# Auto and Cross-Bispectral Analysis of a System of Two Coupled Oscillators With Quadratic Nonlinearities Possessing Chaotic Motion

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*Auto and cross-bispectral analyses of a two-degree-of-freedom system with quadratic nonlinearities having two-to-one internal (autoparametric) resonance are presented. Following the work of Nayfeh (1987), the method of multiple scales is used to obtain a first-order uniform expansion yielding four first-order nonlinear ordinary differential equations governing the modulation of the amplitudes and phases of the two modes. The particular case of parametric resonance of the first mode considered in this paper admits Hopf bifurcations and a pure period doubling route to chaos. Auto bicoherence spectra isolate the phase coupling between increasing numbers of triads of Fourier components for a pure period doubling route to chaos for the individual degrees-of-freedom. Cross-bicoherence spectra, on the other hand, yield information about the phase coupling between the two degrees-of-freedom. The results presented here confirm the capacity of bispectral techniques to identify a quadratically nonlinear mechanical system that possesses chaotic motions. For the chaotic case, cross-bicoherence spectra indicate that most of the nonlinear energy transfer between the modes is owing to cross-coupling between phase modulations rather than between amplitude modulations.*

## 1 Introduction

Polyspectral methods present detailed information about the nonlinear modal couplings present in a given system. Bispectral analysis has been used to study a wide variety of quadratic nonlinear systems, including fluid (Yeh et al., 1973; Li et al., 1976; Helland et al., 1977; Van Atta, 1979; Kim et al., 1980; Ritz et al., 1988; Choi et al., 1984), mechanical (Sato et al., 1977), and a quantum mechanical systems (Miller, 1986). Nikias and Raghuveer (1987) provide a recent review. These higher-order spectral analysis techniques provide information about a chaotic system complementary to that obtained with other methods of dynamical system analyses, such as fractal dimension (Farmer et al., 1983) and Lyapunov exponent calculations (Wolf et al., 1985).

Power spectral techniques are adequate for the analyses of linear systems, but do not, however, provide information about nonlinear interactions between Fourier components in a nonlinear system. Higher-order spectra, on the other hand, can isolate and quantify the phase coupling between nonlinearly interacting Fourier components. Bispectral analyses of the quadratic interactions that produce a pure period-doubling sequence to chaos in the Rossler equations resulted in successful identification of this system for both nonchaotic and chaotic cases (Pezeshki et al., 1990). Further application to mechanical systems, such as the magnetically buckled beam governed by a Duffing equation was useful for parameter ranges where period doubling and other quadratic phenomena dominated the dynamics. In the chaotic regime the bicoherence completely vanished, consistent with the cubic nonlinearity that dominated the system during chaos. The present study presents results of bispectral analyses of quadratically nonlinear mechanical systems with two degrees-of-freedom. Such systems govern the response of many elastic systems such as ships, elastic pendulums, beams, arches, composite plates, and shells (Haddow et al., 1984; Nayfeh, 1986). The bispectrum provides detailed information about the nonlinear mode couplings on a frequency by frequency basis, thus identifying the quadratically interacting Fourier components.

The equations of a system of two coupled oscillators with quadratic nonlinearities which govern the response of a ship whose motion is constrained to pitch and roll are presented in

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Section 2. Following the work of Nayfeh (1983, 1983b), averaged equations are obtained by the method of multiple scales for the case of parametric resonance of the first mode. Trajectories of the modulation equations for the pure period-doubling route to chaos are also presented in Section 2. Definitions of the relevant bispectral quantities and details of the numerics are considered in Section 3. Auto and cross-bicoherence spectra of the two degrees-of-freedom are presented in Section 4. The quadratic interactions resulting in the pure period doubling sequence to chaos are isolated by the auto and cross-bispectra. Conclusions follow in Section 5.

## 2 The Coupled Oscillator

Comprehensive analyses of coupled oscillators with quadratic nonlinearities possessing internal resonances have been considered by numerous authors (Sethna, 1965; Nayfeh and Zavodney, 1986). Froude (1863) observed that ships have undesirable roll characteristics when subjected to internal (autoparametric) resonance. Mook et al. (1974), Nayfeh et al. (1973) and Nayfeh (1983a) provide a detailed analysis of a coupled oscillator with quadratic nonlinearities that governs the response of a ship that is restrained to pitch and roll. Miles (1985) showed that Hopf bifurcations do not exist for the case of an internally resonant, perfectly tuned, pendulum when the lower mode is excited by a principal parametric resonance. Nayfeh (1987) relaxed this assumption, and subsequently found Hopf bifurcations and calculated responses with period multiplying bifurcations leading to chaos.

