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The Fréchet Derivatives of Electromagnetic Induction

ALAN D. CHAVE

Institute of Geophysics and Planetary Physics, Scripps Institution of Oceanography

The Fréchet derivatives of the fundamental toroidal and poloidal magnetic modes of electromagnetic induction are examined in detail. The response functions for both modes are shown to be Fréchet differentiable in an L_2 norm for general conductivity structures and arbitrary source frequency-wavenumber morphology. Perturbation forms of the modal Green functions are derived and used to examine the Fréchet kernels for a seafloor controlled source and a Kelvin wave model. In both cases, the TM mode possesses superior resolution ability, especially for low relative conductivity contrasts at depth. The results suggest that induction by the ocean tides can see details of the lithospheric structure at depths of at least 50 km.

INTRODUCTION

Inverse theory is the use of observational data to make inferences about the physical system producing them. A general mathematical framework for linear inverse problems is given by the work of *Backus and Gilbert* [1967, 1968, 1970]. Most inverse problems of geophysical interest are functionally nonlinear and are usually treated by linearization about some base model. To quantify this statement, suppose the j th component of an N vector $\boldsymbol{\gamma}$ and a possibly infinite dimensional model vector $\mathbf{m}$ are related by a nonlinear functional $\mathbf{F}$ which contains the physics governing the problem:

$$\gamma_j = F_j[\mathbf{m}]$$

If a small perturbation $\delta\mathbf{m}$ to the model produces a small change $\delta\boldsymbol{\gamma}$ in the data, and the relation

$$\delta\gamma_j = \int_M D_j(m,z)\delta m(z)dz + 0[\|\delta m\|^2] \qquad (1)$$

is valid, then $\mathbf{D}$ is called the Fréchet derivative or kernel of the functional $\mathbf{F}$ at $\mathbf{m}$ in the L_2 norm. The concept of a Fréchet derivative lies at the heart of linear inverse theory. It is quite important to actually prove that the remainder term in (1) is second order in the model perturbation; this is problem dependent and often neglected. The linearization implicit in (1) breaks down if the last term is not small, and this is observed in geophysical inverse problems. *Woodhouse* [1976] showed that first- rather than second-order error terms result for the seismic normal mode Fréchet kernels when the model contains discontinuities, invalidating some earlier applications.

Backus-Gilbert linear inverse methods have been widely utilized in all fields of the earth sciences. Their first application in electromagnetic induction was *Parker*'s [1970] inversion of a set of global geomagnetic sounding data. They have subsequently found use for the one-dimensional magnetotelluric problem [*Oldenburg*, 1979, 1981] and on DC resistivity sounding [*Oldenburg*, 1978]. *Parker* [1977] proved that the magnetotelluric problem is Fréchet differentiable in the usual least squares or L_2 norm, but more general electromagnetic induction phenomena, and especially the toroidal magnetic mode of which resistivity methods are a special case, have not been examined in detail. Fréchet kernels are frequently used as sensitivity functions to indicate penetration depth and gain a qualitative feel for resolution ability, especially in the exploration literature [e.g., *Gómez-Treviño and Edwards* 1983]. While considerable progress in solving the fully nonlinear form of the simplest, zero-wave number magnetotelluric inverse problem has been made [*Parker*, 1980; *Parker and Whaler*, 1981], extension of these methods to sources with more complex frequency-wave number structure is not straightforward. This suggests that a more complete examination of the Fréchet differentiability of generalized electromagnetic induction is needed as a prelude to the actual application of linear methods.

In this paper, a similar approach to that of *Parker* [1977] is applied to the electromagnetic response functions for the toroidal magnetic (TM) and poloidal magnetic (PM) modes, as given by *Chave* [1983*a*] (hereinafter called paper 1). Both the magnetotelluric and DC resistivity problems are special cases of PM and TM modes; thus the result is quite general. The two response functions are shown to be Fréchet differentiable in an L_2 norm for a very general class of conductivity models. By the use of a straightforward perturbation procedure on Green function expressions from paper 1, changes in the observed electromagnetic fields at the surface of the earth (or the seafloor) may be related to changes in the subsurface conductivity structure for an arbitrary source wave number morphology. This is illustrated by examining the Fréchet kernels for induction by a seafloor-based horizontal electric dipole controlled source and a Kelvin wave model of the ocean tides. The results suggest greater resolution capability for the TM mode, especially if low rather than high conductivity zones (in a relative sense) exist at depth.

GOVERNING EQUATIONS

The detailed framework for the theory of electromagnetic induction to be used in this paper was constructed in paper 1. The governing equations are those of Maxwell in the quasistatic limit

$$\begin{aligned} \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} + \partial_t \vec{B} &= 0 \\ \nabla \times \vec{B} - \mu\sigma\vec{E} &= \mu\vec{J}^0 \end{aligned} \qquad (2)$$

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where $\vec{E}$ and $\vec{B}$ are the induced electric field and magnetic induction, σ is the electrical conductivity, μ is the magnetic permeability of free space, and $\vec{J}^0$ is the impressed source electric current density. Choose a Cartesian coordinate system with $\hat{z}$ positive upward and zero at the earth's surface. By virtue of the first of (2), the magnetic induction may be written as

$$\vec{B} = \nabla \times (\Pi\hat{z}) + \nabla \times \nabla \times (\Psi\hat{z}) \quad (3)$$

where Π and Ψ are scalar functions which represent the TM and PM modes. The source current in (2) may be expressed as the sum of a vertical component, a tangentially irrotational part, and a solenoidal part

$$\vec{J}^0 = J_z^0\hat{z} + \nabla_h T + \nabla \times (\Upsilon\hat{z}) \quad (4)$$

If σ is taken to vary only with the vertical coordinate, differential equations for Π and Ψ may be derived from first principles

$$\nabla_h^2\Pi + \sigma\partial_z(\partial_z\Pi/\sigma) - \mu\sigma\partial_t\Pi = -\mu J_z^0 + \mu\sigma\partial_z(T/\sigma) \quad (5)$$

$$\nabla^2\Psi - \mu\sigma\partial_t\Psi = -\mu\Upsilon \quad (6)$$

where the electric field is

$$\vec{E} = -(\nabla_h^2\Pi/\mu + J_z^0)/\sigma\hat{z} + \nabla_h[(\partial_z\Pi/\mu - T)/\sigma] - \nabla\times(\partial_t\Psi\hat{z}) \quad (7)$$

Note that the TM mode magnetic and PM mode electric fields have no vertical components. The differential equations (5)–(6) are easily solved if sinusoidal time dependence is assumed and the horizontal components are expressed as spatial Fourier transforms; the conventions to be used are $e^{-i\omega t}$ time dependence and equation (20) in paper 1.

