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Comments on "An Inverse Approach to Signal Correlation" by D. G. Martinson, W. Menke, and P. Stoffa

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In a recent paper by *Martinson et al.* [1982], a quantitative method for recovering the mapping function which makes a distorted data series resemble a reference series using a parameter estimation technique (not an inverse approach) was proposed and illustrated. The mapping function describes the stretching and compressing of one data set with respect to another and may be of considerable geologic interest; for example, the sedimentation rate is the mapping function obtained by correlating a stratigraphic variable with some known measure of time. Since the application of the method has aroused some interest, especially among paleoclimatologists, it seems appropriate to comment on some limitations and omissions in the algorithm of *Martinson et al.* [1982]. Specifically, we show that (1) their use of maximization techniques on a highly nonlinear functional of the data, the coherence, requires considerably more attention to numerical stabilization than the simpler minimization of a nonlinear penalty function, (2) their use of a Fourier series can, under very general circumstances, produce artificial, high-frequency fluctuations in the mapping function, (3) their failure to constrain the derivative of the mapping function to be nonnegative can yield nonphysical results in many cases of geologic interest, especially for a large number of degrees of freedom in the model, and (4) it is difficult to add additional equality constraints on the mapping function at known tie points using their approach. The nonnegativity constraint is not overly restrictive for most interesting applications since, for example, the stratigrapher often cannot differentiate between zero and negative sedimentation rates. The equality constraint can be quite useful; the paleontologist can include the results of radiometric and magnetostratigraphic studies in the mapping function calculation. To accommodate points 3 and 4 and resolve 1 and 2, we choose to recast the problem into one of linearized optimization with linear inequality constraints and to parameterize the mapping function in terms of splines rather than a continuous orthogonal basis like sinusoids. To illustrate the advantages of these constraints and the spline basis, we correlate two oxygen isotope stratigraphies and give examples of both nonphysical mapping functions and artificial wiggliness introduced by a Fourier series representation.

We are given a reference signal $R(t)$ and a data series $D(x)$, from which we wish to recover the mapping function $x(t)$ which makes them, in some sense, most alike. *Martinson et al.* [1982] chose to maximize the coherence

$$C = \frac{\int_0^{t_{max}} R(t) D(x(t))\,dt}{\left[\int_0^{t_{max}} R^2(t)\,dt \int_0^{t_{max}} D^2(x(t))\,dt\right]^{1/2}} \tag{1}$$

which is a highly nonlinear functional of $x(t)$. Their algorithm yields a maximum value of (1) after considerable care is expended to stabilize the problem. A simpler approach is to minimize the penalty function

$$E = \sum_{j=1}^{N} |r(t_j) - d(x(t_j))|^p \tag{2}$$

over the N data points in an appropriate p norm subject to linear inequality and equality constraints on $x(t)$. The lower case letters r and d indicate data self-normalized to unit variance to avoid undue weighting of either variable. While we will deal only with the L_2 (quadratic programing or least squares, $p = 2$) norm in this paper, the use of the L_1 (linear programing, $p = 1$) norm may be more appropriate since it is more robust in the presence of noisy data. The penalty function (2) is nonlinear in $x(t)$ and may be linearized by expanding $d(x(t))$ in a Taylor series to first order:

$$E = \sum_{j=1}^{N} \left| r(t_j) - d(x(t_j, \{\alpha_i^m\})) - \sum_{i=1}^{M} \frac{\partial d(x(t_j, \{\alpha_i^m\}))}{\partial \alpha_i^m} \Delta\alpha_i \right|^p \tag{3}$$

where the superscript on x refers to the m^{th} iterate and the $\{\alpha_i^m\}$ are the M coefficients used to parameterize the mapping function. In addition, we may place an inequality constraint on the slope of the mapping function:

