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Vinod Chandran
Steve Elgar
Barry Vanhoff

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Vinod Chandran, *Member, IEEE*, Steve Elgar, *Member, IEEE*, and Barry Vanhoff

***Abstract*— Statistics of the estimates of tricoherence are obtained analytically for nonlinear harmonic random processes with known true tricoherence. Expressions are presented for the bias, variance, and probability distributions of estimates of tricoherence as functions of the true tricoherence and the number of realizations averaged in the estimates. The expressions are applicable to arbitrary higher order coherence and arbitrary degree of interaction between modes. Theoretical results are compared with those obtained from numerical simulations of nonlinear harmonic random processes. Estimation of true values of tricoherence given observed values is also discussed.**

I. Introduction

TRISPECTRAL analysis is useful in the study of nonlinear random processes characterized by cubic interactions or by non-Gaussian probability density functions that are symmetric about their mean value so that the bispectrum of the process is zero. The trispectrum is one member of a class of higher order spectra [4], [5] that can be defined for a random process and used to identify deviations from linearity and Gaussianity. Higher order spectra are defined as Fourier transforms of higher order cumulants of a random process. Thus, the bispectrum [14], [20], [24] and trispectrum [8], [22] are the Fourier transforms of the third and fourth cumulants, respectively. Brillinger and Rosenblatt [4], [5] provide derivations of higher order spectra in continuous- and discrete-time domains from first principles and also derive asymptotic statistics of higher order spectral estimates. Dalle Molle and Hinich [8] and Lutes and Chen [22] provide similar development of the trispectrum. The trispectrum and the fourth cumulant are also discussed in Lii and Rosenblatt [21], Dwyer [9], Marathay *et al.* [23], and others.

Of particular importance in quantifying nonlinear interactions are normalized (by the power spectrum) higher order spectra, which are termed higher order coherences. The bicoherence and the tricoherence (which are defined in Section II) are measures of the degree of quadratic and cubic phase coupling, respectively. A zero value for tricoherence indicates no cubic interaction and no phase-coupling between quartets of Fourier components, whereas a value of unity indicates perfect phase coupling. For a Gaussian process, it is well known that all higher order spectra, and thus bicoherence and tricoherence, are zero. Owing to statistical fluctuations, the estimate of the bicoherence or tricoherence from a finite record of data will be nonzero even for a Gaussian process. The estimate has a bias and a nonzero variance, both of which depend on the true value of the higher order coherence. To improve statistical reliability, the data record is usually divided into an ensemble of N blocks or realizations, and the ensemble averaged estimate is said to have $2N$ degrees of freedom (dof) because it is the average of N complex quantities. Degrees of freedom can also be obtained by merging neighboring higher order spectral estimates in frequency space. Asymptotic statistics (for large dof) of higher order spectra, including statistics of the real and imaginary parts of the bispectrum, can be found in [3], [4], [27], and others [24]. Estimation and statistical distribution of bicoherence are discussed in [1], [6], [11], [16]–[18], and [20]. Dalle Molle and Hinich [8] discuss the statistics of the trispectrum and its magnitude. Haubrich [16] shows that for a Gaussian process, the bicoherence (which has a true value of zero for infinite dof) is approximately *chi-squared* distributed with 2 dof (χ_2^2), and thus, significance levels for zero bicoherence can be calculated. A similar result is heuristically derived by Dalle Molle and Hinich [8] for trispectral magnitudes. However, statistics of bicoherence or tricoherence for arbitrary true values have never been analytically derived, although the statistics of bicoherence have been reported using numerical simulations [11], [13], and compared with some empirical formulae similar to those proposed for coherence by Jenkins and Watts [19] and Benignus [2]. The objective of the present study is to analytically derive the statistics of the tricoherence for arbitrary true values, assuming a harmonic model for the process. The results are verified by numerical simulations. The statistics are also shown to be identical for all higher order coherences.

Relevant definitions of higher order spectra are reviewed in Section II. In Section III, analytical expressions for the statistical distribution, bias, and variance of estimates of the tricoherence are derived. The theoretical results are compared with those obtained from numerical simulations in Section IV. Procedures to estimate the true value of tricoherence from an observed value are discussed in Section V. An application to observations of nonlinearly interacting ocean waves generated by a hurricane is presented in Section VI. The theoretical results for tricoherence are extended to arbitrary order in Section VII, and conclusions follow in Section VIII.

II. Definitions

Higher order spectra of stationary, ergodic random processes are defined as the Fourier transforms of higher order cumulant functions of the process [3]–[5]. Thus, the trispectrum,

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V. Chandran is with the Signal Processing Research Centre, School of Electrical and Electronic Systems Engineering, Queensland University of Technology, Brisbane, Australia.

S. Elgar and B. Vanhoff are with the School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164-2752 USA.

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$T(f_1, f_2, f_3)$, [8], [22] is defined as the Fourier transform of the fourth cumulant function of the random process $x(n)$ and may also be expressed (reduced from the Stieltjes integral form as in Eq. 8 of [8]) as

$$T(f_1, f_2, f_3) = E[X(f_1)X(f_2)X(f_3)X^*(f_1 + f_2 + f_3)] \quad (1)$$

where

$X(f)$ is the Fourier transform of a realization of $x(n)$;
f is the frequency;
$*$ is the complex conjugation; and
$E[\,]$ is the expectation operator.

The trispectrum is a function of a triad of frequencies, and its definition involves a quartet of Fourier coefficients or modes where the fourth frequency is the sum of the other three. In practice, the expectation operation involves averaging over an ensemble of realizations and/or frequency merging. The principal domain or nonredundant region of computation of the trispectrum is discussed in [8] for continuous, bandlimited, and discrete-time cases. Reference [7] provides a procedure to derive the nonredundant region of computation of periodogram estimates of any higher order spectrum.

When quantifying the degree of phase coupling or nonlinear interaction between Fourier modes of a random process, it is useful to normalize the higher order spectra to remove the dependence on the power at each frequency. The trispectrum can be normalized using the same methods used for the bispectrum [16], [20]. Extending the *Haubrich* [16] bispectral normalization to the trispectrum yields a normalized trispectrum $\mathcal{T}(f_1, f_2, f_3)$ given by

$$\mathcal{T}(f_1, f_2, f_3) = \frac{E[X(f_1)X(f_2)X(f_3)X^*(f_1 + f_2 + f_3)]}{\sqrt{P(f_1)P(f_2)P(f_3)P(f_1 + f_2 + f_3)}} \quad (2)$$

where $P(f) = E[X(f)X^*(f)]$ is the power at frequency f. An alternative normalization given by Kim and Powers [20] for the bispectrum yields

$$\mathcal{T}(f_1, f_2, f_3) = \frac{E[X(f_1)X(f_2)X(f_3)X^*(f_1 + f_2 + f_3)]}{\sqrt{P_{1,2,3}(f_1, f_2, f_3)P(f_1 + f_2 + f_3)}} \quad (3)$$

where

$$P_{1,2,3}(f_1, f_2, f_3) = E[X(f_1)X(f_2)X(f_3)X^*(f_1) \times X^*(f_2)X^*(f_3)].$$

The squared magnitude of the normalized trispectrum is called the *tricoherence*

$$t^2(f_1, f_2, f_3) = |\mathcal{T}(f_1, f_2, f_3)|^2 \quad (4)$$

and it can be shown using the Cauchy–Schwarz inequality (which is similar to the proof for bicoherence [20] that $0 \leq t^2 \leq 1$ if the normalization in (3) is used. If the Fourier components at frequencies f_1, f_2, and f_3 are statistically independent, then the estimates of tricoherence obtained using either normalization are statistically equivalent. The phase of the trispectrum is referred to as the *triphase*. The tricoherence is a measure of the fraction of the total product of powers at the frequency quartet, $(f_1, f_2, f_3, f_1 + f_2 + f_3)$, that is owing to cubicly *phase-coupled* modes. The statistics of tricoherence and computation of the bias and variance for a known true value of tricoherence are discussed in the following section.