The scalar functions T and Υ are solutions of the Poisson equations

$$\nabla_h^2 T = \nabla_h \cdot \vec{J}^0 \quad (8)$$

$$\nabla_h^2\Upsilon = -\nabla \times \vec{J}^0 \cdot \hat{z} \quad (9)$$

Details on their form for electromagnetic induction by moving, conducting seawater are given in paper 1.

The usual conditions on the electric and magnetic fields at horizontal boundaries follow from (3) and (7). In paper 1 the boundary conditions were summarized in the TM and PM mode response functions defined by

$$K = [\sigma\hat{\Pi}/\partial_z\hat{\Pi}]_{z=b} \quad (10)$$

$$\Lambda = [\mu\hat{\Psi}/\partial_z\hat{\Psi}]_{z=b} \quad (11)$$

where the ^ denotes the wave number domain and $z=b$ corresponds either to the earth's surface ($z=0$) or the seafloor ($z=-H$). The response functions are easily computed for layered media by using continued fraction expansions or by solving nonlinear Riccati equations numerically for continuous conductivity profiles. The PM mode function (11) is related to the more familiar E over B magnetotelluric impedance by $Z = i\omega\Lambda/\mu$. The Fréchet kernels for (10) and (11) are of central interest in this paper and are derived in the next section.

The modal equation solutions may be expressed in general terms by using Green functions. In paper 1, Green functions for the Fourier transforms of (5) and (6) and a source confined to the ocean in $-H \leqslant z \leqslant 0$ were given and may be written as

$$\hat{\Pi} = -\mu\int_{-H}^{0} dz'[g_\Pi(z,z')\hat{J}_z^0(z') + \partial_{z'}g_\Pi(z,z')\hat{T}(z')] \quad (12)$$

$$\hat{\Psi} = -\mu\int_{-H}^{0} dz' g_\Psi(z,z')\hat{\Upsilon}(z') \quad (13)$$

where g_Π and g_Ψ are complex expressions involving reflection coefficients containing (10) and (11) and have all of the information on self and mutual induction necessary to solve (5) and (6). These Green functions are easily converted to forms suitable for induction by sources external to the earth by replacing z and z' with $z-H$ and $z'-H$, where H is redefined as the scale distance of the ionospheric electric currents. The TM mode magnetic field vanishes at insulating boundaries, and no TM mode may be induced by a source that is not in electrical contact with the earth, as was first demonstrated by *Price* [1950]. Since the actual electromagnetic fields are related to (12) and (13) by linear differential operators, expressions for perturbations to (12) and (13) caused by subsurface conductivity variations are equivalent to perturbation forms for the fields themselves. These are discussed in a later section.

THE FRÉCHET DERIVATIVES

The derivation of expressions for the Fréchet kernels of (10) and (11) follows the procedures used in *Parker* [1977] quite closely. The PM mode kernel is a straightforward extension of Parker's result when wave number effects are included and will only be outlined. The TM mode result is more complex and has not previously been examined in detail, although the form of the Fréchet derivative is similar to that obtained by *Oldenburg* [1978] for DC resistivity sounding.

TM Mode

The horizontal Fourier transform of (5) for the source-free conducting region below $z=b$ can be written in the self-adjoint form

$$\partial_z(\partial_z\hat{\Pi}/\sigma) - \beta^2/\sigma\hat{\Pi} = 0 \quad (14)$$

where

$$\beta^2(z) = k^2 - i\omega\mu\sigma(z) \quad (15)$$

and k is the composite horizontal wave number. The conducting zone is assumed to be confined to $a \leqslant z \leqslant b$, where a may be taken as arbitrarily large albeit finite. Note that (14) is valid for discontinuous but nonvanishing σ on the open interval (a,b); the more general case where σ and β^2/σ are only required to be Lebesque integrable is treated by *Atkinson* [1964, chap.8]. Define the new variable

$$\alpha(z) = \hat{\Pi}(z)/\partial_z\hat{\Pi}(b)$$

which obeys (14) with boundary conditions $\alpha(a) = 0$, $\alpha'(b) = 1$, and yields the response function (10) as $K = \sigma(b)\alpha(b)$. The lower ($z=a$) boundary condition corresponds to termination in a perfect insulator where the TM mode magnetic field vanishes; if a is taken to be sufficiently large (i.e., many skin depths for the frequency-wave number of interest), then the lower boundary is electrically isolated from the surface and this condition is not restrictive.