Following the work of Nayfeh (1987), let $u_1$ and $u_2$ be two generalized coordinates which describe the motion of the system. Formulating Lagrange's equations and considering simultaneous harmonic parametric and external excitations, yields

$$\begin{aligned}\ddot{u}_1 + \omega_1^2 u_1 + \epsilon[2\mu_1\dot{u}_1 + \delta_1 u_1^2 + \delta_2 u_1 u_2 + \delta_3 u_2^2 + \delta_4 \dot{u}_1^2 \\ + \delta_5 \dot{u}_1\dot{u}_2 + \delta_6 \dot{u}_2^2 + \delta_7 u_1 \ddot{u}_1 + \delta_8 u_2 \ddot{u}_1 + \delta_9 u_1 \ddot{u}_2 \\ + \delta_{10} u_2 \ddot{u}_2 + (f_{11}u_1 + f_{12}u_2)\cos \Omega_1 t] = F_1 \cos(\Omega_2 t + \tau_1)\end{aligned} \quad (1)$$

$$\begin{aligned}\ddot{u}_2 + \omega_2^2 u_2 + \epsilon[2\mu_2\dot{u}_2 + \alpha_1 u_1^2 + \alpha_2 u_1 u_2 + \alpha_3 u_2^2 + \alpha_4 \dot{u}_1^2 \\ + \alpha_5 \dot{u}_1\dot{u}_2 + \alpha_6 \dot{u}_2^2 + \alpha_7 u_1 \ddot{u}_1 + \alpha_8 u_2 \ddot{u}_1 + \alpha_9 u_1 \ddot{u}_2 \\ + \alpha_{10} u_2 \ddot{u}_2 + (f_{21}u_1 + f_{22}u_2)\cos (\Omega_1 t \\ + \tau)] = F_2 \cos(\Omega_2 t + \tau_2)\end{aligned} \quad (2)$$

where $\mu_1$ and $\mu_2$ are the damping coefficients. The $F_n$, $f_{mn}$, $\Omega_n$, $\tau_n$, $\delta_n$, and $\alpha_n$ are constants and $\omega_1$ and $\omega_2$ are natural frequencies. $F_1$, $\Omega_1$ and $F_2$, $\Omega_2$ are the amplitudes and frequencies of the parametric and external excitations, respectively. The parameter $\epsilon$ is a small dimensionless parameter which has been used as a bookkeeping device in the perturbation analysis. If $\delta_1 = \delta_3 = \delta_4 = \phi_6 = \delta_7 = \delta_{10} = \alpha_2 = \alpha_5 = \alpha_8 = \alpha_9 = 0$, the equations that govern the response of a ship constrained to pitch and roll are recovered. Using the method of multiple scales (Nayfeh and Mook, 1979), $u_1$ and $u_2$ are expanded as

$$u_1(t;\epsilon) = u_{10}(T_0, T_1) + \epsilon u_{11}(T_0, T_1) \quad (3)$$

$$u_2(t;\epsilon) = u_{20}(T_0, T_1) + \epsilon u_{21}(T_0, T_1) \quad (4)$$

where $T_0$ is a fast time scale on which the main oscillatory behavior occurs and $T_n (n \geq 1)$ are scales on which amplitude and phase modulations take place.

Time derivatives are given by

$$\frac{d}{dt} = D_0 + \epsilon D_1 + \ldots, \quad \frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + \ldots \quad (5)$$

where $D_n = \dfrac{\partial}{\partial T_n}$.

Substituting Eqs. (3)–(5) into Eqs. (1) and (2), equating like powers of $\epsilon$ and solving for $O(\epsilon^0)$ yields