III. Bias and Variance

Given N independent realizations of a stationary, ergodic random process $x(n)$, say, $x_i(n), i = 1, 2, \ldots N$, the estimate of the trispectrum is

$$\hat{T}(f_1, f_2, f_3) = \frac{1}{N}\sum_{i=1}^{N}[X_i(f_1)X_i(f_2)X_i(f_3)X_i^*(f_1 + f_2 + f_3)] \quad (5)$$

where $X_i(f)$ is the Fourier transform of $x_i(n)$. The estimate of the power at frequency f is given by

$$\hat{P}(f) = \frac{1}{N}\sum_{i=1}^{N}[X_i(f)X_i^*(f)] \quad (6)$$

and the estimate of the tricoherence is

$$\hat{t^2}(f_1, f_2, f_3) = \frac{|\frac{1}{N}\sum_{i=1}^{N}[X_i(f_1)X_i(f_2)X_i(f_3)X_i^*(f_1 + f_2 + f_3)]|^2}{\hat{P}(f_1)\hat{P}(f_2)\hat{P}(f_3)\hat{P}(f_1 + f_2 + f_3)}. \quad (7)$$

Statistical fluctuations of the denominator are small relative to those of the numerator (see [11], [13], and [16] for bicoherence), and thus, to a first approximation, it can be assumed that $\hat{P}(f) = P(f)$, and the denominator in (7) can be assumed constant while computing the statistics of the tricoherence estimate.

A. Gaussian Noise

Claim: The estimate of the tricoherence for Gaussian noise is approximately $\alpha\chi_2^2$ distributed with 2 dof, where α equals $\frac{1}{2N}$, and N is the number of realizations averaged.

Proof: Let $x_i(n)$ be the ith realization of a discrete-time, zero-mean, Gaussian noise process. Let $X_i(f)$ be the Fourier transform of $x_i(n)$. Denote

$$X_i(f_1) = a_{i1} + jb_{i1}$$
$$X_i(f_2) = a_{i2} + jb_{i2}$$
$$X_i(f_3) = a_{i3} + jb_{i3}$$
$$X_i(f_4) = X_i(-f_1 - f_2 - f_3) = a_{i4} + jb_{i4}$$

where $j = \sqrt{-1}$. Since $x_i(n)$ is Gaussian noise, the real and imaginary part of each Fourier coefficient is also Gaussian distributed, and

$$E[a_{ik}] = E[b_{ik}] = 0 \quad (8)$$

$$E[a_{ik}^2] = E[b_{ik}^2] = \frac{E[X(f_k)X^*(f_k)]}{2} = \frac{P(f_k)}{2} \quad (9)$$

for $k = 1, 2, 3, 4$, where $P(f_k)$ denotes the power at frequency f_k. The estimate of the trispectrum can be expanded as

$$\begin{aligned}\hat{T}(f_1, f_2, f_3) &= \frac{1}{N}\sum_{i=1}^{N}(a_{i1}a_{i2}a_{i3}a_{i4} + 7 \text{ other terms}) \\ &\quad + j\frac{1}{N}\sum_{i=1}^{N}(b_{i1}a_{i2}a_{i3}a_{i4} + 7 \text{ other terms}) \\ &= \frac{1}{N}(G_R + jG_I) \end{aligned} \tag{10}$$

where G_R and G_I are sums of i.i.d. random variables and, thus, will be Gaussian distributed for large N according to the central limit theorem. Since the real and imaginary parts of each Fourier coefficient are zero-mean

$$E[G_R] = E[G_I] = 0. \tag{11}$$

Note that

$$\begin{aligned}E[a_{i1}^2a_{i2}^2a_{i3}^2a_{i4}^2] &= E[a_{i1}^2]E[a_{i2}^2]E[a_{i3}^2]E[a_{i4}^2] \\ &= \frac{P(f_1)P(f_2)P(f_3)P(f_4)}{16}\end{aligned} \tag{12}$$

which is also the variance of each term in the summations in (10). Since there are $8N$ terms in each summation, the variances of G_R and G_I are equal and are given by

$$\sigma_G^2 = NP(f_1)P(f_2)P(f_3)P(f_4)/2. \tag{13}$$

The normalized trispectrum can then be written as

$$\begin{aligned}\hat{\mathcal{T}}(f_1, f_2, f_3) &= \frac{\hat{T}(f_1, f_2, f_3)}{\sqrt{\hat{P}(f_1)\hat{P}(f_2)\hat{P}(f_3)\hat{P}(f_4)}} \\ &\simeq \frac{\hat{T}(f_1, f_2, f_3)}{\sqrt{P(f_1)P(f_2)P(f_3)P(f_4)}} \\ &= \frac{1}{\sqrt{2N}}\left(\frac{(G_R + jG_I)\sqrt{2}}{\sqrt{NP(f_1)P(f_2)P(f_3)P(f_4)}}\right) \\ &= \frac{1}{\sqrt{2N}}(\mathcal{G}_R + j\mathcal{G}_I)\end{aligned} \tag{14}$$

where $\mathcal{G}_R$ and $\mathcal{G}_I$ are Gaussian with zero mean and unit variance. It is also known [26] that the sum of the squares of 2 Gaussian random variables of zero mean and unit variance is χ_2^2. Therefore the tricoherence estimate for Gaussian noise can be written as

$$\hat{t}^2(f_1, f_2, f_3) = \frac{1}{2N}C \tag{15}$$

where C is χ_2^2. The mean of a χ_2^2 distribution is 2 and its variance is 4. Therefore, $\hat{t}^2$ is $\alpha\chi_2^2$ distributed with bias $\frac{1}{N}$ and variance $\frac{1}{N^2}$ for large N.

Approximate formulae for significance levels for the tricoherence estimate can then be derived from χ_2^2 statistics and are given in Table I (this is also given for bicoherence in [11]). These formulae are confirmed by numerical simulations in Section IV. The procedure for hypothesis testing of Gaussianity in a stationary random process is explained in [17]. For example, consider a tricoherence test for Gaussianity using an ensemble of $N = 128$ realizations of a stationary, ergodic random process. If the null hypothesis of Gaussianity is true, then 95% of the tricoherence values will be expected to be below $6/256 = 0.234$ (Table I).