Consider two conductivity profiles σ_1 and σ_2, with $\delta\sigma = \sigma_2 - \sigma_1$, and let α_1 and α_2 be the corresponding solutions to the differential equation for the two conductivity structures, with $\delta\alpha$ as their difference. A differential equation for $\delta\alpha$ can be obtained from (14) by subtraction

$$\partial_z(\partial_z\delta\alpha/\sigma_2) - \beta_2^2/\sigma_2\delta\alpha = \partial_z\log(1+\delta\sigma/\sigma_1)\partial_z\alpha_1/\sigma_2 - i\omega\mu\alpha_1\delta\sigma/\sigma_2 \qquad (16)$$

where β_2^2 is given by (15) with $\sigma = \sigma_2$. Equation (16) obeys the homogeneous boundary conditions $\delta\alpha(a) = \delta\alpha'(b) = 0$, and is easily solved by constructing a Green function Γ_π by the method of variation of parameters

$$\delta\alpha(z) = \int_a^b dy\Gamma_\pi(z,y)[\partial_y\log(1+\delta\sigma/\sigma_1)\partial_y\alpha_1/\sigma_2 - i\omega\mu\alpha_1\delta\sigma/\sigma_2] \qquad (17)$$

where

$$\Gamma_\pi(z,y) = \phi_1(z_>)\phi_2(z_<)$$

and $z_>$ ($z_<$) denotes the larger (smaller) of z and y. The two functions ϕ_1 and ϕ_2 are linearly independent solutions of (14) for $\sigma = \sigma_2$ with the Wronskian condition

$$(\phi_1'\phi_2 - \phi_2'\phi_1)/\sigma_2 = 1$$

and the boundary conditions $\phi_1'(b) = \phi_2(a) = 0$. The second term may be taken equal to α_2 since they obey the same equation and boundary values, and substitution into the Wronskian yields

$$\Gamma_\pi(b,y) = -\sigma_2(b)\alpha_2(y)$$

The first term in (17) may be integrated by parts with (14) applied to the integrand to yield

$$\int_a^b dy[\Gamma_\pi\partial_y\alpha_1/\sigma_2]\partial_y\log(1+\delta\sigma/\sigma_1) = \Gamma_\pi\partial_y\alpha_1\log(1+\delta\sigma/\sigma_1)/\sigma_2\Big|_{y=a}^{b} - \int_a^b dy(\partial_y\Gamma_\pi\partial_y\alpha_1 + \beta_1^2\Gamma_\pi\alpha_1)\log(1+\frac{\delta\sigma}{\sigma_1})/\sigma_2 + \tfrac{1}{2}\int_a^b dy[\Gamma_\pi\partial_y\alpha_1/\sigma_2]\partial_y[\log(1+\frac{\delta\sigma}{\sigma_1})]^2$$

The last term may again be integrated by parts, and repeating this yields a convergent power series in $\log(1+\delta\sigma/\sigma_1)$, reducing (17) to

$$\delta\alpha(z) = \partial_y\alpha_1(y)\Gamma_\pi(z,y)\delta\sigma(y)/\sigma_1(y)\sigma_2(y)\Big|_{y=a}^{b} - \int_a^b dy[k^2\alpha_1(y)\Gamma_\pi(z,y) + \partial_y\alpha_1(y)\partial_y\Gamma_\pi(z,y)]\delta\sigma(y)/\sigma_1(y)\sigma_2(y) \qquad (18)$$

This expression is exact. The leading term in (18) may be eliminated by requiring that $\delta\sigma = 0$ at $z = b$; this is equivalent to knowledge of the surface conductivity. To place (18) in a more familiar form, substitute the surface value for Γ_π to yield $\sigma_1(b)\delta\alpha(b) = \delta K + R$, where the first term is

$$\delta K = \sigma_1^2(b)\int_a^b dy[k^2\alpha_1^2(y) + (\partial_y\alpha_1(y))^2]\delta\sigma(y)/\sigma_1^2(y) \qquad (19)$$

and the remainder term is

$$R = \sigma_1^2(b)\int_a^b dy[k^2\alpha_1(\sigma_1\delta\alpha - \alpha_1\delta\sigma) + \partial_y\alpha_1(\sigma_1\partial_y\delta\alpha - \partial_y\alpha_1\delta\sigma)]/\sigma_1^2\sigma_2 \qquad (20)$$

Equation (19) is identical to that obtained by *Oldenburg* [1978] directly from the Riccati equation in K for the DC resistivity problem.

To prove Fréchet differentiability, it is necessary to show that (20) is second order in the conductivity perturbation as $\delta\sigma$ becomes in some sense small. Rewrite (20) as the sum of four terms to simplify the algebra

$$R = R_1 + R_2 + R_3 + R_4$$

where

$$R_1 = \sigma_1^2(b)k^2\int_a^b dy\alpha_1\delta\alpha\delta\sigma/\sigma_1\sigma_2 \qquad (21)$$

$$R_2 = -\sigma_1^2(b)k^2\int_a^b dy\alpha_1^2(\delta\sigma)^2/\sigma_1^2\sigma_2 \qquad (22)$$

$$R_3 = \sigma_1^2(b)\int_a^b dy\partial_y\alpha_1\partial_y\delta\alpha\delta\sigma/\sigma_1\sigma_2 \qquad (23)$$

$$R_4 = -\sigma_1^2(b)\int_a^b dy(\partial_y\alpha_1)^2(\delta\sigma)^2/\sigma_1^2\sigma_2 \qquad (24)$$

The size of functions will be defined by their L_2 norm

$$\|f\| = \left[\int_a^b dy|f|^2\right]^{1/2}$$

By Minkowski's inequality, $\|R\| \leq \|R_1\| + \|R_2\| + \|R_3\| + \|R_4\|$ and the four terms (21)–(24) may be considered individually. Applying Schwarz's inequality to (21) yields

$$\|R_1\| \leq \sigma_1^2(b)k^2\|\alpha_1\delta\sigma/\sigma_1^2\|\,\|\delta\alpha\|$$

and from (18) (less the first term, which has been neglected)

$$\|\delta\alpha\| \leq k^2\|\alpha_1\delta\sigma/\sigma_1^2\|\left[\int_a^b dz\int_a^b dy|\Gamma_\pi(z,y)|^2\right]^{1/2} + \|\partial_y\alpha_1\delta\sigma/\sigma_1\|\left[\int_a^b dz\int_a^b dy|\partial_y\Gamma_\pi(z,y)/\sigma_2(y)|^2\right]^{1/2}$$

