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Recently, it has been shown that the geometrical shape of the ocean's free-surface elevation in a stormy sea is a fractal with dimension $2 < D < 3$ [1]. For the case of unidirectional ocean waves, $D = 4/3$ [1]. The solution spaces of the Zakharov equation [2] and Hasselman's energy transfer equation [3], both of which describe the nonlinear evolution of deep water waves in Fourier space, include the possibility of fractal solutions [1]. Although the fractal dimension of the instantaneous ocean surface in a region is not necessarily related to the dimension of a time series of ocean waves at a particular point within that region, the fractal dimension of ocean wave time series is of interest. Consequently, measurements of deep- and shallow-water ocean surface gravity waves, spanning a variety of oceanic conditions, were analyzed to determine the fractal dimension of time series of sea surface elevation. The measurements reported here suggest the time series are not generated by a low-dimensional attractor.

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Deep water data were obtained from a Datawell Waverider buoy moored in 110 m depth west of San Nicholas Island (33° 24.4′ N, 119° 40.1′ W), off the coast of Southern California. The location is exposed to open ocean wave conditions. Records of 8192 s (2.28 h) duration sampled at 1 Hz were collected eight times per day with gaps of about 45 min (0.72 h) separating each record. Thus, 18.2 h were sampled every day between March 15, 1984 and May 1, 1985. Details of the data collection system, including data quality assurance procedures, are contained in ref. [4]. Four days (65,536 data values each) were selected for processing. The significant wave heights (defined as 4 times the time series' standard deviation, and roughly equal to the average height of the highest 1/3 of the waves) ranged from 1.5 to 4.4 m (table I). The June (not shown) and November (fig. 1) data have narrow band power spectra, representing swell from distant storms. Note the secondary peak at the first harmonic (0.12 Hz) of the power spectral primary peak (0.06 Hz) in fig. 1. On the other hand, the December data have a broad power spectrum (fig. 1) owing to strong local winds. The

Table I
Parameters for the deep-water data. The ± values in the last two columns indicate standard deviations of the estimates [10].

Data set	Number of data (1000s)	H_{sig} [a] (m)	MI_{max} [b]	τ [c]	$MI(\tau)$ [d]	d_{GP}	d_{ap}
15 May 84	35	4.34	0.6	2	0.37	10.1 ± 6.4	9.6 ± 0.95
15 May 84	16						9.3 ± 0.88
4 Nov. 84	35	4.38	1.1	3	0.08	9.1 ± 5.7	9.3 ± 1.34
12 Dec. 84	35	2.94	0.8	3	0.37	10.3 ± 7.8	10.6 ± 1.75
12 Dec. 84	64						9.6 ± 0.70
14 June 85	35	1.5	0.8	3	0.08	9.5 ± 6.4	9.6 ± 1.35
14 June 85	16						8.5 ± 0.62

[a] H_{sig} is the significant wave height.
[b] MI_{max} is the maximum value of the mutual information content.
[c] τ is the delay (in number of samples) that produces the first minimum in the mutual information content.
[d] $MI(\tau)$ is the value of the mutual information content at its first minimum.

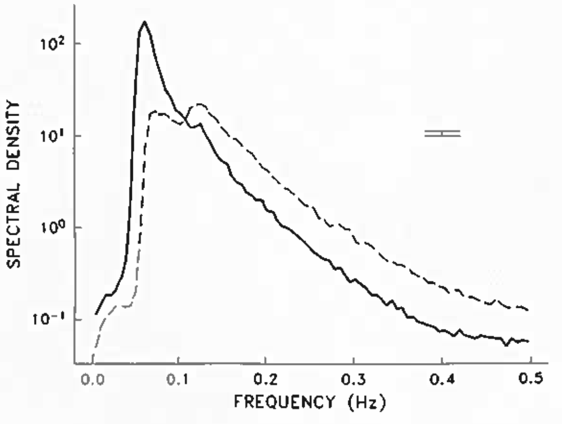

Fig. 1. Power spectra of sea surface elevation measured in deep water (depth = 110 m). There are 768 degrees of freedom, and the bars indicate 99% confidence levels. Solid line, 4 Nov 1984; dashed line, 12 Dec 1984.

Table II
Parameters for the shallow-water data. The significant wave height in 3.7 m depth is 0.9 m. Wave breaking begins at approximately 2.2 m depth, and most of the waves are broken by 0.5 m depth.

