Chapter 11. Spatial instability, absolute and convective instability

11.1 Spatial Instability

Thus far, we have considered problems in which the instability manifests itself as a growth in time, For the normal modes we found exponential increases with time. However, there are cases in which we might expect the instability to appear as a spatial growth of a meander pattern produced in response to a forcing with a real frequency located at, say, x=0 as shown in figure 11.1, .

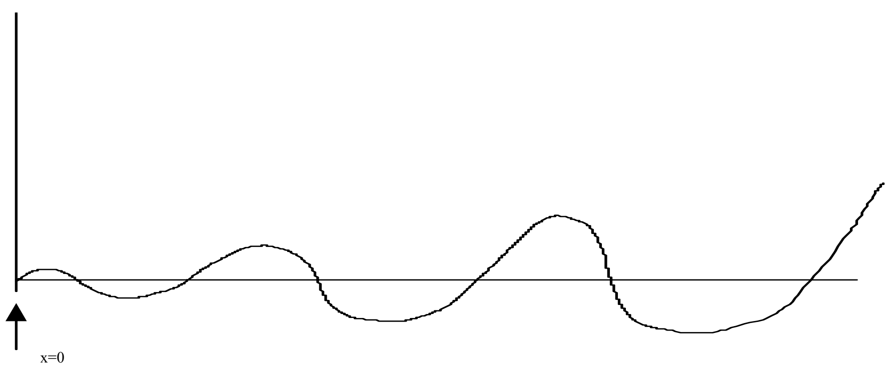


Figure 11.1 A meandering streamline increasing exponentially in amplitude in space.

From the simplest point of view such problems merely requiring examining the same dispersion relation for any problem fixing ω as real and examining the solutions for k (if the growth is expected in the x direction) to see if there are complex k the go along with a real frequency. A little further thought shows that the situation is considerably more obscure. Namely, how do we know when we find an imaginary k for a real ω

whether this represents a physically meaningful spatial growth. An example may be helpful. Consider the simple example of barotropic Rossby waves in two dimensions. The (dimensional) dispersion relation for the Rossby wave is :

$$\omega = -\frac{\beta k}{k^2 + l^2} \tag{11.1.1}$$

As we know, these Rossby waves, existing in a resting fluid are stable. There are no energy sources and ω is real for all real k and l. Now let's solve the problem for k as a function of frequency for real l, There are two roots,

$$k = -\frac{\beta}{2\omega} \pm \left[\frac{\beta^2}{4\omega^2} - l^2 \right]^{1/2} \tag{11.1.2}$$

The maximum (in size) of frequency corresponding to real wavenumbers is -$\beta/2l$ for which k=l. From the waves course you will recall that for a given ω the long wave corresponds to energy propagation to the west (negative x) and the short wave has it group velocity to the east (positive x). Indeed, if there is forcing at x=0 at a given frequency one uses the radiation condition to determine which real root of (11.1.2) is appropriate for the solution in either x>0 or x<0.

Now suppose that ω, exceeds the maximum frequency, i.e. that $-\omega > \dfrac{\beta}{2l}$. The radicand in (11.1.2) is negative and the solutions are

$$k = -\frac{\beta}{2\omega} \pm i \left[l^2 - \frac{\beta^2}{4\omega^2} \right]^{1/2} \tag{11.1.3}$$

If the solutions are of the form $\exp i(kx + ly - \omega t)$, solutions for k which have a negative imaginary part will be exponentially increasing with increasing x while the solutions with a positive imaginary part for k will be exponentially decreasing. Now we "know" in this case that for x >0 the physically meaningful solution is the one that exponentially decreases from the origin. The solution that is exponentially increasing is part of the solution because we must be able to represent a forcing at large x whose response for smaller x is exponentially decreasing as x decreases. But we know this in this problem only because the physical situation is so clear. There are no energy sources and there is no way forcing with real frequency should produce an exponential growth of amplitude with x. In more complicated problems in which there is an energy source, e.g. a shear, it may not be so clear. For example, in the case just discussed we could add a very tiny amount of shear. Clearly we would still have, roughly speaking, the same solutions (11.1.2). Should we reject the spatial growth then? How do we generally decide? If we reject such spatially growing solutions are we rejecting possibly spatial instabilities? This is a serious issue.

