Derivation of ray equations in moving media in angle/depth form

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I. Introduction

In this report the equations for acoustic ray trajectories are derived using Fermat's principle. First, the equations appropriate for the case of a non-moving medium are derived. Next, those appropriate for the case of amoving medium are derived, with special emphasis on motions and sound-speed perturbations typical of the deep ocean.

II. Ray Trajectories in a Motionless Inhomogeneous Medium

Here, the equations for rays through an inhomogeneous medium that is not moving are derived. These can be derived using the calculus of variations and Fermat's principle of least time. This approach is outlined in one reference (Munk, Worcester and Wunsch, 1995), and appears in in the notation used in this paper in a more complete form in a report (Bowlin et al., 1993). To apply Fermat's principle to long-range propagation in the ocean sound channel, we must assume that the ray approximation, as derived from Helmholtz equation via the eikonal equation (Brekhovskikh and Lysanov, 1991, for example ; Munk, Worcester and Wunsch, 1995), a high-wavenumber model, provides a valid description of the physics at the frequency of interest. The ray approximation has been shown to hold at frequencies as low as 75 Hz (Colosi et al., 1999).

Mathematically, Fermat's principle is expressed by setting the variation of the time to zero,

$$\delta T = \delta \int dt = 0 \tag{1}$$

where the integral on the right hand side is over the path of the ray with fixed limits of integration and where δ is any (differentiable) variation in the path that keeps the end points fixed. In order to find the path we convert the integral over dt to an integral over path length ds using the slowness $S = dt/ds = 1/c$, where c is the sound speed, giving

$$\delta T = \delta \int S \, ds \tag{2}$$

The path length can be expressed in terms of Cartesian coordinates x_i as $ds = (dx_i \, dx_i)^{1/2}$ where we use the summation convention of implicitly summing over repeated indices. The variation of this equation is

$$\delta ds = dx_i \, \delta dx_i \, ds^{-1} \qquad (3)$$

$$= \dot{x}_i \, \delta dx_i \qquad (4)$$

$$= \dot{x}_i \, \delta \dot{x}_i \, ds \qquad (5)$$

where a dot over a quantity means the total derivative of that quantity with respect to s. The variation of the time may be written as

$$\delta T = \int (\delta \, S + S\dot{x}_i \delta \dot{x}_i) \, ds \qquad (6)$$

$$= \int (\delta \, x_i \partial_i S + S\dot{x}_i \delta \dot{x}_i) \, ds \qquad (7)$$

The first term in the integrand represents the change in time due to the change in the slowness over a new path. The second term represents the change in time due to the change in the length of the path. Manipulating derivatives, and using integration by parts gives

$$\delta T = \int \delta x_i \left(\partial_i \, S - \frac{d}{ds}(S\dot{x}_i) \right) ds \qquad (8)$$

where the integrated term is zero because the variations are zero at the end points.

The equations of motion can be obtained by demanding that this expression for the variation of the time be zero for any and all variations $\delta \, x_i$. This is possible only if the term that is multiplying the variation vanishes everywhere along the path of integration. This gives the equation for a ray:

$$\frac{d}{ds}(S \, \dot{x}_i) = \partial_i \, S \qquad (9)$$

This may be also be written

$$\ddot{x}_i = \frac{\partial_i S}{S} - \dot{x}_i \left(\dot{x}_j \frac{\partial_j S}{S} \right) \qquad (10)$$

The components $\dot{x}_i$ form a unit vector pointing along the direction of the ray. The second term on the right hand side is the projection of $\nabla S/S$ in the $\dot{x}$ direction. Thus $\ddot{x}$ is equal to that part of $\nabla S/S$ which is perpendicular to the path of the ray.

Putting these into a more familiar form used in ocean acoustics, and applying the typically used restriction of propagation only within a plane, define a horizontal r axis, a vertical z axis, and an angle θ of the ray with respect to the horizontal. Then $\dot{x} = \hat{r} \cos \theta + \hat{z} \sin \theta$, and we have

$$S\left(-\hat{r}\sin\theta+\hat{z}\cos\theta\right)\dot{\theta}=\hat{r}\partial_r S+\hat{z}\partial_z S-\left(\hat{r}\cos\theta+\hat{z}\sin\theta\right)\left(\cos\theta\partial_r S+\sin\theta\partial_z S\right) \tag{11}$$

The r component of this equation gives an expression for $\dot{\theta}$,

$$S\,\dot{\theta}=\partial_z S\cos\theta-\partial_r S\sin\theta \tag{12}$$

Geometry gives $\dot{\theta}=\cos\theta\,d\theta/dr$, and $\partial S/S=-\partial c/c$. This gives one of the two equations often used to define a ray, describing angle versus range

$$\frac{d\theta}{dr}=\frac{\partial_r c}{c}\tan\theta-\frac{\partial_z c}{c} \tag{13}$$

In most ocean applications $\partial_r c/\partial_z c \ll 1$ and $\tan\theta<1$, so the range derivative term can be safely neglected. The definition of θ yields the second ray equation, which gives ray height versus range

$$\frac{dz}{dr}=\tan\theta \tag{14}$$

An ancillary equation for travel time is also useful,

$$\frac{dt}{dr}=\frac{\sec\theta}{c} \tag{15}$$

Expressions (13) and (14) can be integrated from initial conditions to find any ray path, with (15) added if travel time is of interest.