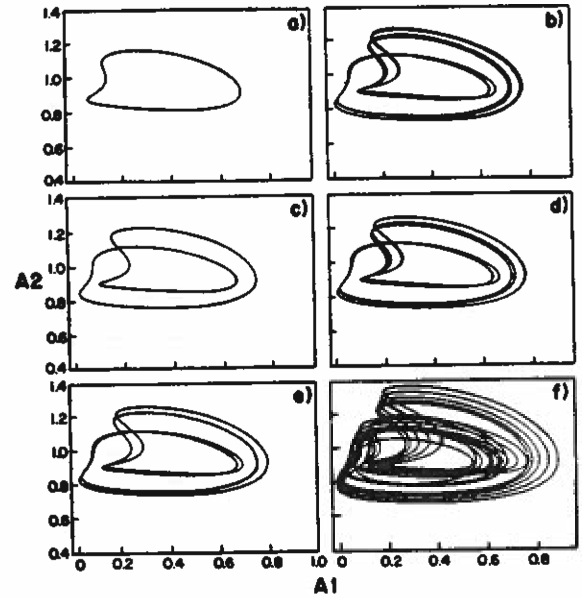


Fig. 1 Projection of the modulation equations on $a_1 - a_2$ plane for $\mu_n/f = 0.02$, $\sigma_2/f = 0.16$; (a) period 1 motion, $\sigma_1/f = 0.205$; (b) period 2 motion, $\sigma_1/f = 0.2025$; (c) period 4 motion, $\sigma_1/f = 0.2014$; (d) period 8 motion, $\sigma_1/f = 0.2013$; (e) period 16 motion, $\sigma_1/f = 0.20126$; (f) chaotic motion, $\sigma_1/f = 0.200$

$$u_{10} = A_1(T_1)e^{i\omega_1 T_0} + cc \quad (6)$$

$$u_{20} = A_2(T_1)e^{i\omega_2 T_0} + cc \quad (7)$$

where $cc$ is the complex conjugate of the preceding terms. $A_n$ can be represented by

$$A_n = 1/2 a_n e^{i\beta_n} \quad \text{for} \quad n = 1,2 \quad (8)$$

where $a_n$ and $\beta_n$ are the amplitude and phase of the $n$th mode, respectively. The particular case of parametric resonance of the first mode given by

$$\Omega_1 = 2\omega_1 + \epsilon\sigma_2 \quad \text{and} \quad \omega_2 = 2\omega_1 + \epsilon\sigma_1 \quad (9)$$

is studied here, where $\sigma_1$ and $\sigma_2$ are detuning parameters. Substituting Eqs. (6)–(9) into $O(\epsilon^1)$ equations and annulling the resultant secular term gives

$$2i(A_1' + \mu_1 A_1) + 4\Lambda_1 A_2 \overline{A}_1 e^{i\sigma_1 T_1} + 2f\overline{A}_1 e^{i\sigma_2 T_1} = 0 \quad (10)$$

$$2i(A_2' + \mu_2 A_2) + 4\Lambda_2 A_1^2 e^{i\sigma_1 T_1} = 0 \quad (11)$$

where the prime denotes $\dfrac{d}{dT_1}$ and $f_{11} = 4\omega_1 f$. $\Lambda_1$ and $\Lambda_2$ are defined by

$$4\omega_1\Lambda_1 = \delta_2 + \delta_5\omega_1\omega_2 - \delta_8\omega_1^2 - \delta_9\omega_2^2 \quad (12)$$

$$4\omega_2\Lambda_2 = \alpha_1 - \alpha_4\omega_1^2 - \alpha_7\omega_1^2. \quad (13)$$

Substituting Eq. (8) into Eqs. (10) and (11) and separating real and imaginary parts yields

$$a_1' = -\mu_1 a_1 - \Lambda_1 a_1 a_2 \sin \gamma_2 \quad (14)$$

$$a_2' = -\mu_2 a_2 - \Lambda_2 a_1^2 \sin \gamma_1 \quad (15)$$

$$\beta_1' = \Lambda_1 a_2 \cos \gamma_1 + f \cos \gamma_2 \quad (16)$$

$$\beta_2' = \Lambda_2 \frac{a_1^2}{a_2} \cos \gamma_1 \quad (17)$$

where

$$\gamma_1 = \sigma_1 T_1 + \beta_2 - 2\beta_1 \quad \text{and} \quad \gamma_2 = \sigma_2 T_1 - 2\beta_1. \quad (18)$$

Equations (14)–(17) were numerically integrated using a fourth-order Runge-Kutta subroutine with a time step of 0.05. (Tests with larger and smaller time steps indicated that 0.05 was sufficient for numerical accuracy and stability.) $\mu_n/f$ was fixed at 0.02 and $\sigma_2/f = 0.16$. Varying the value of $\sigma_1/f$ produces a period doubling sequence leading to chaos, as illustrated in Fig. 1.

Although the averaged equations demonstrate chaotic behavior, an analysis of the original equations has not been performed for the chaotic regime. Indeed, this is not the purpose of the present work. Further work is needed to confirm whether the multiple scales analysis actually represents the physical system in the chaotic regime.