One way to distinguish real from spurious growing solutions in x is to do the initial value problem in time. For all finite t the physical solution originating near x =0 should vanish as x→ infinity. If we then reject solutions that violate that causality condition, and then let t→ infinity we can be assured that a spatially growing solution, so obtained, is physically meaningful. However, that is a rather lot of work to do if all we want to do is to decide on the physical relevance of a modal solution that has $k_i < 0$. This is similar to the prescription for the wave radiation problem for which we substitute a radiation condition for the complete solution of the initial value problem to decide whether one or another steady solution is the relevant one. Is there a similar criterion here? There is one. The successful alternative to the complete initial value problem is also based on the notion of causality but avoids the full complexity of the initial value problem. Unfortunately, the method to be described is, as one author has put it, "simple to state but devilishly complicated to carry out". Hence, in this chapter we will discuss only an equivalent heuristic approach which contains the essence of the method. The classical paper on the subject comes from the literature on plasma instability, [Briggs, R.J. 1964 Electron-stream interaction with plasmas, Chap. 2. 8-46, MIT Press] . Applications of Briggs' method to GFD stability problems are few. In particular, Pierre-Humbert, R. T. 1984 Local and global baroclinic instability of zonally varying flow. J.Atmos. Sci. **41**, 2141-2162.

and,

Pierre-Humbert, R.T. 1986, Spatially amplifying modes of the Charney baroclinic-instability problem. J. Fluid Mech. **170**, 293-317,

and,

Merkine, L.O. 1977, Convective and absolute instability of baroclinic eddies. Geophys. and Astrophys. Fluid Dyn. **9**, 129-157.

The essence of the method requires the joint use of Fourier analysis in x and a Laplace transform in t. The former is used to allow initial conditions that are initially spatially localized and the Laplace transform introduces the causality condition by careful consideration of the position of the singularities of the transform solution in the Laplace transform plane. It requires the explicit or numerical solution of the equation for k as a function of ω. If we note that even for the two layer model this involves the solution of an 8^{th} order equation for k and the need to follow the roots as the real and imaginary parts of ω are altered according to Briggs' recipe, the horrendous nature of the procedure becomes only too clear. In the next section we discuss a heuristic alternative and apply it to an artificial but plausible dispersion relation to illustrate the basic idea. The application to actual meteorological or oceanic problems can be found in the above references.

11.2 A heuristic approach to the criterion.

Instead of considering the initial value problem consider instead adding an artificial damping to the potential vorticity equation of the form,

$$\frac{dq}{dt} = -\sigma q \tag{11.2.1}$$

where σ is a damping constant. As in the wave radiation problems the introduction of such a damping allows us to avoid the initial value approach. One way to think of it is that σ takes the place of the real part of the Laplace transform variable on the Laplace inversion contour.

Now, suppose we have been able to derive the dispersion relation (this may be very implicit and require a numerical approach—think of the Charney problem), i.e. suppose we have,

$$F(\omega, k, \sigma) = 0 \tag{11.2.2}$$

where we have suppressed the dependence on other physical parameters of the problem. For a given ω and σ this yields roots (finding the roots is the hard part—usually requires numerical work) in the complex k plane, i.e. $k = k(\omega,\sigma)$. If the roots are in the lower half plane they would yield solutions exponentially increasing with x. They represent either spatially increasing solutions for x>0 or spatially decreasing solutions for <0. Which?

Similarly, solutions in the upper half plane give growth for x decreasing with positive x from zero or amplifying solutions for x<0. How do we decide? . The trick is to find a situation where it is physically clear which is which.