Continuity of the tangential electric field requires continuity of $\partial_z\hat{\Pi}/\sigma$ even in the presence of discontinuities in σ; the same must hold for $\partial_z\alpha/\sigma$ or the terms $\partial_z\phi_i/\sigma_2$, $i = 1,2$ in the Green function Γ_π. Continuity of the normal electric current and the magnetic induction requires continuity of $\hat{\Pi}$ (or α or ϕ_i, $i=1,2$). Any continuous function on a closed interval is bounded, so that

$$\|\alpha_1\| \leqslant M_1$$
$$\|\Gamma_\pi\| \leqslant M_2$$
$$\|\partial_y \alpha_1/\sigma_1\| \leqslant M_3$$
$$\|\partial_y \Gamma_\pi/\sigma_2\| \leqslant M_4$$

where $M_1 - M_4$ are finite constants, and a finiteness condition will be imposed on the resistivity profile

$$\|1/\sigma\| \leqslant M_5$$

Then (21) is bounded by

$$\|R_1\| \leqslant \sigma_1^2(b) k^2 M_1 M_5^2 (k^2 M_1 M_2 M_5^2 + M_3 M_4)\|\delta\sigma\|^2 \qquad (25)$$

which is $O(\|\delta\sigma\|^2)$ in the limit of small $\delta\sigma$.

The third term (23) can be treated in a similar way to (21), yielding

$$\|R_3\| \leqslant \sigma_1^2(b)\|\partial_y \alpha_1 \delta\sigma/\sigma_1\| \, \|\partial_y \delta\alpha/\sigma_2\|$$

and taking the derivative of (18)

$$\|\partial_z \delta\alpha/\sigma_2\| \leqslant k^2 \|\alpha_1 \delta\sigma/\sigma_1^2\| \left[\int_a^b dz \int_a^b dy \, |\partial_z \Gamma_\pi(z,y)/\sigma_2(z)|^2\right]^{1/2}$$

$$+ \|\partial_y \alpha_1 \delta\sigma/\sigma_1\| \left[\int_a^b dz \int_a^b dy \, |\partial_z \partial_y \Gamma_\pi(z,y)/\sigma_2(z)\sigma_2(y)|^2\right]^{1/2}$$

This reduces to

$$\|R_3\| \leqslant \sigma_1^2(b) M_3 (k^2 M_1 M_4 M_5^2 + M_3 M_1^2)\|\delta\sigma\|^2 \qquad (26)$$

which also is second order in the conductivity perturbation. The second and last remainder terms reduce very simply to

$$\|R_2\| \leqslant \sigma_1^2(b) k^2 M_1^2 M_5^3 \|\delta\sigma\|^2 \qquad (27)$$

$$\|R_4\| \leqslant \sigma_1^2(b) M_3^2 M_5 \|\delta\sigma\|^2 \qquad (28)$$

so that R is $O(\|\delta\sigma\|^2)$ in the limit of small $\delta\sigma$, and Fréchet differentiability of the TM mode response function is proved for an L_2 norm.

PM Mode

The horizontal Fourier transform of (6) for the source-free earth is

$$\partial_z^2 \hat{\Psi} - \beta^2 \hat{\Psi} = 0 \qquad (29)$$

As for the TM mode, the conducting region is assumed to be finite, occupying $a \leqslant z \leqslant b$. Define a new variable

$$\Sigma(z) = \hat{\Psi}(z)/\partial_z \hat{\Psi}(b)$$

and take the boundary conditions to be $\Sigma(a) = 0$, $\Sigma'(b) = 1$. The PM mode response function (11) is given by $\Lambda = \mu\Sigma(b)$. The bottom boundary condition is equivalent to termination by a perfect conductor. Consider two conductivity profiles σ_1 and σ_2, with $\delta\sigma = \sigma_2 - \sigma_1$ and let Σ_1 and Σ_2 be the corresponding solutions of (29), with $\delta\Sigma$ as their difference. The perturbation response satisfies

$$\partial_z^2 \delta\Sigma - \beta_2^2 \delta\Sigma = -i\omega\mu \Sigma_1 \delta\sigma \qquad (30)$$

where β_2^2 is given by (15) with $\sigma = \sigma_2$, and (30) satisfies the homogeneous boundary conditions $\delta\Sigma(a) = \delta\Sigma'(b) = 0$.

Equation (30) is solved by constructing a Green function Γ_Ψ to yield

$$\delta\Sigma(z) = -i\omega\mu \int_a^b dy \, \Gamma_\Psi(z,y)\Sigma_1(y)\delta\sigma(y) \qquad (31)$$

where

$$\Gamma_\Psi(z,y) = \chi_1(z_>)\chi_2(z_<)$$

and χ_1 and χ_2 are linearly independent solutions of (29) with the Wronskian condition

$$\chi_1' \chi_2 - \chi_2' \chi_1 = 1$$

and boundary conditions $\chi_1'(b) = \chi_2(a) = 0$. The second term χ_2 may be set equal to Σ_2, and the surface value of the Wronskian gives

$$\Gamma_\Psi(b,y) = -\Sigma_2(y)$$

The surface solution can be written as the sum of

$$\delta\Lambda = i\omega\mu^2 \int_a^b dy \, \Sigma_1^2(y)\delta\sigma(y) \qquad (32)$$

and a remainder term

$$R = i\omega\mu^2 \int_a^b dy \, \Sigma_1(y)\delta\Sigma(y)\delta\sigma(y) \qquad (33)$$

The remainder (33) is identical to that obtained by *Parker* [1977], and his proof of Fréchet differentiability is valid for nonzero wave numbers. The PM mode Fréchet derivative is defined by (32). A bound on (33) may easily be computed using the eigenfunction approach in *Parker* [1977]

$$|R| \leqslant \frac{\omega |\Lambda_1|^2 \|\delta\sigma\|^2}{\mu \sigma_{\min}}$$

where $\sigma_{\min}$ is the minimum value along the profile $\sigma_2(z)$.