TABLE I
SIGNIFICANCE LEVELS FOR ZERO TRICOHERENCE FOR A GAUSSIAN PROCESS AS A FUNCTION OF N, WHICH IS THE NUMBER OF REALIZATIONS A

Significance level	*Value*
80%	$3.2/2N$
90%	$4.6/2N$
95%	$6.0/2N$
99%	$9.2/2N$

Statistical stability (i.e., increased dof) may also be obtained by merging neighboring trispectral estimates in trifrequency space. If the process is purely Gaussian, than each estimate combined in the frequency merging yields an additional 2 dof (which is similar to the standard procedure for power spectal estimation). On the other hand, for a purely deterministic process, the higher order spectral estimates are not independent, and thus, frequency merging neighboring values does not increase the dof. For an arbitrary random process, the increase in the number of dof attained by frequency merging lies somewhere between these two extremes. A conservative approach is to increase the dof by the number of frequency bands merged (as opposed to the number of actual higher order spectral estimates merged, which equals the cube of the number of bands merged for tricoherence). For unknown true values of higher order coherence, model testing may be necessary to obtain more accurate estimates of the increase in dof resulting from frequency merging.

For testing if an unknown process deviates from Gaussianity, the entire population of tricoherence values computed over the principal domain may be used without regard to the particular quartets of frequencies at which they are defined. Thus, when an overall test for Gaussianity of the process is at issue, the results of this section can be used. The statistical population used for determining the percentage of tricoherence values above a desired significance level may come from different frequency quartets in the principal domain. However, a nonlinearity may be manifested in the statistical dependence of a particular quartet (perhaps in a background of Gaussian noise). A test based on a population of tricoherence values, most of which satisfy the Gaussian hypothesis, may fail to detect the presence of the phase-coupled (i.e., non-Gaussian) quartet. In this case, a significance test must be applied to the quartet in question. In practice, this quartet is often chosen as one whose frequencies are locations of power spectral peaks that satisfy the resonance condition for a sum or difference cubic interaction. The following section is motivated by the need for applying a significance test for phase coupling at a particular quartet.

B. Harmonic Processes

A sinusoidal model for the random process and the normalization of (2) will be used here to simplify the analysis. Thus, the random process $x(n)$ is modeled by

$$x(n) = \sum_{p=1}^{P} A_p \cos(2\pi f_p n + \phi_p) \quad (16)$$

where P is the total number of modes, and f_p, ϕ_p are the frequency and phase of the pth mode, respectively. It is possible to have more than one mode at the same frequency. Although, strictly speaking, the harmonic model assumes that all modes are multiples of some fundamental frequency and is a special case of the more general sinusoidal model, the term *harmonic* model is used here for a sinusoidal model. Unless otherwise stated, it will be assumed that the phases are uniform random in the interval $[0, 2\pi)$ and that the amplitudes A_p are deterministic and constant. The amplitudes are not assumed to be equal but only nonrandom. If the random process $x(n)$ is a harmonic process consisting of cosinusoids

$$\begin{aligned} x(n) = & A_1 \cos(2\pi f_1 n + \phi_1) + A_2 \cos(2\pi f_2 n + \phi_2) \\ & + A_3 \cos(2\pi f_3 n + \phi_3) + A_{4c} \cos(2\pi f_4 n + \phi_4) \\ & + A_{4r} \cos(2\pi f_4 n + \phi_5) \end{aligned} \quad (17)$$

where $f_4 = f_1 + f_2 + f_3$ and the phases are statistically *independent*, the tricoherence at (f_1, f_2, f_3) equals 0. If, on the other hand, the phases are *coupled* such that $\phi_4 = \phi_1 + \phi_2 + \phi_3 + \phi_k$, where ϕ_k is some constant (ϕ_5 remains random and independent), then the tricoherence at (f_1, f_2, f_3) is

$$t^2 = \frac{A_{4c}^2}{A_{4c}^2 + A_{4r}^2}. \quad (18)$$

In practice, however, the tricoherence is estimated from a finite number of realizations, and the estimate of tricoherence will not equal the true value. The estimate is biased and has a nonzero variance, both of which depend on the true value of the tricoherence. The bias and variance become especially significant when the true value is very low (e.g., for weakly nonlinear systems).

1) Harmonic Process with Perfect Phase Coupling: If $x(t)$ is a harmonic random process with modes at frequencies f_1, f_2, f_3, and f_4, where $f_4 = f_1 + f_2 + f_3$, whose phases are perfectly coupled such that $\phi_4 = \phi_1 + \phi_2 + \phi_3 + \phi_k$, where ϕ_k is a constant, then the triphase ϕ_t equals ϕ_k, and the tricoherence is equal to 1. The mean value of the tricoherence estimate is equal to 1, and its variance is zero. The probability density function of the estimate is an impulse at the value 1 because the only randomness in the process is in the phases and for perfect coupling the randomness is cancelled (i.e., $\phi_t = \phi_k$ with probability 1). Thus, $t^2 = 1$ with probability 1. If the phase coupling is not perfect, resulting from independent statistical fluctuations in either the phases or the amplitudes, the tricoherence will be less than 1, and the phase coupling is said to be partial.

2) Harmonic Process with Partial Phase Coupling: Partial phase coupling can be accounted for by considering both phase-coupled and random-phase modes at each frequency. There may be more than one phase-coupled mode at each frequency: one for each phase-coupled quartet. However, when the tricoherence at one particular quartet is being considered, all modes at this frequency other than the one belonging to the phase-coupled quartet can be combined into one random-phase mode. Thus, each Fourier coefficient may be split into a phase-coupled component and a random-phase component. For example, $X_i(f_1) = X_{ic}(f_1) + X_{ir}(f_1)$, where X_{ic} is the phase-coupled component, and X_{ir} is the random-phase component. As shown in Appendix A, an analysis similar to the one for the Gaussian noise case yields

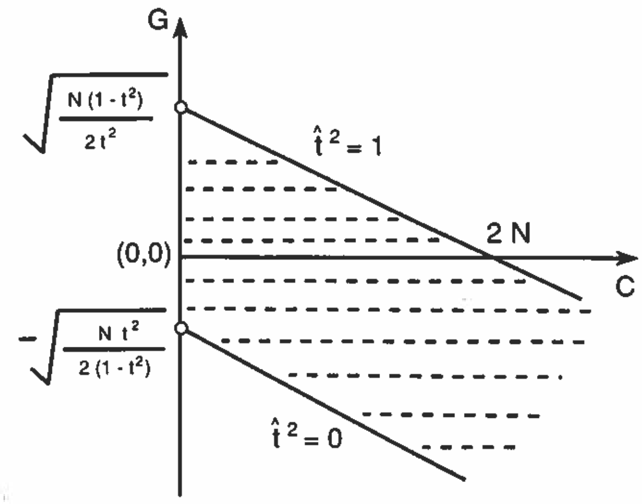

Fig. 1. Region over which the random variables C and G can vary when the true tricoherence is t^2. This region is bounded by the straight lines for which the estimate of the tricoherence is zero (lower line) and one (upper line). N is the number of realizations averaged in the estimation.

$$\hat{t^2}(f_1, f_2, f_3) \simeq t^2 + (1 - t^2)\frac{1}{2N}C + \frac{2t\sqrt{(1 - t^2)}}{\sqrt{2N}}G \quad (19)$$

where C is χ_2^2, G is Gaussian with zero mean and unit variance, and t^2, the true value of tricoherence, is the ratio of the product of the coupled powers at the frequencies f_1, f_2, f_3, and $f_1 + f_2 + f_3$ to the product of the total powers at these frequencies. If $t^2 = 0$ (19) reduces to (15), and if $t^2 = 1$, (19) reduces to the result for a perfectly phase-coupled quartet (zero bias and zero variance).