Depth (m)	d_{GP}	d_{ap}
8.8 [a]	8.9 ± 5.9	8.8 ± 1.07
6.2 [b]	8.7 ± 5.5	8.8 ± 1.04
3.7	8.7 ± 5.4	9.4 ± 1.38
2.9	8.7 ± 5.3	9.2 ± 1.25
2.2	8.6 ± 5.3	8.2 ± 1.14
1.7	8.4 ± 4.3	7.7 ± 0.95
1.1	8.3 ± 3.1	9.2 ± 1.56
0.9	8.4 ± 3.1	9.4 ± 1.28
0.0 [c]	7.1 ± 5.0	8.2 ± 1.04

[a] Measurements are from pressure gages unless otherwise noted.
[b] Current meter measurement.
[c] Runup gage measurement on the beach face.

spectral shape of the May data (not shown) lies between those shown in fig. 1.

The shallow water wave data were obtained at Santa Barbara, California, during the Nearshore Sediment Transport Study experiment conducted in January and February 1980 [5]. The observations used here were obtained primarily from near-bottom-mounted pressure sensors, located along a line perpendicular to the beach (slope = 0.05), from approximately 9 m depth to less than 1 m depth (300 m horizontal extent). Marsh–McBirney electromagnetic current meters and a runup meter on the beach face were also used. Sixteen thousand data values (a sampling rate of 1 Hz) from each of nine sensors on February 4 were processed for the present study (table II). The field data are described in detail elsewhere [6]. Statistics of the time series of sea surface elevation at the 9 m depth sensor are essentially Gaussian (linear), but as the waves propagate into shallower water nonlinear interactions lead to cross-spectral energy transfers and substantial nonlinear phase evolution [7, 8]. The rich harmonic structure typical

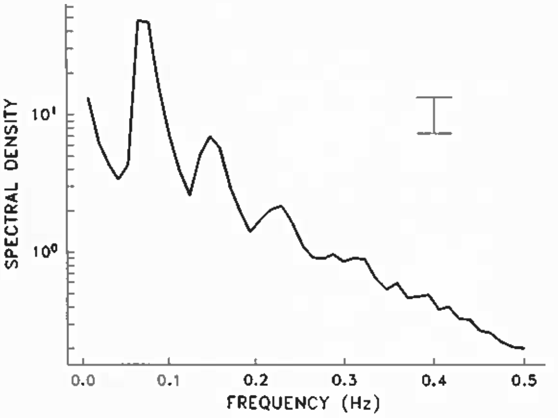

Fig. 2. Power spectrum of sea surface elevation measured in shallow water (depth = 1.7 m). There are 160 degrees of freedom, and the bars indicate 95% confidence intervals.

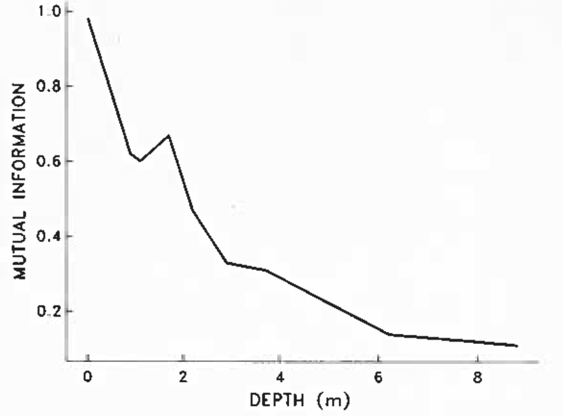

Fig. 3. First minimum of the mutual information content versus depth for the shallow water data.

of unbroken narrow band ocean waves in very shallow water is shown in fig. 2.

The Grassberger–Procaccia dimension [9] (d_{GP}) and the averaged pointwise dimension [10] (d_{ap}) were calculated for each of the time series of sea surface elevation described above. State space dynamics was reconstructed from the sea surface elevation time series by the method of geometrical reconstruction using time delay coordinates [11]. The dimension of this reconstructed state space ("embedding dimension") ranged from 1 to 20. An optimum time delay for each data set was chosen based on the first minimum of the mutual information content obtained from the first 8192 data points of the time series (table I) [12]. Mutual information measures the dependence of two variables (here, sea surface elevation at separate times) in a more general way than the autocorrelation, which measures the linear dependence. The minimum value (which occurred at a delay of 3 seconds) of the mutual information content as a function of depth for the shallow water data is shown in fig. 3. The increase in mutual information as the waves propagate into shallower water is consistent with increasing phase coupling between waves (i.e. increased correlation) owing to nonlinear interactions in shallow water. This phase coupling leads to increases in wave groupiness [13] and changes in wave shape [8].

