Groups of Ocean Waves: Linear Theory, Approximations to Linear Theory, and Observations

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Abstract: Statistics of wave groups observed for a wide range of wave heights, power-spectral shapes, and water depths are compared to the statistics predicted by both direct numerical simulation and analytic approximation of linear wave theory. Comparisons to numerical simulations show that the observed groups are not inconsistent with linear, Gaussian wave fields with the same spectra as the observations. Differences between ocean observations and linear theory are owing to statistical fluctuations in group statistics estimated with the 2.3-h-long data records. One linear analytic model accounts for correlations between two successive waves, and slightly underpredicts the average number of sequential large waves for wave fields with very narrow power spectra. A newly developed extension to Rice better accounts for multiwave correlations, but has only marginally improved accuracy for these data because wave fields with very narrow power spectra rarely occurred. Both approximations overpredict the group lengths for very broad and/or multipeaked power spectra, but are still useful because the errors are small with commonly occurring spectral shapes. Direct numerical simulations, with computational expense between that of the spectral-Kimura and extended-Rice approximations, yield the best predictions of the observed wave group statistics given a power spectrum, and also provide estimates of the statistical fluctuations of group properties about predicted mean values.

Introduction

Statistics of observed groups, or sequences, of waves exceeding a particular height are at least qualitatively consistent with linear theory (Andrew and Borgman 1981; Goda 1983; Elgar et al. 1984, 1985; Battjes and van Vledder 1984; Longuet-Higgins 1984; Thomas et al. 1986; Medina and Hudspeth 1990). Large deviations from linear theory occur only near and within the surf zone (Elgar et al. 1984). These investigations relied on relatively few data records of limited duration (typically less than an hour) and/or data from one particular location. The measurements considered in the present study were made in relatively deep (100 m) and shallow (10 m, seaward of the surf zone) water depths in the Pacific, and in an intermediate depth (19 m) in the Atlantic. The 278 data records span two orders of magnitude in wave energy, and include wave fields with narrow, broad, and multipeaked frequency spectra. Each data record is sufficiently long (2.3 h) to yield statistically stable estimates of the power spectrum and group properties of the wave field (Elgar et al. 1984).

The primary purpose of this study is to compare quantitatively these comprehensive observations of wave groups to direct numerical simulations

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of linear wave fields with the same (statistically) power spectra as the observations. No statistically significant discrepancies between ocean observations and linear dynamics were found.

A second purpose of this study is to compare predictions of two models, based on approximations to linear theory, to wave group statistics obtained from both the numerical simulations and the field observations. One model accounts for correlations between two successive waves (Kimura 1980; Battjes and van Vledder 1984). Van Vledder and Battjes (1992) have proposed that the wave groupiness parameter κ of this model be included in the IAHR list of sea state parameters ("List of" 1989), but verification of the model by comparison to a large range of field data is lacking ("Closure to," 1992). The second model [a newly developed extension to Rice (1944, 1945)] includes correlations between waves separated by an arbitrary number of intervening waves. Although additional assumptions beyond linearity of the wave field lead to biased model predictions in some cases, these errors are usually small for commonly occurring wave fields.

Linear Wave-Group Prediction Models

Approximations to linear theory that assume no correlation between the heights of successive waves (Goda 1970; Nagai 1973) yield inaccurate predictions of wave group statistics (Wilson and Baird 1970; Rye 1974; Thompson 1981; Battjes and van Vledder 1984; Elgar et al. 1984), and will not be considered here. Other approximations which account for the correlation between neighboring waves (Rice 1944, 1945; Kimura 1980), or waves separated by an arbitrary number of intervening waves (extended-Rice model) are described herein.

Kimura's Approximation

Kimura (1980) accounted for the correlation between two successive waves by modeling the time series of wave heights as a discrete, first-order autoregressive process. The partial autocorrelation function [e.g., Jenkins and Watts (1968)] for waves separated by more than one intervening wave is assumed to be zero. For a sea surface with a narrow power spectrum, wave heights are Rayleigh distributed (Rice 1944, 1945; Longuet-Higgins 1952), with the individual probability distribution of wave heights, $Q(H_1)$ and the joint probability distribution, $P(H_1, H_2)$ for two successive wave heights H_1 and H_2 given by

$$Q(H_1) = \frac{2H_1}{H_{rms}^2} \exp\left(-\frac{H_1^2}{H_{rms}^2}\right) \quad \text{(1a)}$$

$$P(H_1, H_2) = \frac{4H_1H_2}{(1-\kappa^2)H_{rms}^4} \exp\left(\frac{1}{1-\kappa^2}\frac{H_1^2+H_2^2}{H_{rms}^2}\right) I_0\left[\frac{2H_1H_2\kappa}{(1-\kappa^2)H_{rms}^2}\right] \quad \text{(1b)}$$

where κ = a correlation parameter; I_0 = modified Bessel function of zeroth order; and H_{rms} = root-mean-square wave height. The correlation coefficient, r_{HH}, between H_1 and H_2 is a function of κ (Battjes 1974; Kimura 1980)

$$r_{HH} = \frac{E(\kappa) - \left[\frac{(1-\kappa^2)K(\kappa)}{2}\right] - \left[\frac{\pi}{4}\right]}{1 - \frac{\pi}{4}} \approx 0.91\,\kappa^2\left(1 + \frac{\kappa^2}{16} + \frac{\kappa^4}{64}\right) \quad (2)$$

where K and E = complete elliptic integrals of the first and second kinds, respectively. Given the observed (or simulated) correlation coefficient r_{HH}, κ can be obtained numerically from (2).