The random variables C and G in (19) are not entirely statistically independent because of the constraint $0 \leq \hat{t^2} \leq 1$. Thus, for any given value of C, G cannot vary from $-\infty$ to ∞ but must remain confined within certain finite limits, as shown in Fig. 1.

However, C and G may be assumed to be statistically independent within this region because a) for any given value of C, the number of possible values of G is still very large for large N, and b) C depends only on the magnitudes of the random phase components, whereas G depends on their phases as well, and the phases of the random phase components can be reasonably assumed to be statistically independent of their amplitudes. This assumption simplifies the computation of the bias and the variance. To compute the mean and variance of tricoherence, the probability density function (see (19)) and the range of permissible values for the random variables involved in this function (see Fig. 1) must be known. The region of permissible values of C and G is bounded by the line $\hat{t^2} = 1$, which intercepts the axes at $C = 2N$ and $G = \sqrt{\frac{N}{2}}\frac{\sqrt{1-t^2}}{t}$

(Fig. 1). Further simplification of the analysis is possible by considering only those values of t^2 (the true value) for which this line is either approximately vertical or approximately horizontal, as described below. The slope of this line is

$$\tan(\theta) = \frac{\sqrt{1-t^2}}{2t\sqrt{2N}}.$$

3) Vertical Line Approximation: Considering $\tan(\theta) \geq 10$ (corresponding to $\theta \geq 84°$) as nearly vertical, the range of true tricoherence for which this approximation holds is given by

$$t^2 \leq \frac{1}{800N+1}.$$

Even for a low value of N such as 8, this range corresponds to tricoherence less than 10^{-4}. Therefore, this is the range for which the true tricoherence is "close to zero." For this assumption, the Gaussian random variable G may be ignored, and the bias and variance depend primarily on C. After some algebraic manipulation, the bias and variance of the estimate of tricoherence can be written as

$$\text{bias}(\hat{t^2}) = \frac{1-t^2}{N} \tag{20}$$

and

$$\text{var}(\hat{t^2}) = \left(\frac{1-t^2}{N}\right)^2. \tag{21}$$

The bias and standard deviation are much larger than the true value of tricoherence itself in this range. Thus, to distinguish estimates of the tricoherence from zero for low values of t^2 requires many dof (large N). The expressions for bias and variance, however, are of interest because a) they reduce to the results for Gaussian noise as $t^2 \to 0$, and b) the expression for bias is identical to that given by Benignus [2] for coherence as a good least-squares fit over a wide range of coherence values.

4) Horizontal Line Approximation: Considering $\tan(\theta) \leq 0.1$ (corresponding to $\theta \leq 6°$) as nearly horizontal, the range of true tricoherence for which this approximation holds is given by

$$t^2 \geq \frac{1}{0.08N+1}.$$

This is the range for which the true tricoherence may be regarded as "significantly greater than zero." The values of tricoherence that may be regarded as "significantly greater than zero" depend on the value of N. For example, if $N = 128$, this range includes tricoherences from 0.1 to 1. For this range of true tricoherence, both random variables C and G must be taken into account in the computation of the bias and variance. The region over which they are defined is now a rectangular strip (Fig. 2) with C defined from 0 to ∞ and G defined from a to b.

After some algebraic manipulation (Appendix B), the bias and variance of the estimate of tricoherence can be written as

$$\text{bias}(\hat{t^2}) = \frac{1-t^2}{N} + \frac{t\sqrt{1-t^2}}{\sqrt{N\pi}}\left[e^{-a^2/2} - e^{-b^2/2}\right] \tag{22}$$

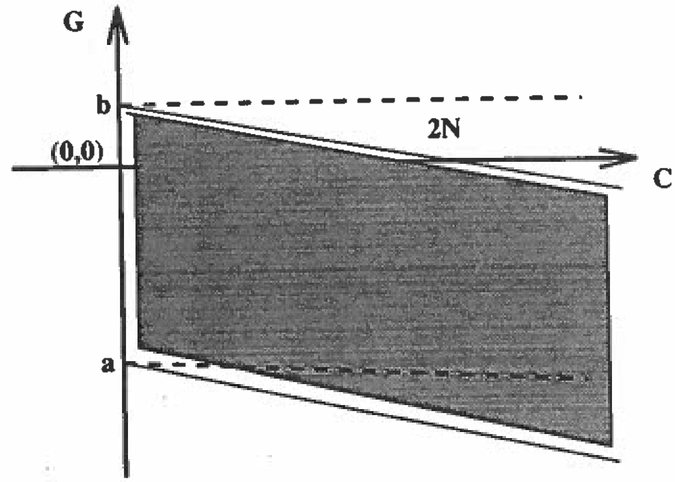

Fig. 2. Region over which the random variables C and G can vary when the true tricoherence is significantly greater than zero is shown shaded. This region can be approximated by a rectangular region bounded by straight lines $G = a$ and $G = b$, $0 < a < b$, to simplify the computation of the bias and variance of the tricoherence estimate. N is the number of realizations averaged in the estimation.

and

$$\text{var}(\hat{t^2}) = \left(\frac{1-t^2}{N}\right)^2 + \frac{2t^2(1-t^2)}{N} \times \left[\frac{1}{\sqrt{2\pi}}\left(ae^{-a^2/2} - be^{-b^2/2}\right) + \text{erf}(b) + \text{erf}(-a) - \frac{1}{2\pi}\left(e^{-a^2/2} - e^{-b^2/2}\right)^2\right] \tag{23}$$

where erf( ) is the error function

$$\text{erf}(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-y^2/2}dy.$$

The bias and variance of the tricoherence estimate may be *underestimated* or *overestimated*, depending on the particular values chosen for a and b. Possible choices for a include $a = -\sqrt{\frac{N}{2}}\frac{t}{\sqrt{1-t^2}}$ and $a = -\infty$, which are at the two extremes of the $\hat{t^2} = 0$ line (see Figs. 1 and 2). Possible choices for b include $b = \sqrt{\frac{N}{2}}\frac{\sqrt{1-t^2}}{t}$, which is the intercept of the $\hat{t^2} = 1$ line with $G = 0$, and $b = 0$, which is the same point in the limit $t \to 1$. Thus

$$a = -\sqrt{\frac{N}{2}}\frac{t}{\sqrt{1-t^2}}$$

and

$$b = \begin{cases} 0 & \text{lower bias} \\ \sqrt{\frac{N}{2}}\frac{\sqrt{1-t^2}}{t} & \text{higher bias} \end{cases}$$

provide two estimates of the bias where $b = 0$ ignores the positive region of the random variable G and is expected to yield a bias that is lower than the true value. Similarly

$$a = \begin{cases} -\sqrt{\frac{N}{2}}\frac{\sqrt{1-t^2}}{t} & \text{lower variance} \\ -\infty & \text{higher variance} \end{cases}$$

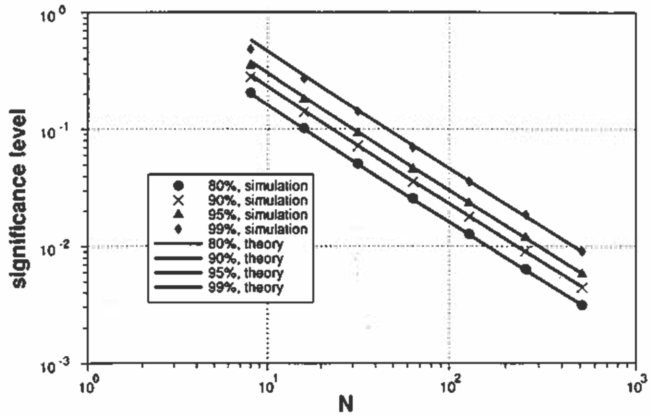

Fig. 3. 80, 90, 95, and 99% significance levels of tricoherence for Gaussian noise as a function of the number of realizations N averaged in the estimate. (The number of degrees of freedom is $2N$).