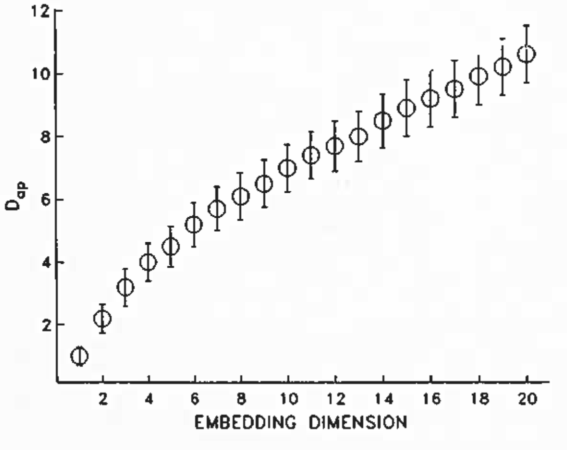

Fig. 4. Averaged pointwise dimension versus embedding dimension for the deep water data measured 12 Dec 1984. The error bars indicate ± 1 standard deviation (see table I).

The results of fractal dimension calculations for the deep water data, based on 35,000 points, are given in table I. Both d_{GP} and d_{ap} for the deep water waves did not converge for embedding dimensions less than 20 (fig. 4), which is consistent with the hypothesis that no low-dimensional attractor is generating the time series. The dimensional complexity measured for embedding dimension = 20 ranged from about 9 to 11. Although the uncertainty in d_{GP} is large, the values obtained are very close to the d_{ap} values, whose uncertainty is relatively small, suggesting the true values of the fractal dimension are well above $D = 9$. The dimensions for a few of the data sets were also calculated from 16,000 and 64,000 data points, with no substantial difference in the results (table I).

The Grassberger–Procaccia dimension of the shallow water data as a function of depth is given in table II. Although there is a trend toward decreasing d_{GP} with decreasing depth, the uncertainty is large. Moreover, no such trend with depth is seen in the d_{ap} values (table II). This depth independence of the observed values of dimension was not a-priori obvious considering that the waves undergo substantial nonlinear evolution as they propagate from deep water (nearly sinusoidal shapes) through the region immediately prior to breaking (asymmetrical wave profiles) to broken bores and finally to swash on the beach face.

The fractal dimensions presented here are not contaminated by noise. The presence of small-amplitude noise (additive, multiplicative, or instrumental) can be distinguished by the algorithms used to calculate dimensions. For length scales below those of the noise level the observed dimension increases with embedding dimension, while for length scales greater than the noise level the observed dimension converges to the dimension of the attractor [12]. This crossover behavior of the dimension, or gauge, function at a small noise level was not observed in the calculations presented here, and is consistent with the very low noise levels of the wave sensors.

In deep water, nonlinear resonant transfers between waves with different frequencies and directions can occur within quartets of waves [3]. The cubic nonlinear interactions of deep water gravity waves are very weak, and for typical ocean conditions hundred of kilometers of evolution distance are required for significant changes in the spectrum of a deep-water wave field to occur [14]. On the other hand, the dispersion relation for shallow-water waves is such that triads of waves are nearly resonant, allowing quadratic nonlinear interactions between modes of the wave field [15]. These interactions lead to substantial cross-spectral energy transfers and nonlinear phase evolution over distances of a few hundred meters. However, it is possible that the nonlinear interactions that occur between ocean surface waves, either deep or shallow, do not effect the state space dynamics in a way detectable by fractal dimensional analysis. In order to address this question, ocean waves with power spectra identical to those observed, but with random Fourier phases were simulated. This simulation procedure [13] produces a time series that is a linear superposition of sinusoids with random phases, and thus has Gaussian statistics. The dimensions (d_{GP} and d_{ap}) for the random phase simulated waves for both deep and shallow water were not statistically different from the dimensions calculated from the corresponding observed wave fields.

In conclusion, the fractal dimension of the time series of sea surface elevation measured in both deep and shallow water for this study is greater than 9. Although the mutual information content of shallow-water waves increases as the wave field propagates towards the beach, the fractal dimension is independent of depth. Numerical simulations of deep- and shallow-water ocean waves with random Fourier phases indicate that the observed values of dimension are not statistically different from the dimension of linear processes with the same power spectra as the ocean observations, even though ocean waves in very shallow water (i.e. immediately before and after breaking) are clearly nonlinear. These results are not necessarily in contradiction with the prediction that $2 < D < 3$ for the geometrical shape of the sea surface, but they do suggest that the time series are *not* generated by a low-dimensional attractor.

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