T-ASSP/36/10//22840

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Reprinted from
IEEE TRANSACTIONS ON ACOUSTICS, SPEECH, AND SIGNAL PROCESSING
Vol. 36, No. 10, October 1988

Statistics of Bicoherence

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Abstract—**Numerical simulations are used to investigate statistics of bicoherence for the special case of a linear random process. Smoothed bicoherence statistics are independent both of the normalization used to form the bicoherence and of whether statistical stability is obtained by ensemble averaging short records, or frequency merging within a long record.**

I. Introduction

The complex-valued bispectrum, $B(w_1, w_2)$ [1], [2], is often recast into its normalized magnitude (the bicoherence) and phase (the biphase). Several forms for the normalization of bicoherence have been used [3]–[5] and others. In particular, Kim and Powers [3] define the bicoherence as

$$b(w_1, w_2) = \frac{|B(w_1, w_2)|}{\left\{E\left[|A(w_1)A(w_2)|^2\right]E\left[|A(w_1 + w_2)|^2\right]\right\}^{1/2}} \qquad (1)$$

where $A(w)$ is the complex Fourier coefficient at radian frequency w, and $E[\ ]$ is the expected value. For a 3-wave system, and this normalization (1), $b^2(w_1, w_2)$ represents the fraction of power at frequency $w_1 + w_2$ owing to quadratic coupling of the 3 modes $(w_1, w_2, w_1 + w_2)$. For a broad-band process, where a particular Fourier component may be involved in many interacting triads, there is no simple interpretation of bicoherence values [6]. An alternate form for the bicoherence denominator is given by Haubrich [4]:

$$b(w_1, w_2) = \frac{|B(w_1, w_2)|}{\left\{E\left[|A(w_1)|^2\right]E\left[|A(w_2)|^2\right]E\left[|A(w_1 + w_2)|^2\right]\right\}^{1/2}}. \qquad (2)$$

For a finite-length time series, even a process with truly independent Fourier components (e.g., a Gaussian process) will have a nonzero bispectrum. Significance levels of zero bicoherence must be known to determine if data are statistically consistent with a linear, random phase process. The present study examines the effects of different normalizing and smoothing (i.e., averaging for statistical stability) procedures on the statistical distributions of bicoherence for a linear process.

II. Numerical Simulation Results

A zero-mean unit-variance, Gaussian distributed time series of 32 768 values was numerically generated. The resulting data have a white power spectrum, but because of the normalization by the power spectrum [(1) and (2)], the bicoherence is independent of spectral shape so long as the true spectrum is relatively smooth over a frequency bandwidth.

Manuscript received March 24, 1987; revised March 10, 1988. This work was supported by the Office of Naval Research under Grants N00014-86-K-0877 (S. Elgar) and N00014-83-C-0182 (R. T. Guza). Computations were performed at the San Diego Supercomputer Center with an allocation provided by the center.

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IEEE Log Number 8822840.

A. Effect of Smoothing Method and Normalization on Bicoherence Distributions

Similar to standard spectral analysis, statistical stability of bispectral estimates is obtained by averaging bispectral values from separate realizations of the process ("ensemble averaging"), by merging neighboring bispectral values (e.g., over squares in bifrequency space), or by combinations of these two smoothing techniques [2]–[8]. For example, an N-point record can be divided into m records, each of length N/m, with frequency resolution $\Delta f = m/N$. Bispectra are calculated for each of the m short records, and ensemble averaged, producing bispectral estimates with $2m$ degrees of freedom (dof). Alternatively, bispectral values from the entire N-point record can be merged over $m \times m$ squares in bifrequency space, again resulting in a final frequency resolution of $\Delta f = m/N$. In the latter case, although m^2 values are merged together, there are still $2m$ dof for each smoothed bispectral estimate. This follows from the result that the variance of both the real and imaginary parts of the bispectrum is (asymptotically) proportional to the number of data points used in estimating the bispectrum [7, and references therein]. That is, for pure frequency smoothing m^2 values of bispectra with variance proportional to N are averaged together, while for pure ensemble averaging m bispectral values with variance proportional to N/m are averaged. The variance of the smoothed bispectral estimates is N/m^2 is both cases. For the general case of averaging over m ensembles, and merging over $n \times n$ bifrequency squares, the record length (and hence the unsmoothed bispectral variance) is N/m. Smoothing reduces the variance to $N/(n^2m^2)$. For a fixed record length, N, and a given final frequency resolution, $\Delta f = nm/N$, the variance of the bispectral estimate is constant, proportional to $1/(N\Delta f^2)$, regardless of what combination of ensemble averaging and frequency merging is used to obtain the final frequency resolution. Each bispectral estimate has $2nm$ degrees of freedom. Thus, asymptotic theory suggests that bispectral statistics should be relatively insensitive to smoothing procedures. Most of the statistical fluctuations of bicoherence stem from variations in the bispectrum, i.e., the numerator in (1) and (2) [3], [4], [7]. Consequently, bicoherence distributions are anticipated to also be insensitive to normalization.