The probability, p_{22} that both H_1 and H_2 exceed a threshold height H_c, given that H_1 exceeds H_c, is

$$p_{22} = \frac{\int_{H_c}^{\infty}\int_{H_c}^{\infty} P(H_1, H_2)\, dH_1\, dH_2}{\int_{H_c}^{\infty} Q(H_1)\, dH_1} \quad (3)$$

The probability of j successive waves greater than H_c (i.e., a run of length j) is

$$P(j) = p_{22}^{j-1}(1 - p_{22}) \quad (4)$$

The mean run length, $E[j]$ and the standard deviation, $\sigma[j]$ about the mean run length, are (Kimura 1980)

$$E[j] = \frac{1}{1 - p_{22}} \quad (5)$$

$$\sigma[j] = \frac{p_{22}^{1/2}}{1 - p_{22}} \quad (6)$$

Spectral Version of Kimura's Approximation

Kimura (1980) determined the correlation coefficient [r_{HH}, in (2)] in the time domain from the observed sequence of wave heights, which is an effort roughly equivalent to determining directly the desired wave group statistics. Therefore, Kimura's time-domain approximation is not considered further. However, drawing on earlier work by Arhan and Ezraty (1978) on correlations of joint probability density functions and by Rice (1944, 1945) on theoretical envelope statistics, Battjes and van Vledder (1984) noted that the correlated parameter κ (1) can be determined directly from the power-spectrum $S(f)$

$$\kappa = \frac{1}{m_0}\left\{\left[\int_0^{\infty} S(f)\cos(2\pi f T_m)\, df\right]^2 + \left[\int_0^{\infty} S(f)\sin(2\pi f T_m)\, df\right]^2\right\}^{1/2} \quad (7)$$

where $T_m = (m_0/m_2)^{1/2}$ = mean wave period; and m_0 and m_2 = zeroth and second moments of the spectrum about the origin, respectively [see also Longuet-Higgins (1984)]. This method of estimating group statistics given a target power spectrum will be referred to as the spectral version of Kimura's approximation. Note that the spectral estimate [(7)] is the average value of κ expected for infinitely many realizations of the target power spectrum while κ from a single time series is only one realization.

Rice's Approximation

Following Rice (1944, 1945), the time series (sea-surface elevation in this case) $I(t)$ is represented as a sum of $N(N >> 1)$ Fourier amplitudes, c_n and phases, ϕ_n

$$I(t) = \sum_{n=1}^{N} c_n \cos(\omega_n t - \phi_n) \quad \text{(8a)}$$

$$c_n = [2S(f_n)\Delta f]^{1/2} \quad \text{(8b)}$$

The random phases are uniformly distributed over the range $(0, 2\pi)$, and the radian frequency of mode n is $\omega_n = 2\pi f_n$ and $f_n = n\Delta f$, where Δf is the frequency resolution. $I(t)$ can be expressed as a wave with a midband frequency ω_m modulated by an envelope $R(t)$ [(10)–(13)]. For narrow-band spectra, the envelope is slowly varying and Rayleigh distributed.

The correlation between points on the envelope separated by a wave period, $R(t)$ and $R(t + \tau)$, corresponds to the correlation between two successive waves, r_{HH}, and is equivalent to (2) (Kimura 1980). However, Rice's results can be extended to include nonzero partial correlations between waves separated by an arbitrary number of intervening waves, as detailed in Appendix I. The probability of a run of length l predicted by the extended-Rice model [(20)] converges to Kimura's prediction [(5)] when κ is not large and multiwave partial correlations are negligible, and in the limit of $\kappa \rightarrow 0$ both models collapse to Goda's (1970) result based on uncorrelated wave heights.

Numerical Simulations

Realizations of time series of a sea-surface with a given target spectrum $S(f)$ can be numerically simulated by coupling random phases with deterministic Fourier amplitudes [(8)], followed by an inverse Fourier transform. Alternatively, the wave field can be represented as

$$I(t) = \sum_{n=1}^{N} a_n \cos(\omega_n t) + b_n \sin(\omega_n t) \quad \text{(9)}$$

where the Fourier coefficients a_n and b_n are Gaussian distributed with zero mean and variance $S(f)\Delta f$. The resulting simulated time series have random phases and spectra with the appropriate statistical fluctuations about $S(f)$. When the number of Fourier modes (N) is large, the two simulation schemes [(8) and (9)] produce time series with nearly identical wave group statistics (Rice 1944, 1945; Elgar et al. 1985). For the comparisons presented here, 1,000 realizations of wave fields were generated for each 2.3 h ocean record using the random Fourier coefficient scheme [(9)], with the target $S(f)$ provided by the unsmoothed, observed ocean $S(f)$. An inverse Fourier transform of each realization of random Fourier coefficients yields a simulated time series with the same time step and 2.3-h-duration as the corresponding ocean observation. The numerical simulations provide estimates of mean group statistics predicted by linear theory by averaging over the 1,000 realizations. Additionally, the simulations provide estimates of the statistical uncertainty of group statistics estimated from a 2.3 h ocean record, allowing it to be shown that differences between ocean observations and mean values predicted by linear theory are probably owing to statistical fluctuations rather than deviations from linearity. The simulations are essentially a brute force calculation of wave group statistics as predicted by linear theory given a target spectrum, without additional approximations.