PERTURBATION FORM OF THE GREEN FUNCTIONS

The Green function expressions (12) and (13) contain information on mutual induction with the earth through the TM and PM mode reflection coefficients

$$R_L^{TM} = \frac{\beta_0 K/\sigma_0 - 1}{\beta_0 K/\sigma_0 + 1}$$

$$R_L^{PM} = \frac{\beta_0 \Lambda/\mu - 1}{\beta_0 \Lambda/\mu + 1}$$

where K and Λ are given by (10) and (11), σ_0 is the conductivity of the ocean, and β_0^2 is given by (15) with $\sigma = \sigma_0$. By considering the response functions corresponding to two conductivity profiles σ_1 and σ_2 and the associated forms for the reflection coefficients, the perturbations to them may be written

$$\delta R_L^{TM} = \frac{2\beta_0 \delta K/\sigma_0}{(\beta_0 K/\sigma_0 + 1)^2} \quad (34)$$

$$\delta R_L^{PM} = \frac{2\beta_0 \delta\Lambda/\mu}{(\beta_0\Lambda/\mu + 1)^2} \quad (35)$$

where δK and $\delta\Lambda$ are given by (19) and (32), and (34)—(35) are accurate to first order in δK and $\delta\Lambda$; since these functions are good to first order in the conductivity perturbation $\delta\sigma$, the remainder is $O(\|\delta\sigma\|^2)$.

Perturbation forms of (12) and (13) are also obtained by subtracting their forms for two conductivity profiles and using (34)—(35). After some algebra, the desired equations are

$$\delta\hat{\Pi}(z) = -\mu \int_{-H}^{0} dz' [\delta g_\Omega(z,z')\hat{J}_z^0(z') + \delta\partial_{z'} g_\Omega(z,z')\hat{T}(z')] \quad (36)$$

$$\delta\hat{\Psi}(z) = -\mu \int_{-H}^{0} dz' \delta g_\Psi(z,z')\hat{\Upsilon}(z') \quad (37)$$

where $\delta g_\Omega = \delta R_L^{TM} \tilde{g}_\Omega$ and $\delta g_\Psi = \delta R_L^{PM} \tilde{g}_\Psi$, with

$$\tilde{g}_\Omega = -\frac{e^{-2\beta_0 H}\left[e^{-\beta_0(z+z')} - \left[e^{-\beta_0|z-z'|} + e^{\beta_0|z-z'|} - e^{\beta_0(z+z')}\right]\right]}{2\beta_0\left[1 + R_L^{TM} e^{-2\beta_0 H}\right]^2} \quad (38)$$

$$\tilde{g}_\Psi = -\frac{e^{-2\beta_0 H}\left[e^{-\beta_0(z+z')} + R_A^{PM}\left[e^{-\beta_0|z-z'|} + e^{\beta_0|z-z'|} + R_A^{PM} e^{\beta_0(z+z')}\right]\right]}{2\beta_0\left[1 - R_A^{PM} R_L^{PM} e^{-2\beta_0 H}\right]^2} \quad (39)$$

The sea surface reflection coefficient for the PM mode is given by equations (22) and (32) in paper 1. The perturbed Green functions for barotropic ocean flow, where the velocity is independent of depth, are obtained by integrating (38), its z' derivative, and (39) across the water column, and are analogous to (36)—(37) in paper 1. These will be denoted by $\tilde{G}_\Omega$, $\tilde{G}_\Xi$, and $\tilde{G}_\Psi$, respectively.

By using (36) and (37) along with (38)—(39), (34)—(35), (19), and (32), the Fréchet derivatives of the electromagnetic fields for any oceanic source type, as specified by $\hat{J}_z^0$, $\hat{T}$, and $\hat{\Upsilon}$, can be obtained. Fréchet kernels for the electromagnetic induction fields produced by sources at or above the earth are easily derived by transforming the z and z' coordinates, and setting $\beta_0 = k$ and $\sigma_0 = 0$ in the atmosphere. H becomes an appropriate vertical scale length and may often be taken as infinite, in which case reflections from an upper boundary, such as the ionosphere, are neglected. Note that the TM mode expression (38), which is proportional to the horizontal magnetic induction, vanishes at the earth's surface, but its vertical derivative, which is proportional to the horizontal electric field, does not.

FRÉCHET KERNELS FOR OCEANIC SOURCES

Seafloor Controlled Source

Chave and Cox [1982] discussed the forward problem and possible geophysical applications for a seafloor-based horizontal electric dipole (HED) source. A detailed look at the inverse problem and a sensitivity analysis for the method is in progress and will be reported elsewhere, and only some of the more fundamental properties will be illustrated.

Consider a point source of moment p A—m, located at $(0,0,z')$, and oriented along the x axis, so that the source current density is

$$\vec{J}^0 = p\delta(x)\delta(y)\delta(z-z')\hat{x}$$

The Fréchet kernels are most easily derived by solving the monopole problem and computing the necessary derivatives at the end: let $\Upsilon = \partial_y\Upsilon'$, $\Psi = \partial_y\Psi'$, $T = \partial_x T'$, $\Pi = \partial_x\Pi'$, and solve (8)—(9) and (36)—(37) for the primed variables. The wave number domain expressions analogous to (36)—(37) are converted to the spatial domain by using the inverse Fourier transform; owing to the cylindrical symmetry of the primed variables, this may be cast into a Hankel transform by converting to cylindrical coordinates. Using these substitutions in (36) and changing back to the unprimed coordinates gives

$$\delta\Pi = \frac{\mu p}{2\pi}\partial_x \int_0^\infty dk\, J_0(k\rho)\delta R_L^{TM}\partial_{z'}\tilde{g}_\Omega(z,z')/k \quad (40)$$

and using (37) yields

$$\delta\Psi = \frac{\mu p}{2\pi}\partial_y \int_0^\infty dk\, J_0(k\rho)\delta R_L^{PM}\tilde{g}_\Psi(z,z')/k \quad (41)$$