The distribution of bicoherence values obtained by ensemble averaging bispectra from 64 records of 512 data points (with no frequency merging, the ensemble averaging in Fig. 1) is very similar to the distribution of bicoherence for the same data processed as one long 32 768-point record, with bispectral values merged over 64×64 squares in bifrequency space (the frequency averaging in Fig. 1). The distributions of bicoherence for other combinations of ensemble averaging and frequency merging (not shown) were comparable to those shown in Fig. 1. Similar comparisons of bicoherence distributions (not shown) were made for estimates with 16, 32, 64, and 256 dof. For each dof value, the distributions of bicoherence were insensitive to the smoothing procedure.

Fig. 1 also demonstrates the insensitivity of bicoherence to the normalization. In the case of pure frequency merging, the Haubrich and Kim and Powers normalizations are identically equal, thus, only the Haubrich normalization for ensemble averaging is shown in Fig. 1. The bicoherence distributions (Fig. 1) and significance levels (Figs. 2 and 3) represent between 4096 and 32 768 bicoherence values (depending on the dof, with fewer bicoherence values for larger dof).

B. Significance Levels for Zero Bicoherence

Significance levels for zero bicoherence were calculated as a function of dof for both normalizations, and for various combinations of ensemble averaging and/or frequency merging. Since the distributions of bicoherence do not depend significantly on the smoothing method or normalization, the significance levels for zero

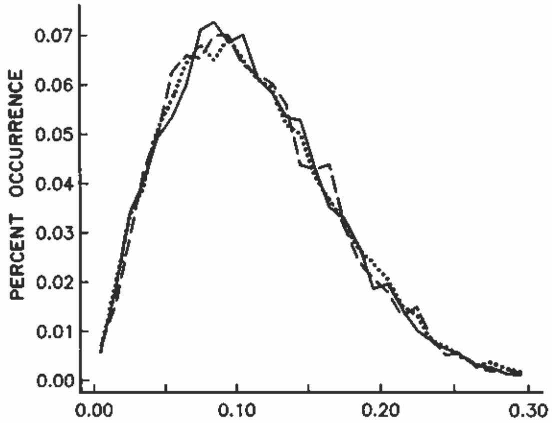

Fig. 1. Frequency distribution of bicoherence (128 dof) for numerically simulated Gaussian data. The bicoherence bin width is 0.01. Statistical stability was obtained by ensemble averaging across 64 records of 512 points each with the Kim and Powers normalization (solid line) and with the Haubrich normalization (dotted line), or by frequency merging over 64 × 64 point squares of the bispectrum from a single 32 768-point record (dashed line, Kim and Powers and Haubrich normalizations are equal). Each distribution contains 8192 smoothed bispectral values.