FIELD DATA

Data were obtained from a bottom-mounted pressure sensor located in approximately 10-m water depth near Santa Barbara, Calif. in January and February 1980 (Gable 1981). The bottom pressures were converted to sea-surface elevation using linear finite depth theory. A second data set was obtained between March 15, 1984 and May 1, 1985 from a Datawell Waverider buoy moored in 110 m depth near Begg Rock, west of San Nicholas Island (33°24.4′ N, 119°40.1′ W) off the coast of Southern California (Seymour et al. 1985). A third data set was obtained between September 1, 1990 and May 31, 1991 from a Waverider buoy located in 19-m water depth off the coast of North Carolina (36°11.5′ N, 75°44.4′ W) (Leffler et al. 1990).

A total of 278 records, each of 8,192 s (2.3 h) duration, were selected for processing. The processed data include one record every other day from Begg Rock and North Carolina, and records near high tide from Santa Barbara. Record lengths of 8,192 s typically contain ≈800 waves, long enough to yield reasonably stable statistics of groups with the threshold height equal to the significant wave height (Elgar et al. 1984).

Each time series was band-passed filtered between 0.04 and 0.3 Hz and 0.04 and 0.4 Hz for the Pacific and Atlantic Ocean data, respectively. Significant wave heights (defined here as $4m_0^{1/2}$) ranged from 20 to 500 cm, and mean periods (corresponding to the centroidal frequency of the power spectrum) from 4 to 15 s. Correlations between successive zero-upcrossing wave heights (r_{HH}) varied between 0.01 and 0.60 (Fig. 1), with the highest values corresponding to $\kappa \approx 0.8$ [(2)] and very highly grouped waves in long travelled swell (Goda 1983; Thomas et al. 1986). The spectral shapes ranged from very narrow band swell [upper panel of Fig. 2(*a*), 10 m depth, Santa Barbara, Calif., February 2, 1980], to broad band, locally generated sea

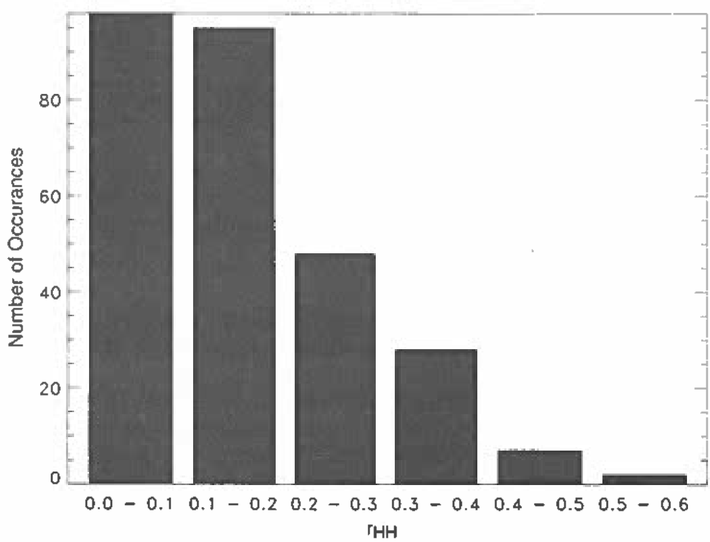

FIG. 1. Number of Occurrences of r_{HH}, One-Wave Correlation

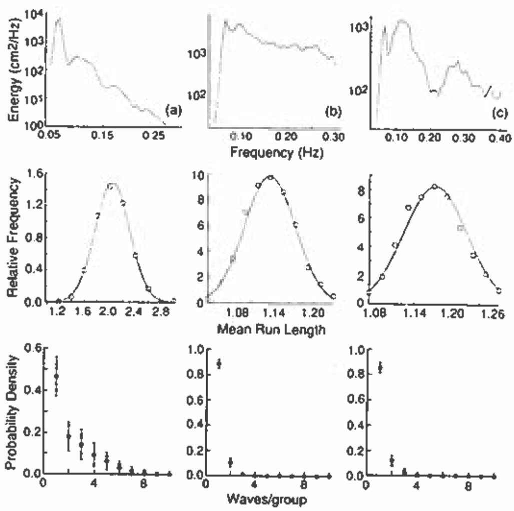

FIG. 2. Representative Power Spectra, Probability Density of Simulated Mean Group Lengths, and Distribution of Number of Waves per Group (Upper, Middle, and Lower Panels, Respectively)

TABLE 1. Parameters for Three Case Studies Shown in Fig. 2

Data set observation date (1)	Correlation parameter κ (2)	Observed mean group length (3)	Simulated mean group length[a] (4)	z statistic for mean group length (5)	Observed variance of group length (6)	Simulated variance of group length (7)	F statistic for group length variance (8)
February 2	0.79	2.000	2.040	0.15	2.200	2.380	1.09
October 17	0.34	1.133	1.129	0.10	0.138	0.141	1.03
January 14	0.41	1.212	1.172	0.83	0.317	0.199	0.63

[a]Averaged over 1,000 realizations.
Note: Threshold wave height is equal to significant wave height.

