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Relationships Involving Third Movements and Bispectra of a Harmonic Process

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Relationships Involving Third Moments and Bispectra of a Harmonic Process

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Abstract—Relationships between third moments of a harmonic random process and its bispectrum are presented. The skewness and asymmetry (with respect to a vertical axis) of the process and its derivative can be obtained by integrating the real, imaginary, and frequency-scaled real and imaginary parts of the bispectrum, respectively. Triads of Fourier coefficients containing a high-frequency component can contribute substantially to the third moments of the derivative of the process.

I. Introduction

LET a discretely sampled, finite length, real-valued harmonic random process be represented as

$$\eta(t) = \sum_{n=1}^{N} A_n e^{i\omega_n t} + A_n^* e^{-i\omega_n t} \quad (1)$$

where t is time, the A_n are complex Fourier coefficients, ω_n is the radian frequency, and N is the number of samples. If the process is linear, there is a random phase relationship between the Fourier components. On the other hand, if the process is nonlinear, the Fourier components become coupled to each other, and the phases are no longer random, resulting in non-Gaussian waveforms. The waveshape can be asymmetrical about the horizontal and/or vertical axis, quantified by nonzero third moments. The wave slopes (i.e., the derivative of the process) can also be skewed and asymmetrical. Examples of skewed and asymmetrical waveshapes are shown in Fig. 1. The purpose of this paper is to relate third moment quantities to the bispectrum of the process. Although the third moments can be obtained directly from the time series or its Hilbert transform, the bispectrum contains additional information about the third moments that cannot be obtained from the time domain. In particular, the bispectrum indicates the contribution to third moment quantities from individual triads of Fourier components.

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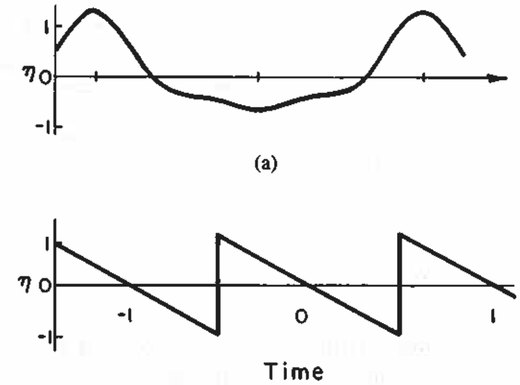

Fig. 1. Waveforms with nonzero third moments. (a) Waveform with nonzero skewness and nonzero derivative asymmetry; (b) waveform with nonzero asymmetry and nonzero derivative skewness. The other third moment quantities for each panel equal zero.

II. Third Moment Quantities in Terms of the Bispectrum

The complex valued autobispectrum is formally defined as the Fourier transform of the third-order correlation of the process [1]. The digital autobispectrum, appropriate for discretely sampled data, is [2]

$$B(\omega_i, \omega_j) = E[A_{\omega_i} A_{\omega_j} A^*_{\omega_i + \omega_j}] \quad (2)$$

where $E[\ ]$ is the expected value, or average, operator. For a digital time series with Nyquist frequency ω_N, the autobispectrum is completely described by its values in a triangle in (ω_1, ω_2)-space with vertices at $(\omega_1 = 0, \omega_2 = 0)$, $(\omega_1 = \omega_{N/2}, \omega_2 = \omega_{N/2})$, and $(\omega_1 = \omega_N, \omega_2 = 0)$ [1].

It follows from (1) and (2) that the normalized third moments, skewness, $S[\eta]$, and asymmetry, $A[\eta]$, are given by

$$S[\eta] + iA[\eta] = \left[12 \sum_n \sum_l B(\omega_n, \omega_l) + 6 \sum_{p=1}^{N/2} B(\omega_p, \omega_p)\right] \Big/ E[\eta^2]^{3/2} \quad (3)$$

where n and l range from 1 to N, with $n > l$ and $n + l \le N$ (i.e., the sum is over the triangle in (ω_1, ω_2)-space described above). $A[\eta]$ measures asymmetry about a vertical axis (fore–aft asymmetry [3], which is related to the

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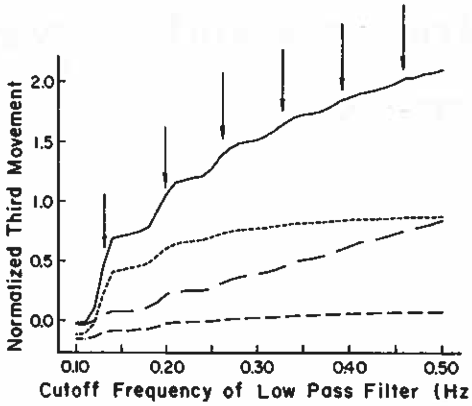

Fig. 2. Third moments versus high-frequency cutoff of a low-pass filter. Solid line, derivative skewness; longest dashes, derivative asymmetry; medium dashes, skewness; short dashes, asymmetry. The vertical arrows indicate locations of harmonics of the narrow power spectral primary peak frequency (0.065 Hz). The data are from a time series of sea surface elevation for near-breaking (i.e., nonlinear) ocean waves measured in 2 m depth.

skewness of the derivative, as described below), while $S[\eta]$ measures asymmetry about a horizontal axis. For example, the bispectrum of the waveform in Fig. 1(a) is purely real, and thus the process has nonzero skewness, but zero asymmetry. On the other hand, the bispectrum of the sawtooth shape in Fig. 1(b) is purely imaginary, and the waveform has zero skewness, but nonzero asymmetry. From (3), the Re (B) is proportional to the contribution to the mean cube of the process from each triad of Fourier components [1], and the Im (B) is proportional to the contribution of each triad to the asymmetry.

For positive frequencies, the Hilbert transform $H[\eta]$ is given by $H[\eta] = i\eta$, and thus, it follows from (2) that bispectrum $\{H[\eta]\} = -i$ bispectrum $\{\eta\}$. Consequently, $A[\eta] = -S[H[\eta]]$, and the asymmetry can be calculated in the time domain directly from the Hilbert transform of the process.

Similar relations can be derived for the derivatives of the process (i.e., the slopes of the waveform), leading to expressions for the skewness and asymmetry of these derivatives in terms of bispectral quantities,

$$S[\eta_t] + iA[\eta_t] = i\left[12 \sum_n \sum_l \omega_n \omega_l \omega_{n+l} \cdot B(\omega_n, \omega_l) + 6 \sum_{p=1}^{N/2} \omega_p \omega_p \omega_{2p} \cdot B(\omega_p, \omega_p)\right] \Big/ E[\eta_t^2]^{3/2} \quad (4)$$

where the ranges of n and l are given above, and the subscript t indicates differentiation with respect to time. The derivative of the waveform in Fig. 1(a) is not skewed ($S[\eta_t] = 0$), while the asymmetrical sawtooth shape in Fig. 1(b) has a skewed derivative ($S[\eta_t] \neq 0$). Equation (4) indicates that triads containing a high-frequency component can contribute substantially to the third moments of the derivative, while the contribution from triads with low frequencies is small. An important consequence of the frequency scaling of the bispectral terms in (4) is that low-pass filtering may substantially bias the estimation of $S[\eta_t]$ and $A[\eta_t]$ even though the estimation of $S[\eta]$ and $A[\eta]$ may not be strongly influenced by the filter, as shown in Fig. 2.

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