where ρ is the horizontal range, δR_L^{TM} and δR_L^{PM} are given by (34)—(35), $\tilde{g}_\Omega$ and $\tilde{g}_\Psi$ are given by (38) and (39), and J_0 is a Bessel function of the first kind of order zero. These results may be cast into the more familiar form of a Fréchet derivative by substituting (19), (32), (34), and (35) into (40)—(41), transforming the z variables to one where $z = 0$ corresponds to the seafloor as in *Chave and Cox* [1982], and neglecting sea surface effects to get

$$\delta\Pi = \int_a^0 ds \left[\frac{\mu p}{2\pi\sigma_0}\partial_x \int_0^\infty dk\, J_0(k\rho) \cdot \frac{\beta_0(k^2\pi^2(s) + (\partial_s\pi(s))^2)}{k\sigma(s)^2(\beta_0\pi(0)/\sigma_0 + \partial_z\pi(0)/\sigma(0))^2}\right]\delta\sigma(s) \quad (42)$$

$$\delta\Psi = \int_a^0 ds \left[\frac{-i\omega\mu^2 p}{2\pi}\partial_y \int_0^\infty dk\, J_0(k\rho) \cdot \frac{\psi^2(s)}{k(\beta_0\psi(0) + \partial_z\psi(0))^2}\right]\delta\sigma(s) \quad (43)$$

where β_0 is given by (15) with $\sigma = \sigma_0$, the conductivity of seawater and π and ψ are solutions of (14) and (29) at

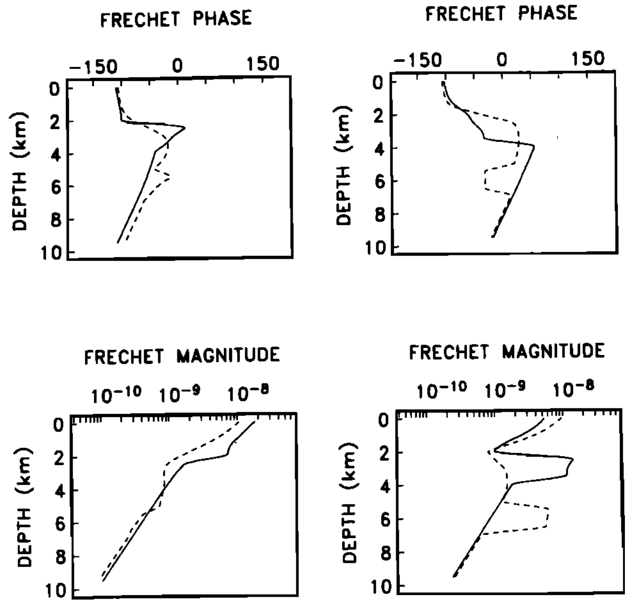

Fig. 1. Fréchet kernels as a function of depth for the horizontal electric field from a unit HED source at a range of 7 km and frequency of 1 Hz. The ocean has a conductivity of 3.2 S/m, while the earth has a conductivity of 0.005 S/m and contains 1-km-thick zones centered on 3 km (solid) and 6 km (dashed). The right panels show results for low conductivity (0.0005 S/m) inclusions, while the left panels show those for high conductivity (0.05 S/m) material.

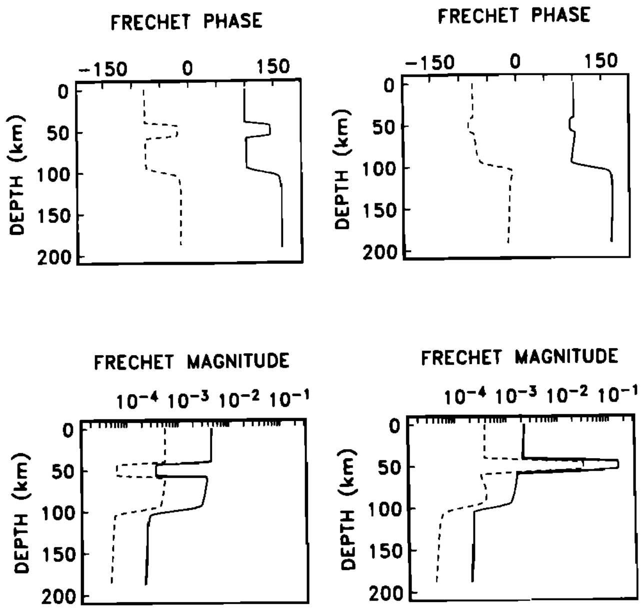

Fig. 3. Fréchet derivatives for the offshore magnetic field at ranges of 250 km (solid) and 3050 km (dashed) for electromagnetic induction by a Kelvin wave. The ocean is a 4-km-thick region of conductivity 3.2 S/m and overlies a 100-km-thick lithosphere of conductivity 0.005 S/m and a 0.05 S/m half space. The lithosphere model contains 10-km-thick zones centered on 50 km and containing either low conductivity (0.0005 S/m) (right panels) or high conductivity (0.05 S/m) (left panels) material.

$a \leqslant z \leqslant b$. The terms in brackets are the actual Fréchet kernels for the Π and Ψ, while those for the electromagnetic fields follow from (3) and (7).

Numerical solution of the Hankel transforms in (42)–(43) can be accomplished by direct integration and special summation methods [*Chave*, 1983*b*]. For each value of the wave number k selected by the quadrature rule, the initial value problems (14) and (29) are integrated from $z=a$ to the depth of interest $z=s$ and to the surface to yield π and ψ for any continuously varying conductivity profile using a predictor-corrector algorithm. The initial conditions are given by a half space solution at $z=a$ at small wave number or a WKB solution (appendix) at a few scale depths for large wave number where penetration is limited.