[Fig. 2(*b*), 100 m depth, Begg Rock, Calif., October 17, 1984], to multi-peaked combinations of sea and swell [Fig. 2(*c*), 19 m depth, North Carolina, January 14, 1991]. Wave group statistics and other parameters of these examples are listed in Table 1.

RESULTS

Observed mean group lengths (with the threshold equal to the significant wave height) are correlated with predictions given by the direct numerical simulations [Fig. 3(*a*), upper panel], with no apparent bias [Fig. 3(*a*), lower panel]. Mean group lengths from individual realizations of each target spectrum are approximately Gaussian distributed about the mean of the 1,000 realizations (Elgar et al. 1984), as illustrated in the center panels of Fig. 2 (simulated values are shown as circles and the solid line is a Gaussian distribution). To test if the observed (ocean) and simulated mean group lengths come from the same statistical population a z statistic for each ocean

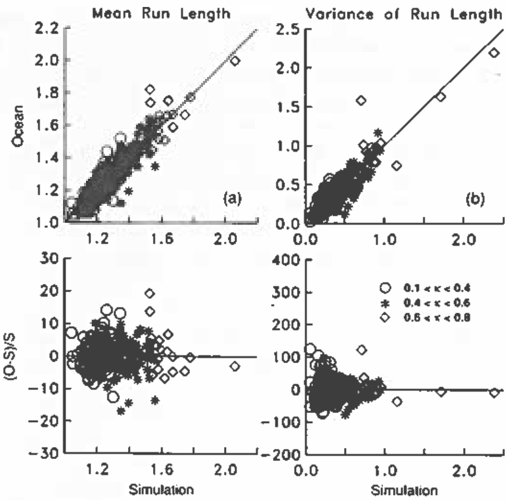

FIG. 3. Observed and Simulated Group Statistics: (*a*) Observed Mean Run Length (Above) and Percent Difference from Simulated (Below) versus Simulated Mean Run Lengths; and (*b*) Observed Variance of Runs (Above) and Percent Difference (Below) versus Simulated Mean Run Length

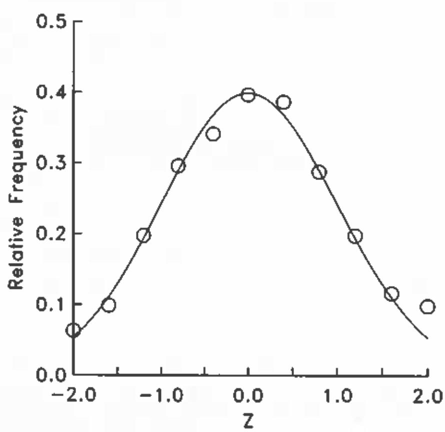

FIG. 4. Frequency of Occurrence of Normalized z Statistic for Mean Group Lengths

data set was calculated. The z statistic is the difference between the observed (ocean) average group length and the global mean (over the 1,000 realizations) of the simulated average group lengths, normalized by the standard deviation of the 1,000 simulated average group lengths from the global mean. The z statistic is thus the deviation of the observed ocean group length from the global mean of the simulations, in units of standard deviations. The observed ocean mean and z statistic are given in Table 1 for each case shown in Fig. 2. For these three cases, observed mean group lengths are within 0.8 standard deviations of the simulated mean group lengths. The entire collection of 278 z values are not inconsistent with the hypothesis that the ocean groups represent linear dynamics (Fig. 4). For example, 89.9% of the z values (shown as symbols in Fig. 4) were less than 1.6 in absolute value, compared to the expected 90.0% (solid line in Fig. 4) if the observed values came from the same population as the simulations and there were a very large number of 2.3-h-long ocean records.

Although it is possible to have similar observed ocean and simulated mean group lengths, but different underlying distributions of group lengths, this is not the case here. The observed and simulated distributions of wave group lengths for the three example spectra are similar (Fig. 2, lower panels, where the circles are the mean from 1,000 simulations, the bars are ± 1 standard deviation from the simulations, and stars are the observed values). The data set with a narrow power spectrum [Fig. 2(*a*)] has a relatively long mean group length, and a fairly wide range of group lengths, including runs as long as six waves. Groups with length 1 dominate the data sets with broad [Fig. 2(*b*)] or multipeaked [Fig. 2(*c*)] spectra. The width of the distribution of group lengths, in a given realization, is quantified by the variance of group length about the mean group length of that realization. If all groups in a realization have the same length, the variance is zero. Thus, the variance of lengths of wave groups for narrow band wave fields is much larger than the variance for broad band or multipeaked wave fields (Fig. 2 and Table 1).