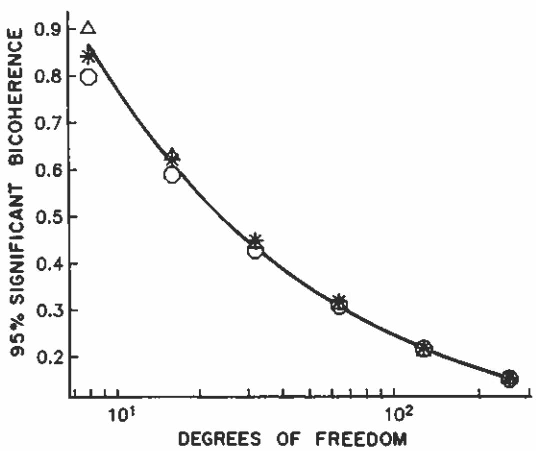

Fig. 2. The 95 percent significance level for zero bicoherence versus dof for different normalizations and smoothing. Triangles, frequency merging, Kim and Powers or Haubrich normalization; asterisks, ensemble averaging, Kim and Powers normalization; octagons, ensemble averaging, Haubrich normalization. The solid line is the theoretical 95 percent significance level.

bicoherence must be independent of smoothing technique. Fig. 2 shows this to be true for the numerically simulated data with the exception of small differences in significance levels at very low dof values (discussed below).

Haubrich [4] demonstrated that, for a true bicoherence of zero and the normalization given in (3), b^2 should be chi-square distributed in the limit of large dof. Thus, for example, the 95 percent significance level for zero bicoherence is approximately $\sqrt{(6/\text{dof})}$. As shown in Fig. 2, the numerical simulations agree with Haubrich's [4] result even with different normalizations and low dof. The agreement of other significance levels with the chi-square distribution is illustrated in Fig. 3. It is clear that many dof are required to distinguish low, but nonzero bicoherence values from truly zero values (e.g., dof > 100 if the true bicoherence = 0.2).

From properties of the chi-square distribution, it can be shown that $E[b] = \sqrt{(2/\text{dof})}$ [4]. Mean values of bicoherence calculated from the numerical simulations were consistent with this predicted bias.

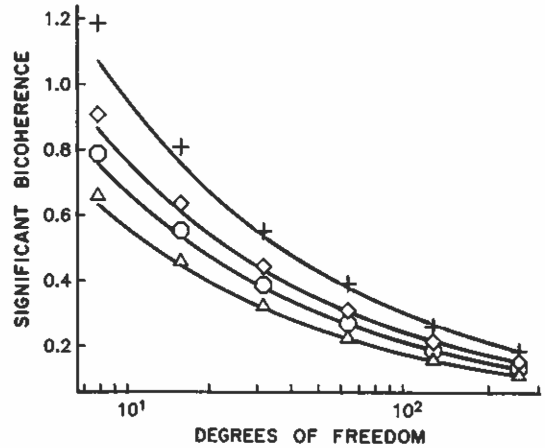

Fig. 3. Significance levels for zero bicoherence versus dof (obtained by frequency merging). Plus, 99 percent level; diamond, 95 percent; octagon, 90 percent; triangle 80 percent. The solid lines are the theoretical significance levels, $\sqrt{9.2/\text{dof}}$, $\sqrt{6.0/\text{dof}}$, $\sqrt{4.6/\text{dof}}$, and $\sqrt{3.2/\text{dof}}$, respectively.

C. Effect of Smoothing Technique at Low dof

For the case of no frequency merging (i.e., statistical stability is obtained solely by ensemble averaging), it can be shown that the Kim and Powers normalization leads to bicoherence values between 0 and 1 [3]. On the other hand, the Haubrich normalization is not bounded above by 1. Furthermore, it can be shown that with frequency merging, the Kim and Powers normalized bicoherences also become unbounded above. In the present numerical simulations with low dof (less than 32), bicoherence values greater than 1 were occasionally obtained for the Haubrich normalization, and for the Kim and Powers normalization when frequency merging was used. This slightly raises the upper tail of the bicoherence distributions. However, the corresponding increases of the significance levels at very low dof is small (13 percent for 8 dof, Fig. 2), and negligible at higher dof.

III. Conclusions

Asymptotic statistical theory [7, and references therein] and numerical simulations of Gaussian data indicate that the distribution of smoothed bicoherence estimates is essentially independent of bicoherence normalization and whether statistical stability is obtained by ensemble averaging short records of data, frequency merging within long data records, or combinations of the smoothing methods. Significance levels for zero bicoherence are therefore also relatively unaffected by the details of the smoothing technique. These significance levels are in agreement with an approximate theory [4].

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