## 3 Spectral Analysis and Numerical Details

Consider a discretely sampled time series $\eta_m(t)$ with the Fourier representation

$$\eta_m(t) = \sum_n A_m(\omega_n)e^{i\omega_n t} + A_m^*(\omega_n)e^{-i\omega_n t} \tag{19}$$

where the subscript $m = 1, 2$ refers to each degree-of-freedom and the asterisk indicates complex conjugation. The power spectrum of mode $m$ is defined as

$$P_m(\omega_1) = E[A_m(\omega_1)A_m^*(\omega_1)] \tag{20}$$

where $E[\ ]$ is the expected value. The auto bispectrum of mode $m$ and the cross-bispectrum between modes $m$ and $n$ are defined, respectively, as

$$B_m(\omega_1,\omega_2) = E[A_m(\omega_1)A_m(\omega_2)A_m^*(\omega_1+\omega_2)] \tag{21}$$

$$XB_{m,n}(\omega_1,\omega_2) = E[A_m(\omega_1)A_m(\omega_2)A_n^*(\omega_1+\omega_2)]. \tag{22}$$

The normalized magnitude of the bispectrum, known as the squared bicoherence, is given by

$$b_m^2(\omega_1,\omega_2) = \frac{|B_m(\omega_1,\omega_2)|^2}{P_m(\omega_1)P_m(\omega_2)P_m(\omega_1+\omega_2)} \tag{23}$$

and the normalized magnitude of the squared cross-bicoherence is given by

$$xb_{m,n}^2(\omega_1,\omega_2) = \frac{|B_{m,n}(\omega_1,\omega_2)|^2}{P_m(\omega_1)P_m(\omega_2)P_n(\omega_1+\omega_2)}. \tag{24}$$

The squared bicoherence represents the fraction of power at the sum frequency ($\omega_1 + \omega_2$) of the triad owing to quadratic interactions between the two other Fourier components ($\omega_1$ and $\omega_2$).

The time series produced by the numerical integrations of Eqs. (14)–(17) were sampled (in dimensional units) at 1 Hz, and subdivided in 32 segments, each of 128 s duration for processing, resulting in a frequency resolution of 0.0078 Hz and 64 statistical degrees-of-freedom. Bicoherence values of $b > 0.40$ are statistically significant at the 95 percent level for 64 degrees-of-freedom (Haubrich, 1979). Although higher frequency resolution was possible, the associated decrease in statistical stability of bispectral estimates was deemed unacceptable. The frequency resolution used here is sufficient to resolve the power spectral primary peak, its super harmonics, and one subharmonic. Finer frequency resolution does not alter any of the conclusions that will be presented as follows.

## 4 Results

Phase-plane portraits of the period-doubling route to chaos are shown in Fig. 1. Power spectra for the first and second degree-of-freedom motions corresponding to period one, two, four, eight, sixteen, and chaotic motions are presented in Figs. 2 and 3, respectively. The harmonic structure is clearly displayed in the power spectra. For period-one motion, the spectrum is dominated by a primary spectral peak at $f = 0.045$ Hz and its higher harmonics for the first-degree-of-freedom motion (Fig. 2$a$). For period two and subsequent period-doubled motion, the subharmonic ($f = 0.0215$) is excited (Fig. 2($b$)-2($f$)). The subharmonics for period four, eight, and sixteen are not resolved owing to the frequency resolution used, as discussed above. Owing to the quadratic nonlinearities, the spectrum contains peaks at frequencies corresponding to sum interactions between the subharmonic, the primary, and their harmonics. For the second degree-of-freedom motion shown in Fig. 3, intermediate frequencies, in addition to the subharmonic, the primary, and their harmonics, are excited by the cross-couplings of both the degrees-of-freedom. Bicoherence spectra quantify the coupling of and energy exchange between triads of Fourier components as the system progresses toward chaos as $\sigma_1/f$ is decreased from $\sigma_1/f = 0.205$ to $\sigma_1/f = 0.200$. Auto-bicoherence spectra for the first and second degree-of-freedom motions are presented in Figs. 4 and 5, respectively. For periodic motions, the coupling is centered about the dominant frequencies composing the limit cycles. Through the period doubling cascade up to period 16 motion (Figs. 4($a$)-4($d$) and 5($a$)-5($d$), additional Fourier components are nonlinearly excited by quadratic interactions between the dominant frequency and itself, as well as smaller interactions among the superharmonics and subharmonics.