The observed (ocean) and simulated (averaged over 1,000 realizations) group length variances are qualitatively similar [Fig. 3(*b*)]. F statistics suggest that the observed (ocean) and simulated variances come from the same population (not shown, similar to the z statistic for means, Fig. 4). Of 278 F values, 29 differed at the 90% level, compared to the expected 28. Previous comparisons between linear simulations and a limited set of about 30 ocean observations (Elgar et al. 1984) demonstrated that, as well as the means and variances considered here, the distributions of observed run length (i.e., the lower panels of Fig. 2) came from the same statistical population as the simulations.

Subsets of the ocean data, as well as the entire data set, pass statistical tests for consistency with linear theory. The data were divided into subgroups based on field site κ and wave steepness parameters, and values of z and F calculated for each subgroup. The data at each of the three individual field sites are consistent with linear theory, as are the data sets subdivided according to κ ($0.10 < \kappa < 0.40$, $0.40 < \kappa < 0.60$, and $0.60 < \kappa < 0.80$). Finally, the effect of wave steepness on observed wave group statistics was investigated by subdividing the data into three groups based on $ak/\tanh(kh)$, and another three groups based on $ak/\tanh(kh)^3$ (for both steepness parameters the three subgroups consisted of values less than 0.05, 0.05–0.10, and greater than 0.10). In all cases, the hypothesis that the data come from the same population as the linear simulations cannot be rejected.

The comparisons of observed and simulated mean and variance of group lengths indicate that for the relatively wide range of conditions observed, ocean wave groups are not inconsistent with linear dynamics. This does not imply that other statistics (both group and otherwise) are necessarily also insensitive to nonlinearity.

Wave group statistics predicted by the spectral-Kimura and extended-Rice approximations were also compared to the numerical linear simulations. Target power spectra (the ocean observations) were used to calculate κ [(7)], and the corresponding spectral-Kimura predictions of mean group length [(5)] and variance [(6)] were compared to the linearly simulated values

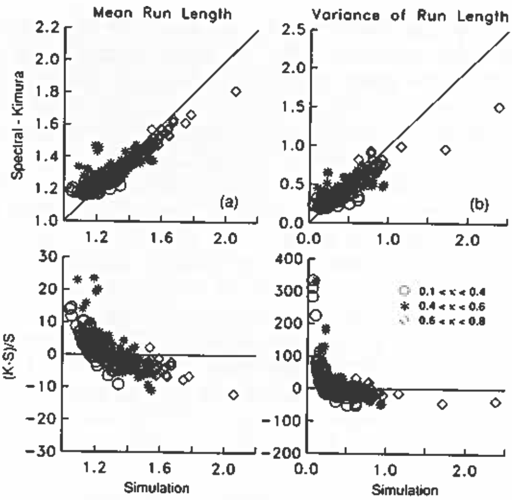

FIG. 5. Same as Fig. 3 Except Comparisons are between Spectral-Kimura and Simulations

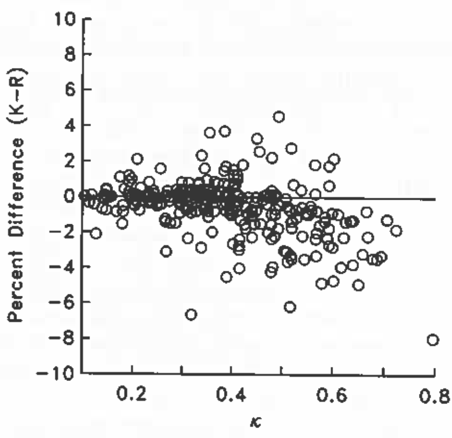

FIG. 6. Percent Difference between Mean Run Length Predicted by Extended-Rice (*R*) and Spectral-Kimura (*K*) versus κ

(Fig. 5). Scatter plot comparisons (not shown) of the linear simulations with predictions of the extended Rice model [obtained from (20)] are very similar to Fig. 5. Differences between extended-Rice and spectral-Kimura predictions of mean group lengths, for the same target spectrum, are shown in Fig. 6. There is less random scatter in the comparisons between linear simulation and analytic model predictions (Fig. 5) than between simulation and ocean observation (Fig. 3) because both the simulations and analytic models predict mean values, while the ocean observations are one realization of a random process. However, the model predictions of mean and variance are clearly biased (Fig. 5).

For low values of κ the mean group lengths predicted by the models differ from each other by less than 2% (Fig. 6), and both models overpredict the simulated mean and variance (Fig. 5). Both Kimura and extended-Rice models approximate the group structure as Rayleigh distributed wave heights modulated by a slowly varying envelope. The underlying assumption of a narrow-banded spectrum is violated for wave fields with small mean group lengths (associated with broad or multipeaked power spectra and/or low values of κ), so the breakdown of model predictions (Fig. 5) for small group lengths is not unexpected. Wave heights from all the data sets separated into each of three categories ($0.10 < \kappa < 0.40$, $0.40 < \kappa < 0.60$, and $0.60 < \kappa < 0.80$) were pooled, and their probability distributions compared to a Rayleigh distribution. There are small, but systematic deviations from a Rayleigh distribution for low κ that may contribute to the observed bias in the group statistics such as that shown in Fig. 5(*a*). A consequence of the assumption of Rayleigh distributed wave heights is that Goda (1970), Kimura (1980), and extended-Rice models all predict a minimum group length (for H_c = the significant wave height) of ≈ 1.16, while the observed mean group length approaches 1.0 for small κ, resulting in the $\approx 20\%$ bias [Fig. 5(*a*)].