$$\frac{\partial x(t)}{\partial t} \geqslant 0 \tag{4}$$

and, possibly, use the L equality constraints

$$x(t_\ell) = c_\ell \qquad \ell = 1, \cdots L \tag{5}$$

The mapping function may be represented in terms of orthogonal functions

$$x(t) = \sum_{j=1}^{M} \alpha_j \phi_j(t) \tag{6}$$

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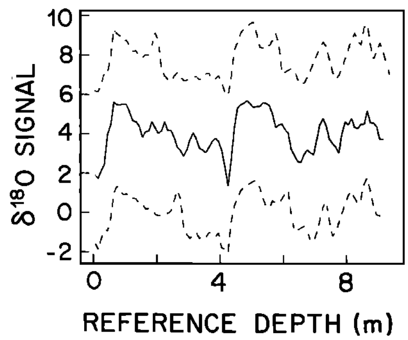

Fig. 1*a*. Mapping functions between cores RC11-120 and V28-238 based on $\delta^{18}O$ stratigraphy. The solid line is the spline fit with 3 knots; the dashed curve corresponds to 4 harmonics plus a ramp, each with inequality constraint (4). Though both mappings produce comparable misfits between the reference and data signals, the Fourier series mapping is considerably more wiggly than the spline curve for the same number of degrees of freedom.

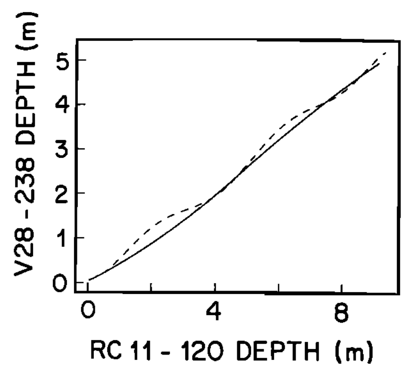

Fig. 1*b*. The normalized reference signal for RC11-120 (solid), and the normalized signal from V28-238 mapped according to the functions in Figure 1*a*. The parabolic spline mapping is beneath the reference while the Fourier series solution lies above. The overall misfits are comparable.

where the ϕ_j may be, for example, Chebyshev polynomials, sinusoids, or spline functions. We note that the Fourier basis used by *Martinson et al.* [1982] is incomplete since the cosine terms were omitted, severely constraining the mapping function to an antisymmetric or odd functional form. In any case, use of a continuous orthogonal basis set will often lead to serious convergence problems, especially in the neighborhood of sharp changes in slope; the discontinuous behavior of finite approximations to Fourier series and the well-known Gibbs phenomenon are discussed in detail by *Edwards* [1967].

The orthogonal functions $\phi_j(t)$ which we prefer are piecewise parabolic splines. We write the mapping function as a *pp* representation.

$$x(t) = a_i + b_i(t-\tau_i) + \frac{c_i}{2}(t-\tau_i)^2 \qquad \tau_i \leqslant t \leqslant \tau_{i+1} \qquad (7)$$

for the M knots τ_i of the spline. Parabolic splines are continuous in both the function and its derivative on the entire interval of interest and do not require the use of a linear ramp as in the method of *Martinson et al.* [1982]. Additional details on splines may be found in the work by *de Boor* [1978]. It can be shown that non-negativity of $x(t)$ at the knots is both necessary and sufficient for its non-negativity everywhere; a similar relation for all continuous orthogonal bases does not exist.

In practice, after normalizing the reference and data signals, we use a guess for the mapping function to initialize the coefficients $\{\alpha_k\}$ and iteratively refine these using (3) subject to (4) and (5) until a convergence criterion is satisfied. This problem is in the exact form required for the application of standard constrained least squares [e.g., *Lawson and Hanson*, 1974] or linear programing [e.g., *Luenberger*, 1973] algorithms. No special care is necessary to ensure the convergence of the solution given an initial guess which is linearly close to the final solution. We note that the inequality constraint (4) can serve to stabilize the solution and speed convergence, especially when dealing with data that are quasi-periodic. Although adding more degrees of freedom for the mapping function $x(t)$ can improve the fit, reflected in the lower residual, the change may not be significant. This can be assessed quantitatively using an F test on the ratio of the residual variances.