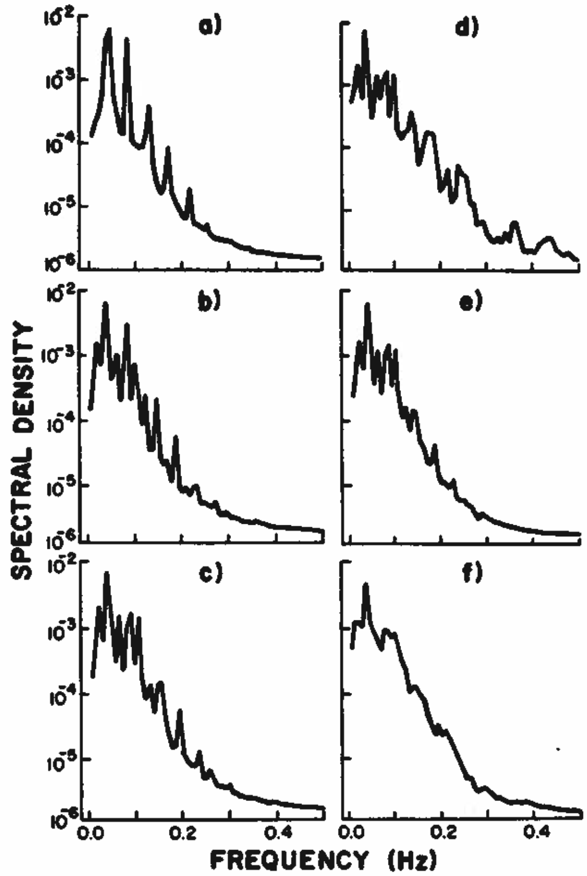


Fig. 2 Power spectra of the first-mode amplitude modulations; ($a$) period 1 motion; ($b$) period 2 motion; ($c$) period 4 motion; ($d$) period 8 motion; ($e$) period 16 motion; ($f$) chaotic motion. The units of power are arbitrary.

The spread of nonlinear interactions to include more Fourier components as $\sigma_1/f$ is reduced from 0.205 to 0.2014 is shown by the increasing number of triads with high auto-bicoherence in Figs. 4($a$)-4($d$), 5($a$)-5($d$). For $\sigma_1/f$ corresponding to 0.2013 and less, the superharmonics decrease in level (Figs. 2($e$)-2($f$) and Figs. 3($e$)-3($f$)) as do the bicoherence of triads containing these superharmonics (Figs. 4($e$)-4($f$); 5($e$)-5($f$)).

Figure 6 shows the Fourier component couplings of the first degree-of-freedom motion to those of the second ($m = 1$, $n = 2$). Figure 7, on the other hand, shows couplings between frequencies of the second degree-of-freedom motion to the first ($m = 2$, $n = 1$). Both the figures display the cross-coupling owing to amplitude modulations of the respective degree-of-freedom motion. The cross-bicoherence increases steadily up to period 8 with the reduction of $\sigma_1/f$ from 0.205 to 0.2013 (Fig. 6(*a*)–6(*d*); Fig. 7(*a*)–7(*d*)). For $\sigma_{1/f} < 0.2013$,

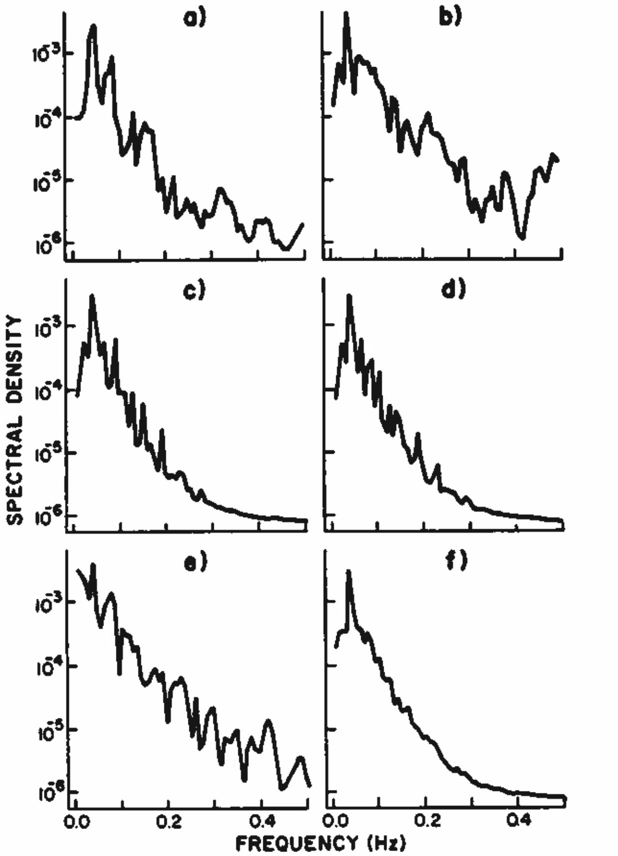


**Fig. 3 Power spectra of the second-mode amplitude modulations; (*a*) period 1 motion; (*b*) period 2 motion; (*c*) period 4 motion; (*d*) period 8 motion; (*e*) period 16 motion; (*f*) chaotic motion. The units of power are arbitrary.**

