutions for small damping are given in Ref. 14; linear drag causes the amplitude to decay at a rate (nondimensional)

$$\frac{1}{\Phi}\frac{d\Phi}{dt^*} = -\frac{a}{2} \tag{51}$$

and the quadratic term causes decay at a rate

$$\frac{1}{\Phi}\frac{d\Phi}{dt^*} = -\frac{8b}{6\pi}\Phi, \tag{52}$$

where again Φ is the slowly varying amplitude. For small damping, these can be added together and evaluated to give an approximate decay rate,

$$\Gamma\sqrt{L/g} = \frac{1}{\Phi}\frac{d\Phi}{dt^*} \approx -5.2\times10^{-4} - 1.6\times10^{-2}\Phi, \tag{53}$$

shown as the dashed line of Fig. 4(b). This approximate model shows very clearly how the decay rate is expected to vary with the parameters that characterize the pendulum and the fluid medium (and gives an excellent account even for quite strong damping). All of the pieces of this model were present in our first attempt at dimensional analysis of the viscous pendulum, Eq. (39), though we had no way to recognize them at the time.

The decay rate can be estimated from the observations and from the numerical solution by a direct (no smoothing required) first differencing [Fig. 4(b)]. A comparison of decay rates makes a much more sensitive test of the drag formulation than does the amplitude itself [see Fig. 4(a)] and reveals that the decay is not a simple exponential as it first appears. There is a significant dependence of the decay rate upon the amplitude, which in the approximate model follows from the quadratic drag term, Eq. (58). Thus the hydrodynamic drag on this pendulum appears to be mainly inertial (Sec. IV), though viscous drag is important too, especially at smaller amplitudes.

Although the modeled decay rate is fairly accurate, there is at least a hint that the appropriate drag law for this pendulum overall (that is, the entire system, including the supporting structure) has a somewhat greater linear drag than is found in the models, and slightly less quadratic drag. This behavior is found consistently over a range of conditions, but further study of drag phenomena is outside the scope of this paper.[18]

VI. CONCLUDING REMARKS

The claim was made in Sec. I that dimensional analysis was occasionally quite powerful. With some experience we can see that dimensional analysis is most useful in cases where the mathematical model is either not known or cannot be solved usefully. Dimensional analysis can always make a little progress toward a solution merely by showing the form that variables must take in an equation that is invariant to a change of units. That, in a nutshell, is what dimensional analysis does. In the case that there are only two or three nondimensional variables in a problem, dimensional analysis can be an immensely powerful tool leading almost directly to a solution (the inviscid pendulum) or an efficient way to correlate a large data set (drag on a moving sphere). If the problem has many variables (the viscous pendulum), then dimensional analysis alone will probably not suffice, and further analysis or simplification will be required.

We have emphasized that an equation written in nondimensional variables, for example Eq. (17), is more efficient than its dimensional counterpart, Eq. (5). There is something to keep in mind, however. An equation written in nondimensional variables must be accompanied by a definition of the nondimensional variables. Better yet is an explanation of just why a particular definition was used, and what its advantages and limitations may be. The thoughtful use of dimensional analysis is a hallmark of insightful analysis, while the cavalier use of nondimensional variables can obscure what might otherwise have been a valuable message.

The mathematical steps that produce a nondimensional basis set are certain and quick (indeed, automatic), and the physical model is a finite list of variables. The ease with which a dimensional analysis can be done might generate confidence that the procedure is without risk of error. When dimensional analysis is applied to a mathematical (or numerical) model, this may well be true. But when dimensional analysis is meant to describe a real, physical system, that is not the case. Though the mathematical analysis is certain, it remains that the definition of an appropriate physical model is seldom as straightforward as the examples here might suggest. The absolute requirement that the physical model be complete is always at odds with the practical need to keep the physical model concise. The success of a dimensional analysis depends upon finding a satisfactory compromise; this requires judgment that comes with experience and from continual reference to relevant observations and numerical integrations.

ACKNOWLEDGMENTS

The author was supported primarily by the Seward Johnson Chair in Physical Oceanography, administered by the Massachusetts Institute of Technology—Woods Hole Oceanographic Institution Joint Program in Oceanography. Additional support during the period of manuscript preparation was provided by the U.S. Office of Naval Research. Thanks to James Girton, Eric Kunze and Andy Solo for their comments on a draft manuscript, and to Jim Hansen for comments and suggestions that have helped to clarify this manuscript.