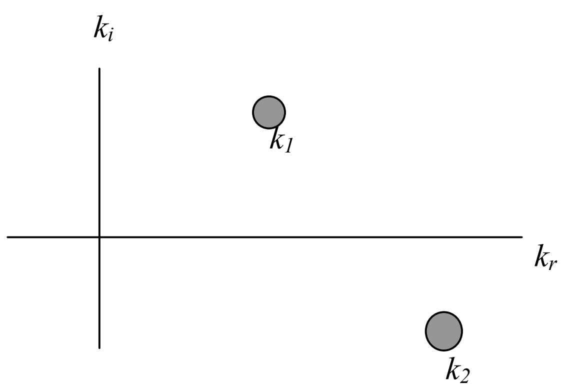


Figure 11.2.1 The roots for *k* in the complex plane for given frequency and damping.

For very large values of σ we expect all roots to have an imaginary part different from zero and to definitely represent decay away from a source region. This is the frictional equivalent of the causality condition. It assures us that the exponential behavior we observe for large σ is physical and identifies the proper direction. That is, a root for large damping in the upper half plane must represent decay of the solution for increasing x and

k_i

not exponential growth for negative x. Consider such a root that for fixed ω and large σ is in the upper half plane.

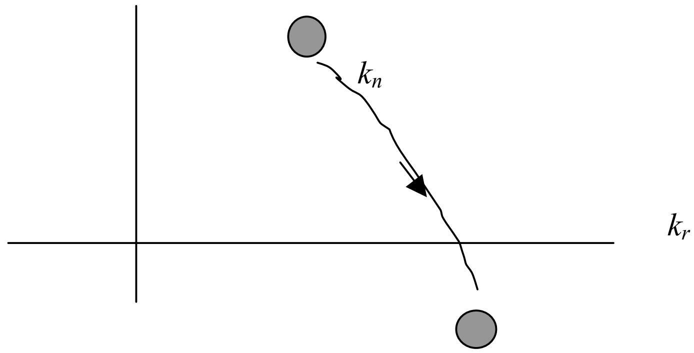


Figure 11.2.2 A root crossing the real k axis representing spatial growth for x > 0.

If, as σ decreases to zero Im(k) crosses the real axis and ends up in the lower half plane ($k_i < 0$) that root would correspond to real spatial growth for x>0. The basic idea being that we can associate this root with one that is decaying away from x=0 if there is sufficient friction in the system to overpower the natural instability. Allowing $\sigma \rightarrow 0$ then uncovers that instability. The same holds true for roots that start in the lower half plane for large damping. If the root does not cross the real axis the exponential behavior continues to represent decay from the appropriate direction.

There is an important and comforting corollary to this condition. As the root (if it does) crosses the real k axis it corresponds at that point to a wave with a real wavenumber, a non zero damping, σ, but a real frequency. With the form of damping we have introduced in (11.2.1) this is equivalent to a solution with no friction growing in time with a growth rate equal to the value of σ at the point where the root crosses the axis. This implies that to have the root cross the axis, i.e. to have the possibility of spatial growth, <u>the flow must be unstable in time for a real wavenumber.</u> That is, spatial growth can only occur for flows that are subject to temporal instability. If you have a flow that is stable according to the treatment of our earlier work in the time domain you can be assured it can not have real spatial growth. Any such growth is guaranteed to be spurious. We will discuss later the relationship between growth in x and t.

Roots which start in the lower half plane for large σ and remain there as $\sigma \to 0$ do not represent spatial growth for positive x but represent decay in the negative x direction. Let's see how this works for the Rossby wave example we started with. In order to add the damping term implied by (11.2.1) we need only make the transformation $\omega \to \omega + i\sigma$ in (11.1.2),

$$k = -\frac{\beta}{2(\omega + i\sigma)} \pm \left[\frac{\beta^2}{4(\omega + i\sigma)^2} - l^2\right]^{1/2} \tag{11.2.3}$$

For very large σ the roots are,

$$k_1 = \frac{i\beta}{2\sigma} + i\left[\frac{\beta^2}{4\sigma^2} + l^2\right]^{1/2},$$