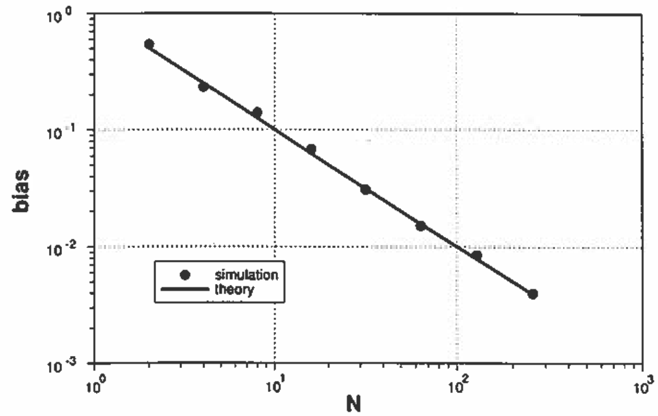

Fig. 4. Bias of tricoherence for Gaussian noise as a function of the number of realizations N averaged in the estimate. The solid line connects theoretical values, and the dots represent numerically simulated values.

and

$$b = \begin{cases} 0 & \text{lower variance} \\ \sqrt{\frac{N}{2}}\frac{\sqrt{1-t^2}}{t} & \text{higher variance} \end{cases}$$

provide two estimates of the variance where the larger region of integration yields the higher variance. The lower value of bias and the higher value of variance are closer to the results of numerical simulations described in the next section. Hence, these values are proposed here as approximate theoretical values. Because of the bias-variance tradeoff in spectral estimation, it can be expected that the higher estimate of variance will be closer to the true variance when the lower estimate of bias is closer to the true bias. The lower bias is closer to the values obtained from numerical simulations, probably owing to the fact that the region shown in Fig. 1 is larger over the negative half of the G.

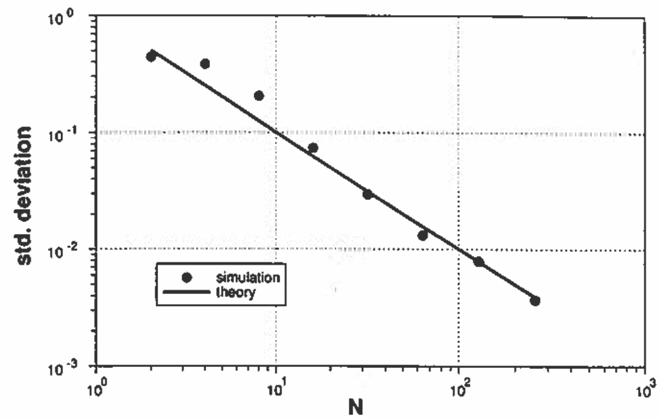

Fig. 5. Standard deviation of tricoherence for Gaussian noise as a function of the number of realizations N averaged in the estimate. The solid line connects theoretical values, and the dots represent numerically simulated values.

IV. Numerical Simulations

Numerically simulated time series were used to verify the results obtained analytically in Section III. Details of the simulation procedures are explained below for each of the tests performed.

A. Significance Levels of Zero Tricoherence for Gaussian Noise

Tricoherence values were computed for numerically simulated, zero-mean, unit variance Gaussian noise by averaging over N 128-point realizations. N was varied from 8 to 512 in multiples of 2, and for each case, a population of 7106 tricoherence values was sorted and used to determine the 80, 90, 95, and 99% significance levels. These levels are shown in Fig. 3 along with the theoretical values computed using the analytical expressions in Table I. The theoretical results agree well with those obtained from numerical simulations even for $N = 8$.

B. Bias and Standard Deviation for Gaussian Noise

The tricoherence for Gaussian noise was shown in Section III to be χ_2^2 with bias $\frac{1}{N}$ and variance $\frac{1}{N^2}$, which was numerically verified using 256-point realizations of simulated, zero-mean, unit variance Gaussian noise. Tricoherence values were computed at an arbitrarily chosen triad (f_1, f_2, f_3) of frequencies, averaging over N realizations, and varying N from 2 to 256 in multiples of 2. For each N, a population of 100 tricoherence values was obtained for the same triad and used to compute the bias and standard deviation. Figs. 4 and 5 show the numerically simulated bias and standard deviation, respectively, along with the theoretical values, as a function N. There is good agreement between theoretical and numerically simulated values even for low values of N.

C. Harmonic Random Process with Arbitrary True Tricoherence

Equations (22) and (23) for the bias and variance of the tricoherence for a harmonic random process were also tested using numerically simulated, 256-point realizations of a five-mode harmonic random process as defined in (17). Three modes with normalized frequencies, $f_1 = 0.0625$, $f_2 = 0.1875$, and $f_3 = 0.25$ each had unit amplitudes ($A_1 = A_2 = A_3 = 1$) and random phases ϕ_1, ϕ_2, ϕ_3. It is not necessary to have unit amplitudes, which were chosen here to simplify the specification of true tricoherence by adjusting the ratio of the amplitudes of the phase-coupled and random modes at the sum frequency. A fourth mode with frequency $f_4 = f_1+f_2+f_3$ had

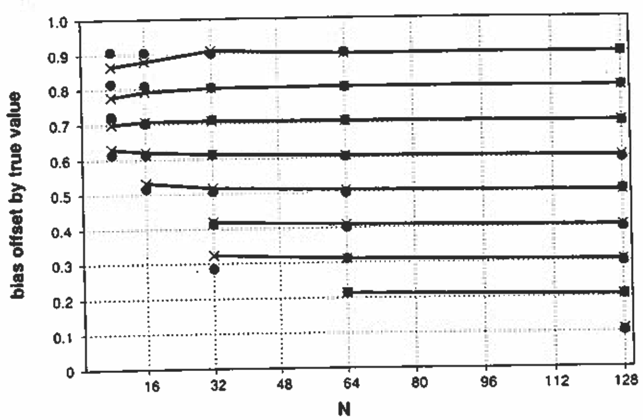

Fig. 6. Bias of tricoherence for a harmonic random process as a function of the number of realizations N averaged in the estimate for arbitrary true values ranging from 0.1 to 0.9 in steps of 0.1. Each function is offset vertically by the true value. Theoretical values are represented by X marks, and those corresponding to the same true value are connected by solid lines. Numerically simulated values are represented by dots. Only those combinations that satisfy the assumption in Section III-C-2 are plotted.