Figure 1 shows the horizontal electric field Fréchet ker-

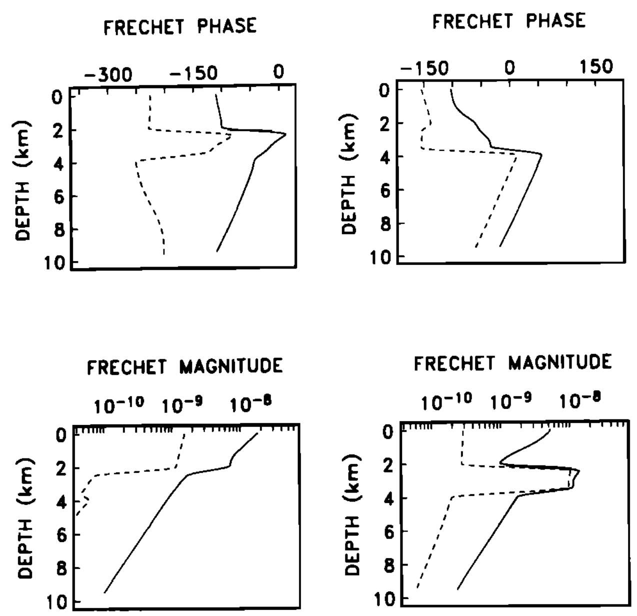

Fig. 2. Fréchet kernels for the 3 km model of Figure 1 shown at source-receiver ranges of 7 km (solid) and 21 km (dashed). See Figure 1 caption for details.

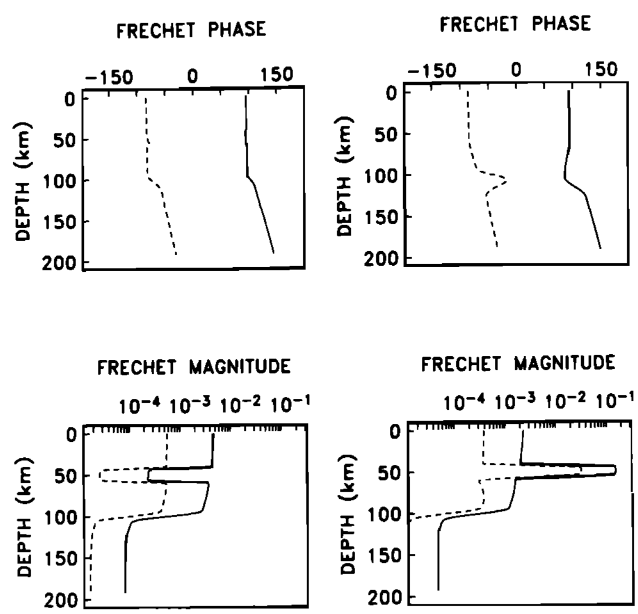

Fig. 4. The TM mode part of Figure 3. See Figure 3 caption for details.

nel along the x axis for a unit (1A–m) dipole at a range of 7 km and a frequency of 1 Hz. The ocean conductivity is 3.2 S/m, and the lithosphere is modeled as a half space of conductivity 0.005 S/m containing 1 km thick resistive (0.0005 S/m) or conductive (0.05 S/m) zones centered at depths of 3 and 6 km and joined to the half space in a smooth fashion. The horizontal range is equivalent to one skin depth in the lower half space. General attenuation of the kernel with depth is apparent, but a marked preference for low conductivity material even at depths as large as the source-receiver offset is also seen. By contrast, conductive zones yield only attenuation across them. Examination of the individual modes shows that the detail in Figure 1 is due entirely to the TM mode, and the PM mode displays only monotonic exponential attenuation with depth. The two modes react to the buried regions in an opposite sense and with very different amplitudes: The TM mode is strongly affected by resistive but indifferent to conductive material, while the PM mode is unaffected by resistive and only mildly influenced by conductive material.

Figure 2 shows the same models with only the 3 km deep conductivity contrasts at horizontal ranges of 7 and 21 km. The preference for resistive material becomes more pronounced with range, suggesting the use of very long source-receiver offsets to detect such zones.

Kelvin Wave

Electromagnetic induction by a Kelvin wave model of the ocean tide was first investigated by *Larsen* [1968] and was subsequently pursued in paper 1. Kelvin waves are coastally trapped, nondispersive, free progressive wave solutions of the linearized shallow water equations whose characteristics closely match coastal observations of sea level. Since their frequency is close to that of the rotating earth at mid-latitudes, TM and PM modes in similar proportions are produced; see paper 1 for details.

Expressions for the Fréchet kernels for Kelvin wave induction follow by combining equations (45)–(47) of paper 1 for J_z^0, T, and Υ with (36) and (37). As is shown in paper 1, the term in (12) containing J_z^0 is $O(kH)$ when compared to the terms containing T, and may be neglected. The Kelvin wave model used here is identical to that of paper 1, consisting of a semidiurnal type at 30°N off California in a 4 km deep ocean. The resulting Fréchet derivative forms are

$$\delta\hat{\Pi} = \int_a^{-H} ds\ [2\beta_0\mu/\sigma_0 \tilde{G}_\Xi(z)\hat{T}(k^2\pi^2(s)+(\partial_s\pi(s))^2)/\sigma^2(s) \cdot (\beta_0\pi(-H)/\sigma_0+\partial_z\pi(-H)/\sigma(-H))^2]\delta\sigma(s) \quad (44)$$

$$\delta\hat{\Psi} = \int_a^{-H} ds\ [i2\beta_0\omega\mu\tilde{G}_\Psi(z)\hat{\Upsilon}\psi^2(s)/ (\beta_0\psi(-H)+\partial_z\psi(-H))^2]\delta\sigma(s) \quad (45)$$

where π and ψ are solutions of (14) and (29) in the conducting earth below $z=-H$ for the conductivity profile about which the problem is linearized. The Fréchet derivatives are the terms in brackets and are computed as a function of offshore distance by inverse transforming