[a]Present address: Office of Naval Research, Code 322OM, 800 N. Quincy St., Arlington, VA 22217-5660; electronic mail: jprice@whoi.edu

[1]E. Thurairajasingam, E. Shayan, and S. Masood, ''Modeling of a continuous food pressing process by dimensional analysis,'' Comput. Ind. Eng. (in press); J. R. Hutchinson and M. Garcia, ''Tyrannosaurus was not a fast runner,'' Nature (London) **415**, 1018–1022 (2002).

[2]F. Wilczek, ''Getting its from bits,'' Nature (London) **397**, 303–306 (1999).

[3]This essay builds upon the introduction to dimensional analysis that can be found in most comprehensive fluid mechanics textbooks. Recent examples include P. K. Kundu and I. C. Cohen, *Fluid Mechanics* (Academic, New York, 2001); B. R. Munson, D. F. Young, and T. H. Okiishi, *Fundamentals of Fluid Mechanics* (Wiley, New York, 1998), 3rd ed.; D. C. Wilcox, *Basic Fluid Mechanics* (DCW Industries, La Canada, CA, 2000); F. M. White, *Fluid Mechanics* (McGraw–Hill, New York, 1994), 3rd ed. An older but very useful reference is by H. Rouse, *Elementary Mechanics of Fluids* (Dover, New York, 1946). A particularly good discussion of the relationship between dimensional analysis and other analysis methods is by C. C. Lin and L. A. Segel, *Mathematics Applied to Deterministic Problems in the Natural Sciences* (MacMillan, New York, 1974).

[4]An algorithm for computing nondimensional variables has been implemented in MATLAB and can be downloaded from the author's web page, ⟨http://www.whoi.edu/science/PO/people/jprice/misc/Danalysis.m⟩ or from the MATLAB archive (the file name is Danalysis.m). Also available is a manuscript that treats aspects of dimensional analysis that are not touched on here, ⟨http://www.whoi.edu/science/PO/people/jprice/class/ND.pdf⟩.

[5]The simple pendulum is the starting point for most discussion of dimensional analysis including the classic text by P. W. Bridgman, *Dimensional Analysis* (Yale U.P., New Haven, CT, 1937), 2nd ed., which is an excellent introduction to the topic, and the more advanced treatment by L. I. Sedov, *Similarity and Dimensional Methods in Mechanics* (Academic, New York, 1959). Still more advanced is G. I. Barenblatt, *Scaling, Self-Similarity and Intermediate Asymptotics* (Cambridge U.P., Cambridge, 1996).

[6]An excellent discussion of physical measurement and much else that is relevant to dimensional analysis is given by A. A. Sonin, ''The physical basis of dimensional analysis,'' 2001. This unpublished manuscript is available from ⟨http://me.mit.edu/people/sonin/html⟩.

[7]What would be the result if the acceleration of gravity, g, was omitted, that is, what phenomenon would that entail? What if g were omitted, but an initial angular velocity $d\phi/dt$ was included? What if in place of g we used the acceleration due to the earth's rotation, $\Omega^2 R$? ($\Omega \doteq m^0 l^0 t^{-1}$ is the rotation rate of the earth and R is the distance normal to the rotation axis.)

[8]T. Szirtes, *Applied Dimensional Analysis and Modeling* (McGraw–Hill, Englewood Cliffs, NJ, 1997).

[9]S. Brückner and the University of Stuttgart Pi-Group ⟨http://www.pigroup.de/⟩, is an excellent resource for advanced applications of dimensional analysis.

[10]The calculation of a null space basis is, in effect, what all computational methods accomplish, and was noted by E. A. Bender, *An Introduction to Mathematical Modelling* (Dover, New York, 1977). We delegate the calculation to the computer, and emphasize those properties of the null space basis that are essential for the present purpose.

[11]For example, suppose that X_5 is a speed in British engineering units, feet/second, and we wish to compute X_5' in SI units, meters/second. This variable has dimensionality, $D_{15}=0$ (X_5 does not have units of mass), $D_{25}=1$ for length, and $D_{35}=-1$ for time. The appropriate scale change factors are $\alpha_1=0.435$ (pounds to kilograms for nominal g), $\alpha_2=0.3048$ (feet to meters), and $\alpha_3=1$ (seconds to seconds). Thus $X_5'=0.3048 X_5$.

[12]G. Strang, *Introduction to Linear Algebra* (Wellesley–Cambridge Press, Wellesley, MA, 1998).