the superharmonics are suppressed, resulting in the reduction of the cross-bicoherence (Figs. 6(*e*)–6(*f*) and 7(*e*)–7(*f*)). The auto and cross-bicoherence spectra suggest that there is very little nonlinear energy transfer between amplitude modulations during chaos; in other words, there is very little cross-coupling owing to amplitude modulations in the chaotic regime. If both the modes are decoupled (i.e., if $\omega_2 \neq \omega_1$), then the system of averaged equations cannot exhibit chaos, as is true for a second-order homogeneous system. Consequently, in order to admit chaos, there must be a strong cross-coupling between the phase and amplitude modulations or between the phase modulations of the two modes. Cross-bispectra between amplitude and phase modulations are small (Figs. 8(*a*), 8(*b*)), thus negating the first possibility. On the other hand, there is very strong cross-coupling of the phase modulations, as demonstrated by high values (as great as $xb = 0.8$) of cross-bicoherence between the phases of the two modes of motion (Fig. 8(*a*)–8(*c*)). Thus, the entire source of coupling, or the nonlinear energy transfer is through the cross-coupling of phase modulations in the chaotic regime.

## 5 Conclusions

Auto and cross-bicoherence calculations were performed for a system of two coupled oscillators with quadratic nonlinearities possessing chaotic motions. Since the nonlinear interactions among the Fourier components are quadratically nonlinear, they are characterized by the auto and cross-bicoherence both outside and inside the chaotic regime. The period one, two, four, eight, sixteen, and chaotic trajectories all possessed strong auto bicoherence, originating primarily from interactions involving the fundamental frequency of oscillation. There is negligible cross-bicoherence between the amplitude modulations in the chaotic regime. On the other hand, there is a strong cross-coupling of the phases. In the averaged equations, the cross-couplings of both the amplitude and phase modulations must be assessed (e.g., with bispectral analysis) in order to understand the dynamics of nonlinear energy transfer.

The different bispectral analyses presented here clearly show the coupling mechanisms responsible for chaotic motion of

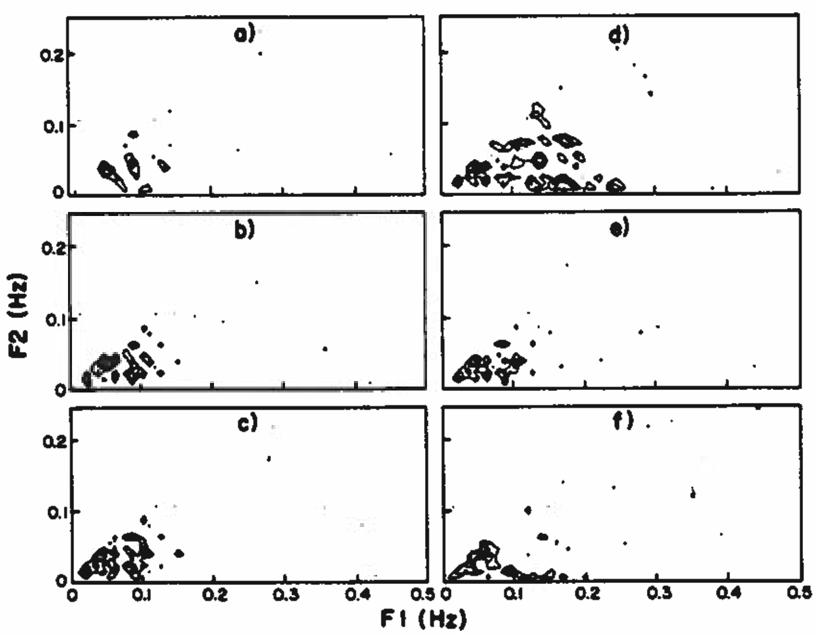


**Fig. 4 Contours of auto-bicoherence of the first-mode amplitude modulations. $f_1$ and $f_2$ are shown, while the sum frequency $f_1 + f_2$ is implied. The minimum contour plotted is $b = 0.40$, with contours every 0.1. Panels (*a*)–(*f*) are described in the caption to Fig. 1.**