To illustrate the properties of the parabolic spline basis, we recover the mapping function between cores RC11-120 and V28-238 using published $\delta^{18}O$ stratigraphies. Figure 1*a* shows $x(t)$ for both splines and a complete Fourier series with ramp and subject to (4), each with 9 degrees of freedom (i.e., 3 knots or 4 harmonics plus ramp). The spline curve is considerably smoother than the Fourier series solution, although the overall misfits are comparable, as revealed using an F statistic on the residuals. Figure 1*b* shows the reference signal and mapped data series for the two methods. Visual similarity is apparent, although spectral analysis of the Fourier result will be affected by the wiggliness in the mapping function [*Schiffelbein and Dorman*, 1982]. In addition, the variations in the Fourier mapping function correlate qualitatively with glacial-interglacial changes in the data, a relationship which must be regarded with suspicion due to the known properties of finite Fourier approximations. Figure 2 illustrates the importance of the nonnegativity constraint as well as the effect of too many degrees of freedom in the mapping function. The $x(t)$ relations for 8 knots (24 degrees of freedom) and 12 harmonics plus ramp (25 degrees of freedom), with and without (4), show artificial wiggliness that cannot be physical. The unconstrained Fourier curve contains regions of unreasonable

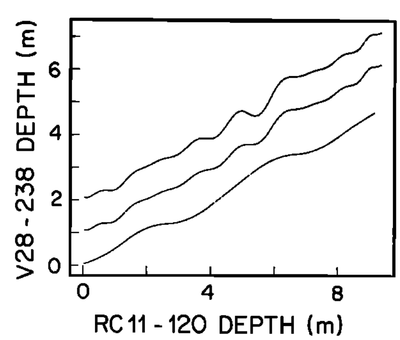

Fig. 2. Mapping functions, offset by 100 for convenience, between RC11-120 and V28-238. The bottom curve is the spline fit with 8 knots; the middle curve is the Fourier series solution with 12 harmonics plus a ramp. Both have nonnegativity of the slope imposed as a constraint. The upper curve is the 12 harmonic Fourier series without this extra constraint. This last function had not converged after 15 iterations with the misfit oscillating widely as the mapping function jumped between neighboring peaks.

negative slope. We note that this example is unduly kind to the Fourier method, and a higher number of harmonics will be required if any kinks occur in the mapping function, exacerbating the problem. By contrast, a few strategically placed knots can yield the required detail with a more realistic number of degrees of freedom. A similar approach was used by *Parker and Shure* [1982] to avoid spurious detail in maps of the magnetic field at the core-mantle boundary that always occur for orthogonal function (spherical harmonic) bases.

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REFERENCES

de Boor, C., *A Practical Guide to Splines*, Springer-Verlag, New York, 1978.

Edwards, R. E., *Fourier Series: A Modern Introduction*, vols. I and II, Holt, Rinehart and Winston, New York, 1967.

Lawson, C. L., and R. J. Hanson, *Solving Least Squares Problems*, Prentice-Hall, Englewood Cliffs, N.J., 1974.

Luenberger, D. G., *Introduction to Linear and Nonlinear Programming*, Addison-Wesley, Reading, Mass., 1973.

Martinson, D. G., W. Menke, and P. Stoffa, An inverse approach to signal correlation, *J. Geophys. Res.*, *87*, 4807–4818, 1982.

Parker, R. L., and L. Shure, Efficient modeling of the earth's magnetic field with harmonic splines, *Geophys. Res. Lett.*, *9*, 812–815, 1982.

Schiffelbein, P. A., and L. M. Dorman, Climatically controlled differential sedimentation rates in deep sea cores and time depth relationships, *Geol. Soc. Am. Abst. Programs*, *14*, 610, 1982.

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