Since Kimura's approximation is based on the assumption that the partial correlation is nonzero only for neighboring waves the predictions are expected to become less accurate for wave fields with high values of κ. In these cases with narrow power spectra and longer group lengths, nonzero partial correlations extend beyond neighboring waves, and the spectral-Kimura model tends to underpredict the mean group length, as shown in Fig. 5(a) and observed by Battjes and van Vledder (1984). In addition, errors in the spectral-Kimura model may occur because κ is systematically underestimated by the spectral method (Closure to 1992) as was demonstrated for the present data by comparing κ calculated from the spectrum [(7)] with κ calculated by first determining r_{HH} from discrete counting of waves in the time series and then solving (2) for κ (not shown). Although the extended-Rice approximation accounts for partial correlations between waves separated by intervening waves, and predicts slightly (1–6%, Fig. 6) longer group lengths than spectral-Kimura for $\kappa > 0.60$ (simulated group length greater than 1.6), wave fields with these very narrow spectra occurred rarely in these data. Systematic differences in modeled mean group lengths with increasing κ are barely detectable. Consistent with these small differences, van Vledder (1983) suggested that the mean group length depends only on κ. Even for the largest value of κ in these data, the distribution of the number of waves per group (as in the lower panels of Fig. 2) predicted by spectral-Kimura and extended-Rice models are very similar.

Although the spectral-Kimura (Fig. 7) and Rice (not shown, very similar to Fig. 7) predictions are clearly biased when compared to ocean data, these biases are not large compared to the variations in mean group length associated solely with random fluctuations of group statistics occurring in time series of 2.3 hour duration (compare the lower panels of Figs. 3 and 7). The root mean square deviation (RMS, see Table 2) between numerically simulated mean group lengths and spectral-Kimura (Rice) model predictions (differences caused by model bias only) is 0.062 (0.051), about the same as the RMS difference between numerically simulated and observed group lengths (0.061) caused by statistical fluctuations only. RMS differences between the analytic models and ocean data (owing to both model bias and statistical fluctuations, Table 2) are only about 50% larger than the purely statistical deviations between numerical simulations and observations. Com-

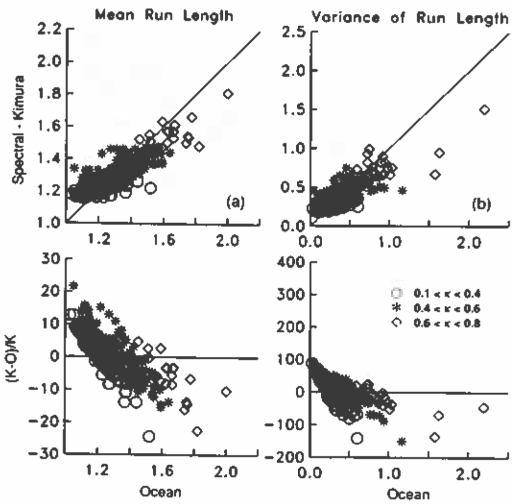

FIG. 7. Same as Fig. 3 Except Comparisons are between Spectral-Kimura and Ocean Observations

TABLE 2. RMS Differences between Group Statistics Observed and Predicted by Models

Items compared (1)	RMS (mean) (2)	RMS (variance) (3)
Observation versus simulation	0.061	0.118
Kimura versus simulation	0.062	0.132
Rice versus simulation	0.051	0.096
Kimura versus observation	0.088	0.159
Rice versus observation	0.076	0.131

Note: Columns labeled RMS (mean) and RMS (variance) are RMS differences between mean group lengths and variance of group length, respectively.

parison of simulated, ocean, and model variances (Table 2) yields similar results. Thus, although the bias at low (for spectral-Kimura and Rice) and high (spectral-Kimura) values of κ degrades the model predictions of the ocean observations, the bias is not substantially greater than the statistical fluctuations associated with records of 2.3-h duration.

The statistics discussed above are for a threshold level equal to the significant wave height. Predicted (by spectral-Kimura) and observed mean group lengths as functions of κ for three thresholds (H_{mean}, H_{sig}, and $H_{1/10}$) are shown in Fig. 8. The increased scatter of $H_{1/10}$ about the spectral-Kimura theory [Fig. 8(*c*)] relative to that of H_{sig} and H_{mean} [Fig. 8(*a*) and (*b*)] is expected because the relatively small number of groups of ocean waves larger than $H_{1/10}$ in a 2.3-h record increases the statistical uncertainty of the observed mean. Overall, the agreement is comparable for all three thresholds.