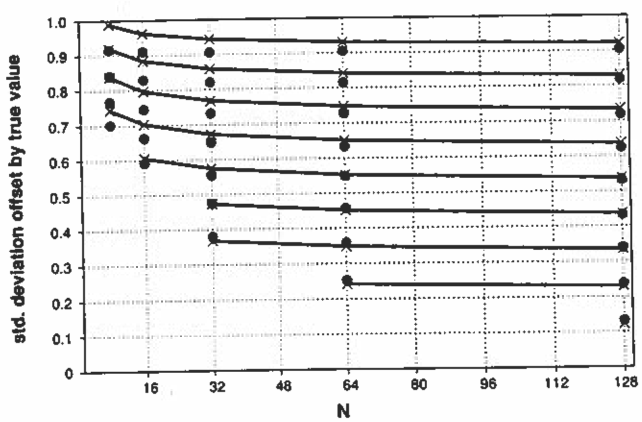

Fig. 7. Standard deviation of tricoherence for a harmonic random process as a function of the number of realizations N averaged in the estimate for arbitrary true values ranging from 0.1 to 0.9 in steps of 0.1. Each function is offset vertically by the true value. Theoretical values are represented by X marks, and those corresponding to the same true value are connected by solid lines. Numerically simulated values are represented by dots. Only those combinations that satisfy the assumption in Section III-C-2 are plotted.

phase $\phi_4 = \phi_1+\phi_2+\phi_3$ and thus was coupled to the phases of the other three modes in every realization. An additional mode at frequency f_4 had a uniform, random phase. The amplitudes of the phase-coupled and random phase modes at frequency f_4 were $A_{4c} = t$ and $A_{4r} = \sqrt{1-t^2}$, respectively, where t^2 is the true tricoherence desired. Tricoherence values were computed at (f_1, f_2, f_3) using (7), averaging over N realizations for each estimate, with N varying from 8 to 128 in multiples of 2 and t^2 varying from 0.1 to 0.9 in steps of 0.1. A population of 100 tricoherences was obtained for each case and used to compute the bias and the standard deviation. Analytical estimates of bias and standard deviation were computed using (22) and (23), respectively, and *Mathematica*.[1] The values from the numerical simulations are compared with theoretical values in Figs. 6 and 7.

When $N \geq 64$, the theoretical values agree very well with the numerically simulated values. For low values of N, the agreement is not as good because the horizontal line approximation (Section III-C) involved in computing the theoretical estimate starts to break down.

[1] a software package for symbolic and numeric mathematical computation

V. Estimation of True Tricoherence

The results of the previous sections describe the statistics of estimates of tricoherence given the true value t^2. This facilitates the design of experiments to measure tricoherence. On the other hand, it may be desired to estimate the true value of tricoherence t^2 given an observed value $\hat{t}^2$ estimated from a limited set of data. In this section, the maximum and the mean of the likelihood function [19] are discussed as estimates of the true tricoherence given an observed value. Likelihood estimation of the true value given an observed value or a few observed values consists of determining a parameter of the underlying probability distribution function (pdf) from samples of the distribution. The pdf is reformulated as a function of the parameter to be evaluated called the likelihood function. The value of the parameter that maximizes this likelihood function is called the maximum likelihood estimate (MLE). For the tricoherence, the likelihood function $L(t^2)$ is given by (19), where $\hat{t}^2$, which is the observed value, is fixed. This is a straight line in the 2-D space over which random variables C and G are defined (see Fig. 1 for the region of definition). To obtain the MLE of the true value, the probability that C and G lie on this line is computed by integrating their joint pdf along the line. This probability is then differentiated with respect to t^2 to find the MLE of t^2. The MLE of the true tricoherence is therefore difficult to compute analytically, in general. However, if the true tricoherence is "significantly greater than zero" as stated in Section III, this line is nearly horizontal and given by

$$G = \frac{(\hat{t}^2 - t^2)\sqrt{2N}}{2t\sqrt{1-t^2}}. \tag{24}$$

The likelihood function is then given by

$$L(t^2) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{(\hat{t}^2 - t^2)\sqrt{2N}}{2t\sqrt{1-t^2}}\right)^2\right] \tag{25}$$

and the MLE of the true tricoherence is found to be the observed value itself. Thus, for large N (with true values of tricoherence significantly greater than zero as stated in Section III-C), the MLE of the true tricoherence given an observed value is the observed value.

The mean likelihood estimate [19] (MELE) of the true tricoherence can be obtained by equating the mean of the likelihood function to the observed value, $\hat{t}^2$, or equivalently, equating the bias (22) to $\hat{t}^2 - t^2$ and solving for t^2. Again, it can be seen from Fig. 6 that when N is large enough for the true value to be significantly greater than zero, the bias tends to zero, and therefore, the MELE estimate also tends to the observed value $\hat{t}^2$.

VI. Application

Perturbation expansion solutions to the equations of motion for ocean waves predict energy transfer between, and phase

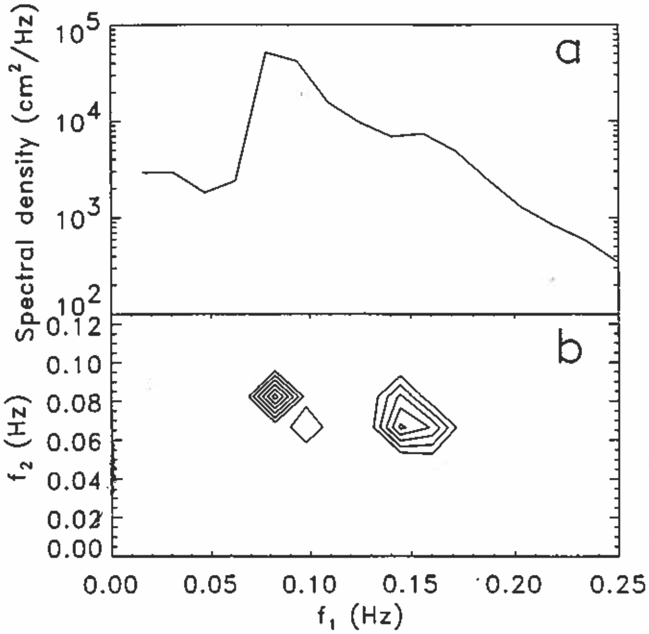

Fig. 8. (a) Power spectrum and (b) contours of tricoherence from ocean waves observed during Hurricane Lili. The tricoherences are for a fixed value of f_4, and thus, the contours indicate possible phase coupling between waves with frequencies f_1, f_2, f_4, and $f_3 = f_4 - (f_1 + f_2)$. The power spectral peak is approximately (a) $f_p = 0.08$ Hz and (b) $f_4 = 0.24$ Hz, which is approximately $3f_p$. The minimum value of $\hat{t}^2$ plotted is $\hat{t}^2 = 0.00025$, with contours every 0.00025. There are at least 3200 dof in the estimates, and the 90% significance level is thus about 0.0014. The measurements are discussed in more detail in [26].

coupling among, triads and quartets of waves owing to nonlinear quadratic and cubic wave-wave interactions, respectively [15], [25]. These nonlinear interactions result in non-Gaussian statistics of the ocean surface and are important in remote sensing applications. Detection of phase coupling between waves in the ocean is complicated because the wind-wave frequency-directional spectra are broad, and there can be a mix of independent and much less energetic phase-coupled waves at any particular frequency. Second-order waves resulting from quadratic interactions have been observed by many investigators (see [10]). Using the statistics of tricoherence derived above, the question of the existence of much weaker tertiary waves resulting from cubic nonlinear wave-wave interactions can be addressed. Tricoherences for large ocean surface gravity waves produced by Hurricane Lili (October 12, 1990) are shown in Fig. 8.