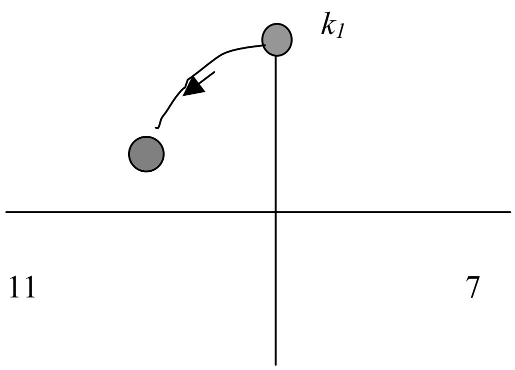
$$k_2 = \frac{i\beta}{2\sigma} - i\left[\frac{\beta^2}{4\sigma^2} + l^2\right]^{1/2} \tag{11.2.4 a,b}$$

As $\sigma \to 0$ the two roots move in the complex plane to the points,

$$k_1 = -\frac{\beta}{2\omega} + i\left[l^2 - \frac{\beta^2}{4\omega^2}\right]^{1/2},$$

$$k_2 = -\frac{\beta}{2\omega} - i\left[l^2 - \frac{\beta^2}{4\omega^2}\right]^{1/2} \tag{11.2.5 a,b}$$

and ,as shown in the figure, <u>neither root crosses the real axis</u>.



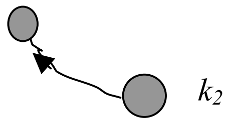


figure 11.2.3 Trajectory of the roots for k for the Rossby wave as the friction is diminished to zero. Note that neither root crosses the real axis.

This verifies our physical intuition that neither root can represent real spatial instability.

In some exceptional cases roots from opposite sides of the real axis will cross as $\sigma \to 0$ and coalesce at some complex value of k, e.g. ko. In that case we have a double root in k. This implies that for σ=0 and k near ko the dispersion relation must be,

$$F(\omega,k,0) \approx \overbrace{F(\omega(k_o),k_o,0)}^{=0} + \frac{\partial F}{\partial \omega}(\omega-\omega(k_o)) + \underbrace{\frac{\partial F}{\partial k}(k-k_o)}_{=0} + \frac{\partial^2 F}{\partial k^2}\frac{(k-k_o)^2}{2} + ... = 0 \quad (11.2.6)$$

so that locally,

$$\omega-\omega(k_o) = const.(k-k_o)^2, \qquad (11.2.7)$$

$$\Rightarrow \frac{\partial \omega}{\partial k}(k_o) = 0$$

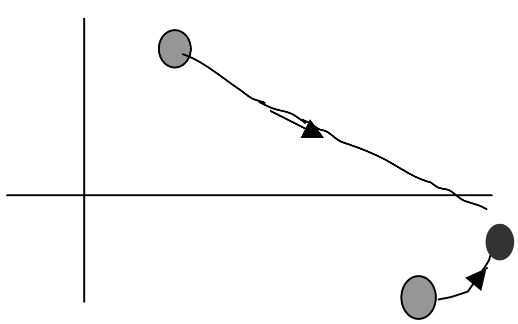


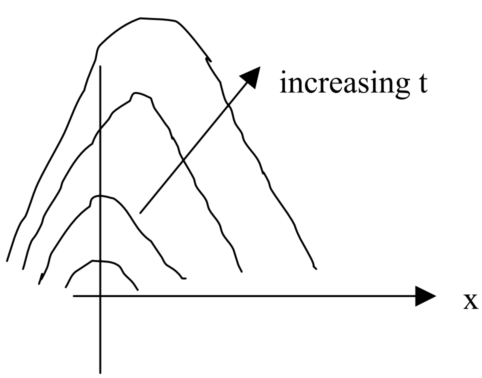
Figure 11.2.4 The coalescence of two roots.