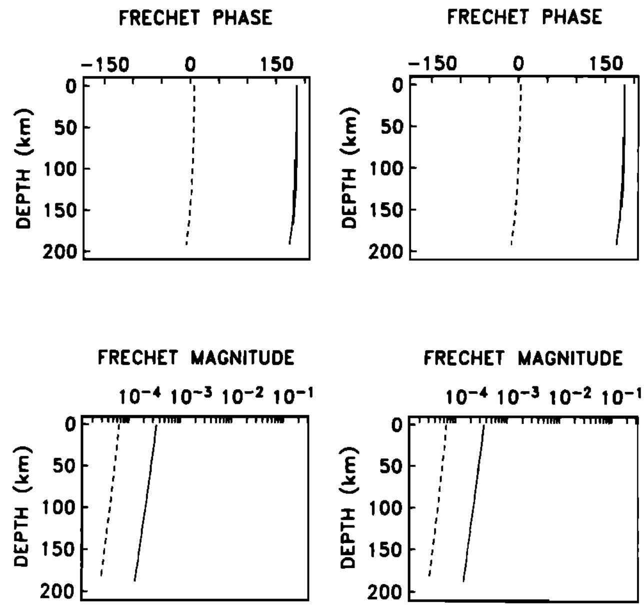

Fig. 5. The PM mode part of Figure 3. See Figure 3 caption for details.

(44) and (45) at each depth of interest after solving (14) and (29) numerically at each wave number. The WKB approximation is poor for this problem, and penetration depths are quite large, so the initial conditions are computed from a half space model below $z=a$.

Figure 3 shows the offshore component of the Fréchet derivative of the seafloor magnetic field at offshore distances of 250 and 3050 km. The lithosphere is modeled as a 100 km layer of conductivity 0.005 S/m overlying a 0.05 S/m half space; these values are compatible with upper lithospheric estimates by *Chave and Cox* [1984] and magnetotelluric measurements such as those of *Filloux* [1981]. The left panels show this base model with the addition of a 10-km thick zone of high conductivity (0.05 S/m) centered on 50 km, while the right panels have a substitute low conductivity (0.0005 S/m) region; in all cases the zones are interconnected in a smooth manner. The magnitude of the Fréchet kernels clearly mimics the conductivity profile, with larger values where the conductivity is lower. The amplitude shows a general offshore variation which is consistent with the behavior of the magnetic field itself. Figures 4 and 5 show the kernels broken down into TM and PM modes. The detail seen in Figure 3 is due almost entirely to the TM mode, which shows a factor of 100 change in amplitude for a factor of 10 decrease in conductivity in the right panels. The PM mode kernel is essentially featureless, and this mode yields very little information on lithospheric conductivity. Similar results can be obtained for the long shore magnetic and both horizontal electric field components.

The total change in the seafloor magnetic field caused by a conductivity change at depth may be calculated either from Figure 3 by integration or by forward modeling. In either case, the addition of the 10 km thick low conductivity zone at 50 km yields an increase of 0.9 nT at the seafloor, while the high conductivity case yields a decrease of 0.2 nT. The M_2 lunar semidiurnal tidal magnetic field at the seafloor typically has an amplitude of 1–4 nT, with statistical uncertainties of 3–10% [*Chave and Filloux*, 1984], and the effect of the low conductivity zone should be detectable unless other sources of noise are very large. This suggests that the study of electromagnetic induction by ocean tides, using current global tide models, has enormous potential for investigating ocean lithospheric structure.

APPENDIX: WKB SOLUTIONS TO MODAL EQUATIONS

WKB theory yields a global approximation to the solution of a linear differential equation whose highest derivative is multiplied by some small parameter ϵ. Mathematical details may be found in *Bender and Orszag* [1978, chap. 10]. First-order WKB solutions to (14) and (29) are derived in this appendix and find application in initializing numerical solutions of the modal equations. Note that first-order WKB theory corresponds to the physical optics approximation and gives the leading asymptotic behavior of the solution as $\epsilon \rightarrow 0$.

Consider the PM mode equation (29) for the source-free conducting region $a \leqslant z \leqslant b$ in the limit of large wave number

$$\epsilon^2 \partial_z^2 \hat{\Psi} - Q(z)\hat{\Psi} = 0 \qquad \text{(A1)}$$

where $\epsilon = k^{-1}$, $Q(z) = \beta^2(z)/k^2$, and β^2 is given by (15). The first-order WKB solution has the form

$$\hat{\Psi} \sim \exp[S_0/\epsilon + S_1] \qquad \text{(A2)}$$

where nonlinear equations for S_0 and S_1 follow by matching orders and are called the eikonal and transport equations, respectively. Let (A1) satisfy initial conditions at $z = a$ that correspond to a half space of conductivity σ_H for $z < a$. Then the solution of (A1) with (A2) is

$$\hat{\Psi}(z) \sim \sqrt{\frac{\beta_H}{\beta(z)}} \exp\left[\int_a^z ds\, \beta(s) + \beta_H a\right] \qquad \text{(A3)}$$

where β_H is given by (15) with $\sigma = \sigma_H$.

The TM mode equation (14) must be placed in normal form using the substitution $\hat{\Pi} = \sqrt{\sigma}\,\hat{\xi}$, where $\hat{\xi}$ satisfies

$$\epsilon^2 \partial_z^2 \hat{\xi} - R(z)\hat{\xi} = 0 \qquad \text{(A4)}$$

and $R(z) = \zeta^2(z)/k^2$ with

$$\zeta^2(z) = \beta^2(z) + 3/4(\partial_z\sigma/\sigma)^2 - ½\partial_z^2\sigma/\sigma \qquad \text{(A5)}$$

The first-order WKB solution for $\hat{\xi}$ follows from (A2) and (A3), and after conversion back to the $\hat{\Pi}$ scalar

$$\hat{\Pi}(z) \sim \sqrt{\frac{\beta_H \sigma(z)}{\beta(z)\sigma_H}} \exp\left[\int_a^z ds\, \zeta(s) + \beta_H a\right] \qquad \text{(A6)}$$

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A. D. Chave, Institute of Geophysics and Planetary Physics, Scripps Institution of Oceanograhy, University of California at San Diego, La Jolla, CA 92093.

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