According to theory, a tertiary wave with $f = 0.24$ Hz ($\approx 3 f_p$, where $f_p = 0.08$ Hz is the frequency of the power spectral primary peak; Fig. 8(a)) could be generated by cubic nonlinear interactions between energetic waves with frequencies near f_p. As shown in Fig. 8(b), $\hat{t}^2(f_p, f_p, f_p) = 0.0017$, which is significantly different than zero at the 90% level for the (conservatively estimated) 3200 dof used in the estimates. Similarly, $\hat{t}^2(0.15, 0.07, 0.02) = 0.0013$ (significant at the 80% level), indicating possible phase coupling between waves with frequencies corresponding to nearly $2f_p$, f_p, a low frequency ($f_3 \approx 0.02$ Hz), and $3f_p$. On the other hand, the tricoherence peak corresponding to (0.10, 0.07, 0.07, 0.24) is not significantly different from zero. The oceanographic significance of these interactions is discussed in [12].

VII. Extensions

The estimate of the pth-order coherence of a stationary, ergodic random process may be defined as

$$\hat{p}^2(f_1, \dots f_p) = \frac{|\frac{1}{N}\sum_{i=1}^{N}[X_i(f_1)\cdots X_i(f_p)X_i^*(f_1 + \cdots + f_p)]|^2}{\hat{P}(f_1)\cdots \hat{P}(f_p)\hat{P}(f_1 + \cdots + f_p)}. \tag{26}$$

The proof for Gaussian noise in Section III can be easily generalized to show that estimates of all higher order coherences for Gaussian noise are approximately $\alpha\chi_2^2$, where $\alpha = \frac{1}{2N}$. The only modifications necessary to the proof are that

- there are 2^p terms instead of eight terms in the summations in (10)
- the product of powers is divided by 2^{p+1} instead of 16 in (12)
- consequently, (13) and the rest of the proof remain unchanged.

The proofs in the appendices and the equations derived for the statistics, the bias, and the variance of tricoherence, viz., (19), (22), and (23), respectively, are also applicable to any higher order coherence, with t^2 denoting the true value of the appropriate higher order coherence.

VIII. Conclusion

Statistics of the tricoherence estimate and expressions for bias and variance are derived analytically. The tricoherence for Gaussian noise is shown to be approximately $\alpha\chi_2^2$ distributed. The distribution of tricoherence for a harmonic random process with arbitrary true tricoherence is expressed as a function of the true value and two random variables, one of which is χ_2^2, whereas the other is Gaussian. Significance levels for zero tricoherence are shown to be the same as those for corresponding zero bicoherence levels. Numerical simulations confirm the validity of the analytically derived results. MLE's of the true tricoherence are shown to approach the observed values when the true value is above a certain level, which depends on the number of realizations averaged. The results are generalized to any higher order spectrum, showing that all higher order coherences will be approximately $\alpha\chi_2^2$ distributed for Gaussian noise, and they have similar statistical distributions for phase-coupled inputs.

Appendix A
Proof for Partially Phase-Coupled Harmonic Input

To analyze the partially phase-coupled case, split each Fourier coefficient included in the quartet of interest into a phase-coupled and a random-phase component

$$\begin{aligned} X_i(f_n) &= X_{ic}(f_n) + X_{ir}(f_n) \\ &= a_{icn} + jb_{icn} + a_{icr} + jb_{icr} \end{aligned} \tag{27}$$

where the subscripts c and r refer to the coupled and random components, respectively, subscript i refers to the ith realization, and n refers to the nth frequency. The periodogram estimate of the trispectrum, averaging over N realizations, is then

$$\begin{aligned}\hat{T}(f_1,f_2,f_3) &= \frac{1}{N}\sum_{i=1}^{N}[X_{ic}(f_1)X_{ic}(f_2)X_{ic}(f_3) \\ &\quad \times X_{ic}^*(f_1+f_2+f_3)] \\ &\quad + \frac{1}{N}\sum_{i=1}^{N}[X_{ir}(f_1)X_{ir}(f_2)X_{ir}(f_3) \\ &\quad \times X_{ir}^*(f_1+f_2+f_3) \\ &\quad + \text{similar terms in each of which at least} \\ &\quad \text{one factor is a random component}]. \\ &= \hat{T}_c(f_1,f_2,f_3)+\hat{T}_r(f_1,f_2,f_3)\end{aligned}$$

where $\hat{T}_c$ arises from only the phase-coupled components, whereas $\hat{T}_r$ includes contributions from random phase components as well. Further, assume that the random phase part has a uniform random phase in $[0, 2\pi)$. The proof for a Gaussian noise process in Section III also holds for a harmonic process with independent uniform random phase modes. If the true triphase is ϕ_k and $P_c(f_p)$ represents the coupled component of power at frequency f_p, then since

$$|\hat{T}_c(f_1,f_2,f_3)|^2 = P_c(f_1)P_c(f_2)P_c(f_3)P_c(f_4)$$

the phase-coupled component of the trispectrum estimate is

$$\hat{T}_c(f_1,f_2,f_3) = \sqrt{P_c(f_1)P_c(f_2)P_c(f_3)P_c(f_4)}e^{j\phi_k}. \quad (28)$$

where $f_4 = f_1 + f_2 + f_3$. Invoking the central limit theorem for a sum of i.i.d. random variables, when N is large

$$\hat{T}_r(f_1,f_2,f_3) = G_R + jG_I \quad (29)$$

where G_R and G_I are Gaussian random variables. They are zero-mean because every random phase component is assumed to be zero mean. Assuming that the phase of each individual mode is also symmetric (although sums of two or more phase-coupled components may have a nonzero mean, asymmetric distribution yielding nonzero biphase, triphase, etc.), the variances $\sigma_{G_R}^2$ and $\sigma_{G_I}^2$ of G_R and G_I, respectively, must be equal. Therefore

$$\begin{aligned}\sigma_{G_R}^2 = \sigma_{G_I}^2 &= \frac{E[\hat{T}_r(f_1,f_2,f_3)\hat{T}_r^*(f_1,f_2,f_3)]}{2} \\ &= \frac{1}{2N}[P_r(f_1)P_r(f_2)P_r(f_3)P_r(f_4)+\cdots \\ &\quad \text{all products of 4 powers with at least one factor} \\ &\quad \text{from a random phase mode}] \\ &= \frac{1}{2N}[P(f_1)P(f_2)P(f_3)P(f_4) \\ &\quad - P_c(f_1)P_c(f_2)P_c(f_3)P_c(f_4)]\end{aligned}$$