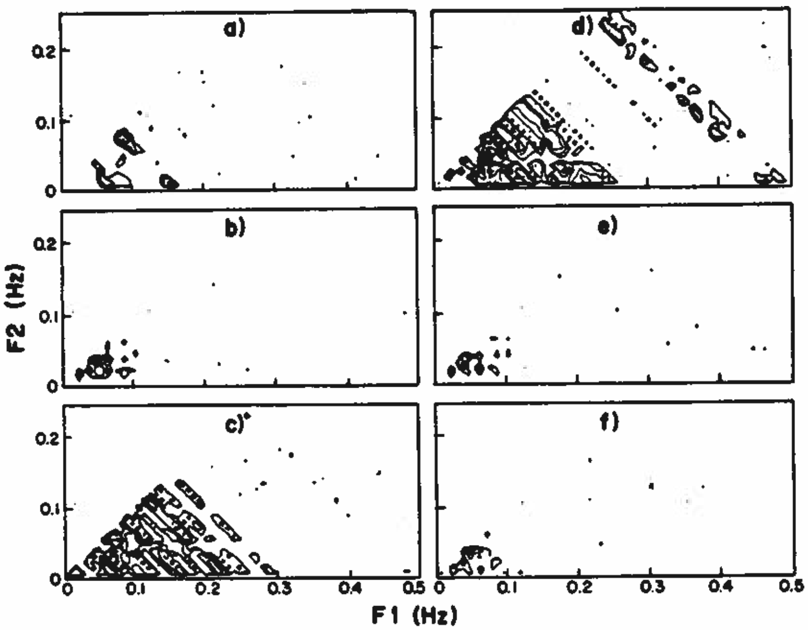


**Fig. 5** Contours of auto-bicoherence of the second-mode amplitude modulations. The format is the same as Fig. 4.

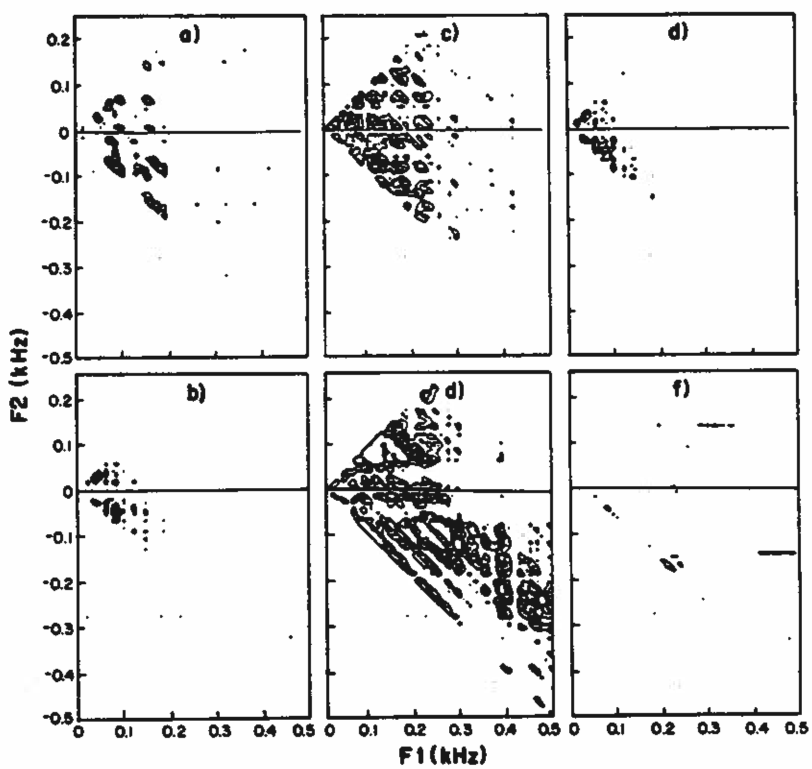


**Fig. 6** Contours of cross-bicoherence between the first-mode amplitude modulations ($f_1$, $f_2$) and the second-mode amplitude modulations ($f_1 + f_2$). The minimum contour plotted is $xb = 0.4$, with contours every 0.1. The panels (*a*)–(*f*) are described in the caption to Fig. 1.

this oscillator. Moreover, since they illustrate the wave coupling mechanisms and the resulting chaotic transitions, bispectra provide a powerful tool for understanding other nonlinear, quadratic systems, including those possessing chaotic motion. These methods operate solely on time series information, and thus, they offer a way to distinguish chaotic motion from linear random motion because the bicoherence of a linear random time series is zero.