II. Ray Trajectories in a Moving Inhomogeneous Medium

To include medium motion in the propagation plane, simply replace $S=dt/ds=1/c$ in (2) with

$$Q=dt/ds=1/(c+u_s)=1/(c+u_r\cos\theta+u_z\sin\theta) \tag{16}$$

where u_r and u_z are horizontal and vertical current in the previously used geometry and u_s is the velocity in the direction of the ray. The r-component ray equation (12) becomes

$$Q\,\dot{\theta}=\partial_z Q\cos\theta-\partial_r Q\sin\theta \tag{17}$$

The substitution of Q for S implies that an advective push of the sound does not change the physics, which may be intuitive, and that Fermat's principle therefore holds true in a moving medium. Intuition does not constitute a proof however, but fortunately a proof that Fermat's priciple is valid for a nonstationary moving medium has recently appeared (Godin and Voronovich, 2004).

Converting to $d\theta/dr$ as before and using $R = Q^{-1}$ instead of Q,

$$\frac{d\theta}{dr} = R\left(\partial_z R^{-1} - \partial_r R^{-1}\tan\theta\right) \tag{18}$$

Differentiating with the chain rule yields

$$\frac{d\theta}{dr} = R(-R^{-2})\partial_z R - R\tan\theta(-R^{-2})\partial_r R \tag{19}$$

which is

$$\frac{d\theta}{dr} = -Q\left(\partial_z c + \partial_z u_r \cos\theta + \partial_z u_z \sin\theta\right) \\ +Q\tan\theta\left(\partial_r c + \partial_r u_r \cos\theta + \partial_r u_z \sin\theta\right) \tag{20}$$

Now, expand Q with u_i/c as a small parameter, giving

$$\frac{d\theta}{dr} = -c^{-1}(1 - u_r\cos\theta/c - u_z\sin\theta/c)\left(\partial_z c + \partial_z u_r \cos\theta + \partial_z u_z \sin\theta\right) \\ +c^{-1}\tan\theta(1 - u_r\cos\theta/c - u_z\sin\theta/c)\left(\partial_r c + \partial_r u_r \cos\theta + \partial_r u_z \sin\theta\right) \tag{21}$$

Eighteen terms remain on the right hand side. With no motion this is the basic ray angle equation (13). The other terms show the small effects of advection. If the sound speed is written as a basic profile plus perturbations $c(r, z) = c_o(z) + c'(r, z)$, and if we delete most of the small range derivatives because the ocean is known to have anisotropic perturbations, then we can write an expression that includes the effects of velocity and sound speed perturbations c',

$$\frac{d\theta}{dr} = -c_o^{-1}(1-c'/c_o-u_r\cos\theta/c_o-u_z\sin\theta/c)\left(\partial_z c_o + \partial_z c' + \partial_z u_r\cos\theta + \partial_z u_z\sin\theta - \partial_r c_o\tan\theta\right) \tag{22}$$

where terms of higher order in c'/c_o have been omitted. Further analysis in the next section shows the relative influences of the anomaly c', the shear $\partial_z u_r$, the strain $\partial_z u_z$, and the velocities u_r and u_z.

The ray height equation is unchanged from the motionless case. The travel time equation changes slightly. After expansion assuming small u_1/c it becomes

$$\frac{dt}{dr} = \frac{\sec\theta}{c}\left(1 - u_r\cos\theta/c - u_z\sin\theta/c + ...\right) \tag{23}$$

where the advective effect of velocity enters in an intuitive way.

Note that (21), (14), and (23) have essentially the same meaning as the set of equations appearing in other works (Lamancusa and Daroux, 1993 c.f.), which might have been chosen for analysis instead of (22) (derived from (21)). However, the form used here has been found to be convenient.

III. Perturbation Terms

To first order in u_i/c_o and c'/c_o, rewrite (22) as

$$\frac{d\theta}{dr} = -c_o^{-1} \left[\partial_z c_o(1 - c'/c_o - u_r \cos\theta/c_o - u_z \sin\theta/c_o) + \partial_z c' + \partial_z u_r \cos\theta + \partial_z u_z \sin\theta - \partial_r c_o \tan\theta\right] \tag{24}$$

Here, we have taken advantage of the fact that small-scale ocean sound-speed and velocity perturbations have red spectra (Garrett and Munk, 1975; Pinkel, 1984), so that most of the energy is at low wavenumbers, below order $2\pi/(500\text{m})$ in the horizontal and order $2\pi/(100\text{m})$ in the vertical. This means that the differentiation operation on perturbations (multiplication by wavenumber for members of a Fourier expansion) reduces magnitude, so that many derivative terms have been omitted.

There are many terms included in (24) that are typically quite small and could sensibly be omitted for almost all ocean acoustic situations. However, there is no danger in retaining these terms, although their presence may obscure the essential physics; the true danger lies in incorrectly omitting significant terms. The first and last terms in (24) represent the effect of the unperturbed sound channel. There are six remaining perturbation terms. The first three terms are corrections to the rate of change of θ versus r that arise because the sound is not traveling at the background speed c_o. These are very small and will not be considered any further. Term 7 is expected to be much less than term 6.

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