after some simplifications using statistical independence and zero-mean properties. The normalized trispectrum (dividing by the square root of the product of total powers as in (2)) can then be expressed as

$$\hat{T}(f_1,f_2,f_3) = te^{j\phi_k} + \frac{\sqrt{1-t^2}}{\sqrt{2N}}\left(\hat{G_R}+j\hat{G_I}\right) \quad (30)$$

where

$$t^2 = \frac{P_c(f_1)P_c(f_2)P_c(f_3)P_c(f_4)}{P(f_1)P(f_2)P(f_3)P(f_4)} \quad (31)$$

is the true value of tricoherence, and $\hat{G_R}$ and $\hat{G_I}$ are zero-mean, unit variance Gaussian random variables. The real and imaginary parts of the normalized trispectrum are

$$\Re[\hat{T}(f_1,f_2,f_3)] = t\cos\phi_k + \frac{\sqrt{1-t^2}}{\sqrt{2N}}\hat{G_R} \quad (32)$$

and

$$\Im[\hat{T}(f_1,f_2,f_3)] = t\sin\phi_k + \frac{\sqrt{1-t^2}}{\sqrt{2N}}\hat{G_I} \quad (33)$$

respectively. Using these expressions to deduce the statistical distribution of the estimates of tricoherence and triphase and making the substitution $C = \hat{G}_R^2 + \hat{G}_I^2$

$$\hat{t^2} = t^2 + \frac{(1-t^2)}{2N}C + \frac{2t\sqrt{1-t^2}}{\sqrt{2N}}(\cos\phi_k\hat{G_R}+\sin\phi_k\hat{G_I}) \quad (34)$$

where C is a χ_2^2 random variable, and

$$\hat{\phi}_k = \arctan\left(\frac{t\sin\phi_k + \frac{\sqrt{1-t^2}}{\sqrt{2N}}\hat{G_I}}{t\cos\phi_k + \frac{\sqrt{1-t^2}}{\sqrt{2N}}\hat{G_R}}\right). \quad (35)$$

Since a linear combination of Gaussian random variables is also Gaussian, the expression for the estimate of the tricoherence can be further simplified to

$$\hat{t^2} = t^2 + \frac{(1-t^2)}{2N}C + \frac{2t\sqrt{1-t^2}}{\sqrt{2N}}G \quad (36)$$

where G is a zero-mean, unit variance Gaussian random variable.

Appendix B
Derivation of Expressions for Bias and Variance

Assume the following:

1) The region over which random variables C and G are defined in (36) is rectangular and bounded by $C = 0$, $C = \infty$, $G = a$, and $G = b$, where $a < 0 < b$.
2) C and G are statistically independent.

Let p_C and p_G represent the probability density functions of C and G, respectively. The density p_C is chi-squared with two degrees of freedom, and p_G is Gaussian with zero mean and unit variance. The means and variances of C and G computed by integrating over the rectangular region described above are

$$\begin{aligned}E[G] &= \left(\int_a^b g p_G dg\right) \\ &= \frac{1}{\sqrt{2\pi}}\left(e^{-a^2/2} - e^{-b^2/2}\right) \quad (37)\end{aligned}$$

$$E[G^2] = \left(\int_a^b g^2 p_G dg\right) = \frac{1}{\sqrt{2\pi}}(ae^{-a^2/2} - be^{-b^2/2}) + \text{erf}(b) + \text{erf}(-a) \tag{38}$$

$$E[C] = \left(\int_0^\infty c p_C dc\right) = 2 \tag{39}$$

$$E[C^2] = \left(\int_0^\infty c^2 p_C dc\right). = 8 \tag{40}$$

Then, the expected value of the tricoherence estimate is

$$E[\hat{t^2}] \simeq t^2 + (1-t^2)\frac{1}{2N}\left(\int_0^\infty c p_C dc\right) + \frac{2t\sqrt{(1-t^2)}}{\sqrt{2N}}\left(\int_a^b g p_G dg\right). \tag{41}$$

Using (39) and (37), the bias of the estimate of the tricoherence can be written as

$$\text{bias}(\hat{t^2}) = E[\hat{t^2}] - t^2 \tag{42}$$

$$= \frac{1-t^2}{N} + \frac{t\sqrt{1-t^2}}{\sqrt{N\pi}}\left[e^{-a^2/2} - e^{-b^2/2}\right]. \tag{43}$$

The variance of the estimate of the tricoherence is

$$\text{var}(\hat{t^2}) = \left(\frac{1-t^2}{2N}\right)^2 \text{var}(C) + \left(\frac{2t\sqrt{1-t^2}}{\sqrt{2N}}\right)^2 \text{var}(G) \tag{44}$$

using the statistical independence of C and G. Using (37)–(40), the variance of the tricoherence estimate can be written as

$$\text{var}(\hat{t^2}) = \left(\frac{1-t^2}{N}\right)^2 + \frac{2t^2(1-t^2)}{N}\left[\frac{1}{\sqrt{2\pi}}(ae^{-a^2/2} - be^{-b^2/2}) + \text{erf}(b) + \text{erf}(-a) - \frac{1}{2\pi}(e^{-a^2/2} - e^{-b^2/2})^2\right]. \tag{45}$$

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Vinod Chandran (S'85-M'90) received the B.Tech. degree in electrical engineering from the Indian Institute of Technology, Madras, India, in 1982, the M.S.E.E. degree from Texas Tech University, Lubbock, in 1985, and the Ph.D. degree in electrical and computer engineering and the M.S. degree in computer science, both from Washington State University, in 1990 and 1991, respectively.

He is currently a lecturer with the School of Electrical and Elecronic Systems Engineering, Queensland University of Technology, Brisbane, Australia. His research interests include image processing, computer vision, higher order spectral analysis, nonlinear system theory and chaos.

Dr. Chandran is a member of the Optical Society of America, the Tau Beta Pi and Phi Kappa Phi, honor societies, and is an associate member of Sigma Xi.

Steve Elgar (M'86) received the Bachelors degrees in mathematics and civil engineering from the University of Idaho, Moscow, in 1980 and the Masters and Ph.D. degrees in oceanography from the Scripps Institution of Oceanography, University of California, San Diego, in 1981 and 1985, respectively.

In 1986, he joined the School of Electrical Engineering and Computer Science at Washington State University, Pullman. His current research interests involve the study of nonlinear waves and, in particular, ocean surface gravity waves and signal processing techniques necessary for analyzing ocean field measurements. In addition, his eclectic interests have included studies of irrigation canals, chaos in fluid, mechanical, and electrical systems, image processing, sediment motion near the beach, and global temperatures during the last 2–3 million years.

Dr. Elgar is a member of the American Geophysical Union (he is Ocean Sciences editor for *Eos, Transactions of the AGU*), the American Physical Society, the American Meteorological Society, the American Association for the Advancement of Science, and was an associate editor for IEEE TRANSACTIONS ON SIGNAL PROCESSING.

Barry A. Vanhoff received the B.S. and the M.S. degrees in electrical engineering from Washington State University, Pullman, in 1988 and 1989, respectively.

Since 1990, he has been working towards the Ph.D. degree in electrical engineering in the area of nonlinear time series analysis, primarily using higher order spectral analysis methods to study the evolution of ocean waves.