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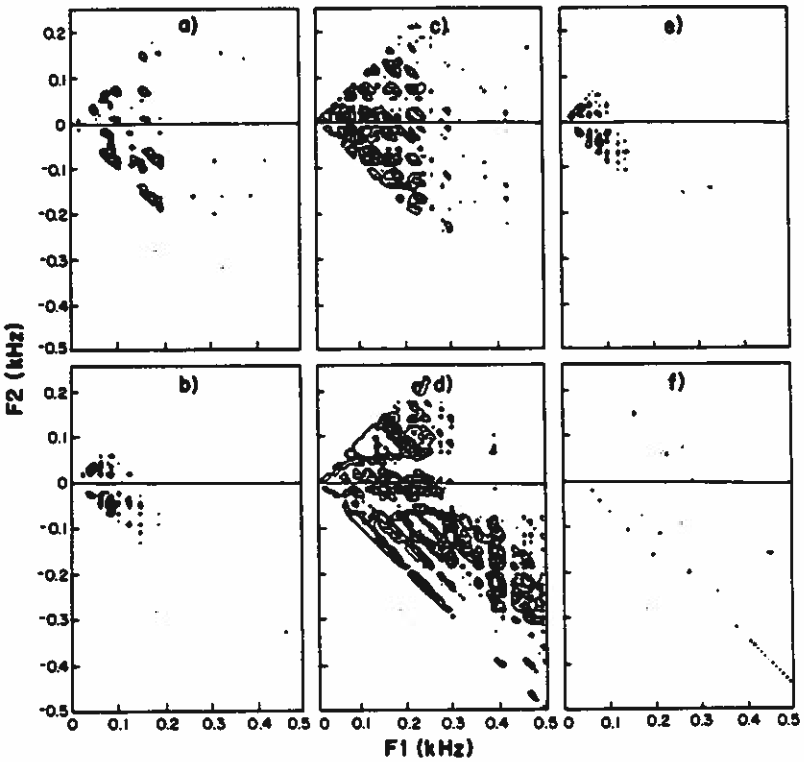


Fig. 7 Contours of cross-bicoherence between the second-mode amplitude modulations ($f_1$, $f_2$) and the first-mode amplitude modulations ($f_1 + f_2$). The format is the same as Fig. 6.

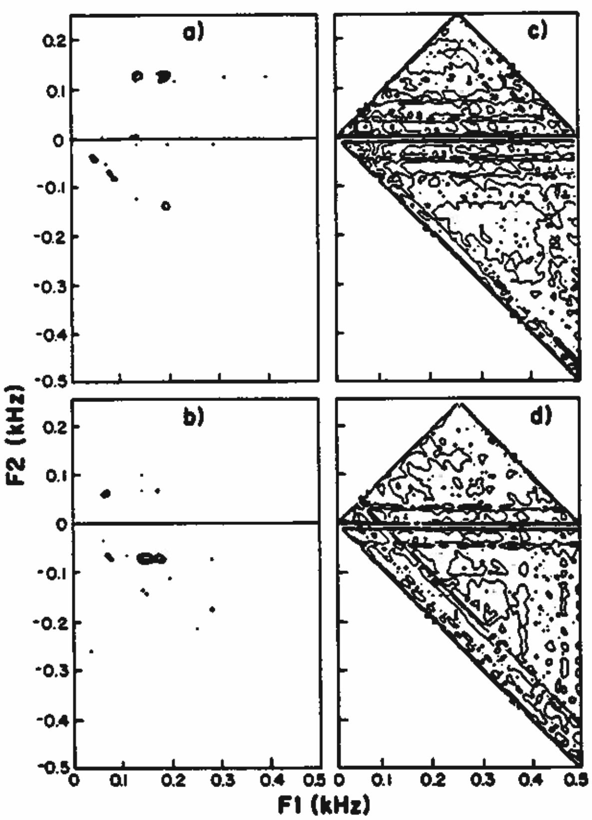


Fig. 8 Contours of cross-bicoherence between; (*a*) first-mode amplitude modulations ($f_1$, $f_2$) and second-mode phase modulations ($f_1 + f_2$); (*b*) second-mode amplitude modulations ($f_1$, $f_2$) and first-mode phase modulations ($f_1 + f_2$); (*c*) first-mode phase modulations ($f_1$, $f_2$) and second-mode phase modulations ($f_1 + f_2$); (*d*) second-mode phase modulations ($f_1$, $f_2$) and first-mode phase modulations ($f_1 + f_2$). The minimum contour plotted is $xb = 0.4$ with contours every 0.1.

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