[13]There is no doubt that dimensional analysis has just added something significant to what we knew from numerical integrations (that is, the maximum tension is independent of L). Does this result from dimensional analysis constitute a satisfactory *explanation*? This is clearly a matter of degree and opinion, but my opinion is that it does not. On the one hand, dimensional analysis has deduced a very clear statement of the observation from a general principle (invariance to the choice of units) and a set of specific conditions (the physical model). This is a form of explanation, but one that seems shallow and unsatisfying; there is no connection to a physical principle, and not the slightest hint of quantitative limits. In this instance and frequently, we will have to look beyond the immediate problem at hand or use something more than dimensional analysis when we seek explanations with enough depth to confer a useful understanding. Consider the following: The period of a simple (inviscid) pendulum undergoing small amplitude motion is independent of the amplitude, and yet increases with the square root of the length. Can you explain these facts? One approach might be to use dimensional analysis to analyze oscillators that have a restoring force that is proportional to some arbitrary power of the displacement. A salient fact for the maximum tension shown in Fig. 3(b) is that the maximum value is exactly 5 (nondimensional units) and occurs at $\phi_0=\pi$. Is dimensional analysis of any further use for explaining this? Consider energy conservation.

[14]Detailed treatment of damping processes are by P. T. Squire, ''Pendulum damping,'' Am. J. Phys. **54**, 984–991 (1986) and R. A. Nelson and M. G. Olsson, ''The pendulum: Rich physics from a simple system,'' *ibid.* **54**, 112–121 (1985).

[15]One criterion is to follow conventions of the field. In this case Π_1 is a drag coefficient, usually defined as $C_d = H/\frac{1}{2}\rho A U^2$, where A is the frontal area of the object. For the purpose of this essay we will consider other possible forms for Π_1.

[16]More recent textbooks (Ref. 3), like this article, show only the curve that runs through the middle of a tight cloud of data points that have accumulated from many laboratory experiments, see for example, Rouse (Ref. 3). What is most important, but not evident from this kind of presentation, is that drag coefficients inferred from experiments made using a very wide range of spheres and cylinders moving at widely differing speeds and through many different viscous fluids (Newtonian fluids) do indeed collapse to a well-defined function of Reynolds number alone, just as dimensional analysis had indicated. This is a result, characteristic of dimensional analysis generally, that is at once profound and trivial. One might say trivial because, after all, dimensional analysis told us that the drag coefficient must depend upon Re alone. From this perspective, an effective collapse of the data verifies that carefully controlled laboratory conditions can indeed approximate the idealized physical model. It is profound in that

dimensional analysis has shown the way to a useful result (Fig. 5), where there would otherwise have been be an unwieldy mass of highly specific data (as in going from Fig. 1 to Fig. 2). An open question of considerable practical importance is whether the steady drag laws are robust in the sense of giving useful estimates in practical problems, say our pendulum, in which the idealized conditions are inevitably violated. Other data sets have been developed to define the effects of idealized surface roughness, for example, but our pendulum has a long list of violations—time-dependence, a nearby solid boundary (the floor), slight surface roughness, etc., all present at once, so that we are on our own. About all that can be said is that it is important to understand the full set of assumptions under which a similarity law has been defined, and to be skeptical of applications outside of those bounds.

[17] Even at very large Re it does not follow that viscosity is entirely irrelevant. Significant changes in the drag coefficient occur at around $\mathrm{Re} \approx 2 \times 10^5$ due to changes in the viscous boundary layer and the width of the wake behind a moving sphere. This is the Re range of a well-hit golf ball or tennis ball, and is part of the reason that aerodynamic drag on these objects has a surprising sensitivity to surface roughness or spin. For much more detail on these phenomenon see S. Vogel, *Life in Moving Fluids* (Princeton U.P., New York, 1994), and P. Timmerman and J. P. van der Weele, ''On the rise and fall of a ball with linear and quadratic drag,'' Am. J. Phys. **67**, 538–546 (1999).

[18] Can you calculate a Reynolds number for the bob and the line from the original six nondimensional variables of Eq. (39)? Which nondimensional variable is present in Eq. (39) but not in Eqs. (57) and (58)? How or why was it omitted? Under what conditions (what parameter range) would you expect to see a significant effect of the time-dependent motion? How could you test (in principle and in practice) that the steady drag formulations really are appropriate for modeling the damping of a simple pendulum? You might, for example, consider that the fluid medium was water in place of air (the approximate density and kinematic viscosity of water are $\rho = 1.0 \times 10^3$ kg m^{-3} and $\nu = 1.8 \times 10^{-6}$ m^2/s at a temperature $= 0\,°\mathrm{C}$, and $\rho = 1.0 \times 10^3$ kg m^{-3} and $\nu = 0.7 \times 10^{-6}$ m^2/s at a temperature $= 40°\mathrm{C}$). Given these results, can you think of a name more apt than ''viscous'' pendulum?