This is the generalized form of the statement that the group velocity is zero (both frequency and wavenumber are generally complex at k_o. If (emphasized if) the imaginary part of ω is positive at this coalescence point it yields a disturbance growing with time and not propagating. Any spatial growth would then be overwhelmed by the local growth with time and spatial instability, even if it exists would be irrelevant. Such perturbations which satisfy (11.2.7) with $\omega_i(k_o) > 0$ yield what are called *absolute instability*. These are disturbances that grow such at fixed x the disturbance amplitude grows exponentially. There are other modes, as we shall see, in which the flow is temporally unstable but the disturbance moves away from its initial position rapidly enough so that at fixed x the disturbance decays with time; such instabilities are called *convective instabilities*. (It has nothing to do with thermal convection; it signifies that the disturbance rapidly convects away from its origin). We can contrast the two types as: $\sigma \rightarrow 0$,

i) $\frac{\partial \omega}{\partial k} = 0, k = k_o, \quad \omega_i(k_o) > 0$ (absolute instability)

ii) $\frac{\partial \omega}{\partial k} \neq 0, \omega_i(k) > 0, \textit{ for } \quad x/t = \mathrm{Re}(\omega_k)$ (convective instability)

We can distinguish the two types in the figures below. Each figure shows the amplitude as a function of x as t increases.



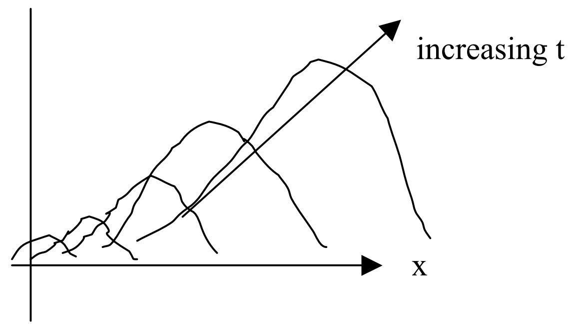
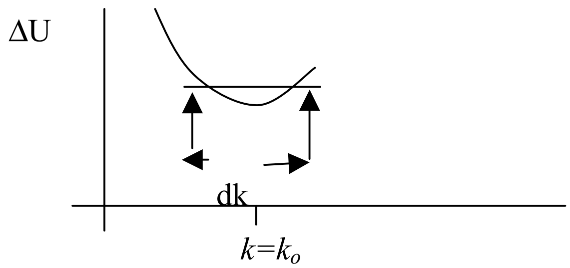


Figure11.2.5 a,b a) absolute instability, b) convective instability.

To examine this further, consider the instability which occurs in the vicinity of the minimum critical shear of, say, the two layer model. The imaginary part of ω is non zero for a small range of k around k_o the wavenumber of minimum critical shear.



If the dispersion relation is approximately, where U is a constant advective velocity,

$$\omega = Uk + \omega_r(k) + i\,\omega_i\,(k) \tag{11.2.8}$$

and suppose that that the growth rate has a maximum at k=k_o. The wave phase is

$$\theta = kx - \omega t = k(x - Ut) - \omega_r t - i\omega_i t \tag{11.2.9}$$

The phase will have a stationary point at the real wavenumber k_o ,

$$\frac{d\theta}{dk} = x - Ut - \frac{\partial \omega_r}{\partial k}(k_o)t - \overbrace{\cancel{i\frac{\partial \omega_i}{\partial k}(k_o)t}}^{=0}=0 \tag{11.2.10}$$

for $x/t = U + \frac{\partial \omega_r}{\partial k}(k_o)$ since k_o is the wavenumber which maximizes the growth rate. In the vicinity of k_o we can expand the phase and phase gradient,

$$\theta \approx k_o x - \omega(k_o)t + (k-k_o)[x - t(U+c_g)]$$
$$-\frac{(k-k_o)^2}{2}\left[\omega_r'' + i\omega_i''\right]t + \ldots$$

(11.2.11a,b)

$$\frac{d\theta}{dk} = [x - t(U+c_g)] - i(k-k_o)\left[\omega_i'' - i\omega_r''\right]t + \ldots$$

where $c_g = \partial\omega_r / \partial k$,

and so for k in the vicinity of k_o, the stationary point of the phase is at ,

$$k - k_o = -i\frac{\left[x/t - (U+c_g)\right]}{\left[\omega_i'' - i\omega_r''\right]} \qquad (11.2.12)$$