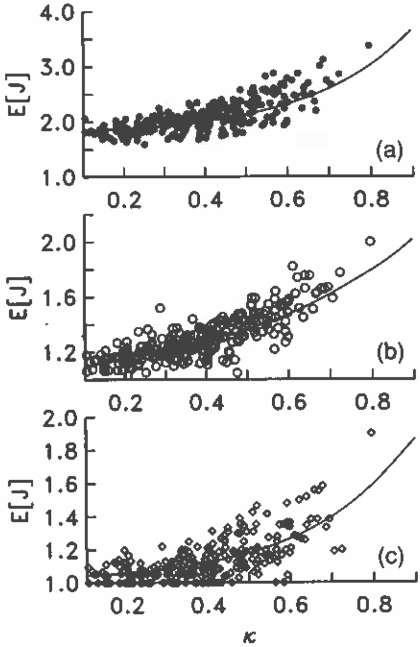

FIG. 8. Mean Length of Runs $E(j)$ versus Correlation Parameter κ for Thresholds: (a) H_{mean}, (b) H_{sig}, and (c) $H_{1/10}$

Analysis

In addition to extended-Rice and spectral-Kimura models, observed (ocean) mean group lengths were compared to predictions based on the average duration the envelope is above the threshold (Rice 1958; Vanmarke 1972; Ewing 1973). According to this model the mean group length depends on m_2/μ_2 where m_2 and μ_2 are second moments of the spectrum about the origin and about the centroid, respectively. However, the observed mean group lengths (for thresholds corresponding to H_{mean}, H_{sig}, and $H_{1/10}$) were practically uncorrelated with both m_2/μ_2 and the associated model predictions. Although the m_2/μ_2 model is valid for wave fields with narrow, unimodal power spectra, and is equivalent to Kimura's theory for pathologically narrow spectra (Longuet-Higgins 1984) m_2/μ_2 model predictions are significantly corrupted by typical deviations from this idealized spectral shape (Elgar et al. 1984). In contrast, κ is robust.

It has been suggested [Nolte and Hsu (1974); Longuet-Higgins (1984); and elsewhere] that simulated or observed wave data must be band-passed filtered over a relatively narrow frequency range around the spectral peak prior to considering wave group statistics. If "groups" are strictly defined as conforming to the theoretical construct of a slowly modulated wave train with a distinct carrier frequency, then indeed "groups" do not even exist for wave fields with broad or multipeaked frequency spectra. On the other hand, the average number of consecutive high waves (defined here as the mean group length) is of practical interest for both broad- and narrow-band wave fields. Although some filtering is usually necessary (e.g., high-frequency capillary waves are unimportant in many applications), the appropriate filter width cannot be determined a priori, but depends on the application for which group statistics are desired. Here, the filter low-frequency cutoff was 0.04 Hz and the high-frequency cutoff (0.3 or 0.4 Hz) corresponds to about two to five times f_p (f_p is the frequency corresponding to the centroid of the power spectrum). Application of the narrower $0.5-1.5f_p$ filter recommended by Longuet-Higgins (1984) effectively removes any nonlinear effects owing to bound waves at harmonic frequencies $2f_p$, increases κ and the average group length, produces time series with slowly varying wave heights, and usually yields improved agreement between data (or simulations) and analytic models because the models assume narrow-band (but not too narrow) power spectral shapes. On the other hand, in some cases the narrowly filtered time series would bear little resemblance to the original unfiltered ocean data.

Conclusions

Comparisons of statistics of wave groups observed in the ocean with those predicted by numerical simulations demonstrate that ocean wave group statistics are not inconsistent with a linear, Gaussian sea surface. The deviations between observed and predicted statistics are expected owing to the finite record length (2.3 h) of the observations (Figs. 2–4).

Wave group statistics predicted by models based on approximations to linear theory are biased for some parameter ranges. Kimura's approximation, based on the assumptions of Rayleigh distributed wave heights and nonzero partial correlation only between neighboring waves, underpredicts (by roughly 15% when $\kappa \approx 0.65$) the mean group lengths for very narrow spectra (Fig. 5). In these relatively rare cases, waves separated by more than one intervening wave have significant partial correlations, and the extended-Rice approximation provides slightly better predictions. For very broad-band power spectra the spectral-Kimura and extended-Rice approximations are similar (small κ in Fig. 6), and overpredict the mean and variance of group length (Fig. 5). However, the errors are not very large (Fig. 5) and when $\kappa > 0.3$ the biases are approximately the same size (<10% error in mean group length) as statistical fluctuations in wave group statistics from observed time series of 2.3-hour duration (Fig. 3). The field observations support the use of κ [(7)] as a simple and unbiased parameter relating wave spectra and groupiness, as suggested by van Vledder and Battjes (1992).

Rice's approximation is computationally expensive, while the spectral-Kimura approximation is numerically simple. Direct numerical simulations, with computational expense between that of the spectral-Kimura and Rice approximations, provide the most accurate predictions of wave group statistics given a power-spectral shape, and also provide information about the expected statistical fluctuations about the predicted mean values.

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Appendix I. Extension of Rice Theory

In the following, Rice's theory (1944, 1945) is extended to establish the probability distribution of the number of sequential large waves for the case of nonzero partial correlations between waves separated by an arbitrary number of intervening waves. Beginning with equation (8) random noise with a narrow band spectrum can be represented by (Rice 1944, 1945)

$$I(t) = \sum_{n=1}^{N} c_n \cos(\omega_n t - \omega_m t - \phi_n + \omega_m t) \quad \text{(10a)}$$

$$I(t) = I_c(t)\cos(\omega_m t) - I_s(t)\sin(\omega_m t) \quad \text{(10b)}$$

where

$$I_c(t) = \sum_{n=1}^{N} c_n \cos((\omega_n - \omega_m)t - \phi_n) \quad \text{(11)}$$

$$I_s(t) = \sum_{n=1}^{N} c_n \sin((\omega_n - \omega_m)t - \phi_n) \quad \text{(12)}$$

and the envelope $R(t)$ is given by

$$R(t) = [I_c^2(t) + I_s^2(t)]^{1/2} \quad \text{(13)}$$

For convenience, let

$$I_c(t) = R(t)\cos\theta(t) \quad \text{(14a)}$$

$$I_s(t) = R(t)\sin\theta(t) \quad \text{(14b)}$$

The correlation between $R(t)$ and $R(t + \tau)$ (where τ is the wave period) is equivalent to the correlation between successive waves. To extend Rice's (1944, 1945) results to the case of nonzero partial correlations between nonneighboring waves the probability density of the envelope at times t, $t + \tau$, $t + 2\tau$, $\cdots$, is needed. First, consider the joint probability of I_{c1}, I_{c2}, $\cdots$, I_{cn}, I_{s1}, I_{s2}, $\cdots$, I_{sn}, where the subscripts 1, 2, $\cdots$, n refer to $I_c(t)$, $I_c(t + \tau)$, $\cdots$, $I_c(t + (n - 1)\tau)$, etc. The $2n$ random variables, I_{c1}, I_{s1}, I_{c2}, I_{s2}, $\cdots$, I_{cn}, I_{sn}, have a $2n$-dimensional normal joint density given by

$$P(I_{c1}, I_{s1}, \cdots, I_{cn}, I_{sn}) = \frac{1}{(2\pi)^{2n/2}|M|^{1/2}} \exp\left(-\frac{1}{2} I^T M^{-1} I\right) \quad \text{(15)}$$

where $I = (I_{c1}, I_{s1}, \cdots, I_{cn}, I_{sn})^T$ is a column vector. The second moment matrix M (with size $2n * 2n$) is

$$M = \begin{Bmatrix} \overline{I_{c1}I_{c1}} & \overline{I_{c1}I_{s1}} & \cdots & \overline{I_{c1}I_{cn}} & \overline{I_{c1}I_{sn}} \\ \overline{I_{s1}I_{c1}} & \overline{I_{s1}I_{s1}} & \cdots & \overline{I_{s1}I_{sn}} & \overline{I_{s1}I_{sn}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{I_{sn}I_{c1}} & \overline{I_{sn}I_{s1}} & \cdots & \overline{I_{sn}I_{cn}} & \overline{I_{sn}I_{sn}} \end{Bmatrix} \quad \text{(16)}$$

Note that

$$\overline{I_{c1}^2} = \overline{I_{s1}^2} = \cdots = \overline{I_{cn}^2} = \overline{I_{sn}^2} = \int_0^\infty S(f)\, df = m_0 \quad \text{(17a)}$$

$$\overline{I_{c1}I_{s1}} = \overline{I_{c2}I_{s2}} = \cdots = \overline{I_{cn}I_{sn}} = 0 \quad \text{(17b)}$$

$$\overline{I_{ci}I_{cj}} = \overline{I_{si}I_{sj}} = \int_0^\infty S(f)\cos 2\pi f(j-i)\tau\, df \quad \text{(17c)}$$

$$\overline{I_{ci}I_{sj}} = -\overline{I_{si}I_{cj}} = \int_0^\infty S(f)\sin 2\pi f(j-i)\tau\, df \quad \text{(17d)}$$

where $S(f)$ is the energy density spectrum.

Changing variables from I_c and I_s to R and θ in (15), yields

$$P(R_1, R_2, \cdots, R_n, \theta_1, \theta_2, \cdots, \theta_n) = \frac{1}{(2\pi)^{2n/2}|M|^{1/2}} \exp\left(-\frac{1}{2} R_\theta^T M^{-1} R_\theta\right) \quad \text{(18)}$$

where $R_\theta = (R_1 \cos\theta_1, R_1 \sin\theta_1, \cdots, R_n \cos\theta_n, R_n \sin\theta_n)^T$. The probability that any n successive waves exceed the critical value R_c is given by

$$P(R_1 > R_c, R_2 > R_c, \cdots, R_n > R_c)$$

$$= \int_{R_c}^\infty \int_{R_c}^\infty \cdots \int_{R_c}^\infty \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} R_1 R_2 \cdots R_n P(R_1, R_2, \cdots,$$

$$\cdot R_n, \theta_1, \theta_2, \cdots, \theta_n)\, dR_1\, dR_2 \cdots dR_n\, d\theta_1\, d\theta_2 \cdots d\theta_n \quad \text{(19)}$$

The probability of a group with length l is equal to the probability of groups that have only l waves exceeding the critical value divided by the probability of groups with one or more waves exceeding the critical value

$$P(l) = \frac{P(R_1 > R_c, \cdots, R_l > R_c) - P(R_1 > R_c, \cdots, R_l > R_c, R_{l+1} > R_c)}{P(R_1 > R_c)} \quad \text{(20)}$$

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