Now reconsider the phase (11.2.11a) at the wavenumbers given by (1.2.12) near the point k_o,. A little algebra shows that,

$$\theta \approx \overbrace{k_o x - \omega(k_o)t}^{\text{growth and propagation of plane wave}} - \frac{i}{2}\frac{\{x-(U+c_g)t\}^2}{t\left(\omega_i'' - i\omega_r''\right)} \qquad (11.2.13)$$

The form of the wave is therefore,

$$\phi = Ae^{i(k_o x - \omega(k_o)t)} e^{\left\{\frac{\left(x-[U+c_g]t\right)^2}{t[\omega_i''^2+\omega_r''^2]}\right\}\omega_i''} e^{\left\{\frac{\left(x-[U+c_g]t\right)^2}{t[\omega_i''^2+\omega_r''^2]}\right\}i\omega_r''} \qquad (11.2.14)$$

Since $\omega_i''(k_o) < 0$ (the growth rate is a maximum at k_o) the second exponential factor represents a Gaussian envelope moving with the group velocity associated with the real wavenumber of the maximum growth rate. Note also that the width of the envelope increases like $t^{1/2}$ but that the sharper the peak of the growth rate curve the broader the Gaussian becomes. As time goes on this packet moves away from the origin *as long as* *the total group velocity* $U+c_g \neq 0$. This is the convective instability and it is important to note that at the center of the traveling packet the amplitude is growing with the maximum growth rate $\omega_i(k_o)$. As (and if) the packet moves off, it leaves a tail behind that can be interpreted as a spatially developing instability. In general, to find the growth rate at a fixed and arbitrary x for large t one has to consider the points where $\partial\omega/\partial k = 0$ and these, in general will be complex. Only the particular x moving with the group velocity will see real *k* and growth at the normal mode frequency at that *k*.

So, in addition to checking whether the spatial instability in real by examining the movement of the roots as σ goes to zero, one also has to determine that there are no root coalescence points that have imaginary $\omega > 0$.

11.3 An example

The hard and annoying but absolutely essential part of the process is the finding of the roots and tracking their positions as σ is altered. For the actual stability problems this is a big task and you are referred to the references to see the process at work for the two-layer (Merkine) and Charney (Pierrehumbert) problems. To illustrate the process more simply we will discuss an example taken from the paper of Briggs. Let's suppose the dispersion relation for real wavenumbers is,

$$\omega^2 - k(V_1+V_2)\omega + V_1V_2(k^2-k_o{}^2) = 0 \qquad (11.3.1)$$

where V_1, V_2, k_o are parameters of the problem. (11.3.1) is an example of (11.2.2). If we consider *k* to be real, we can solve for ω in terms of *k*.

$$\omega = \frac{V_1+V_2}{2}k \pm \frac{1}{2}\left\{k^2(V_1-V_2)^2 + 4V_1V_2k_o{}^2\right\}^{1/2} \qquad (11.3.2a)$$

while we can also solve for *k* in terms of ω,

$$k = \frac{\omega(V_1+V_2)}{2V_1V_2} \pm \frac{1}{2}\left\{\frac{\omega^2(V_1-V_2)^2}{V_1{}^2V_2{}^2} + 4k_o{}^2\right\}^{1/2} \qquad (11.3.2\text{ b})$$

We now examine some particular cases.

A) Suppose $V_1 > 0,\ V_2 > 0,\ k_o{}^2 > 0$.

Then ω is real for all real wavenumber. Similarly k is real for all real ω. There is clearly neither temporal or spatial instability.

B) $V_1 > 0,\ V_2 < 0,\ k_o{}^2 < 0$.

Again, ω is real for all real wavenumber so there is no temporal instability. There is apparent spatial instability for small ω but our previous results tell us that the apparent spatial instability is illusory. There is no spatial instability. You may check the trajectory of the roots to verify this.

C) $V_1 > 0,\ V_2 > 0,\ k_o{}^2 < 0$

Here there is both temporal instability for

$$k^2 < \frac{4V_1V_2(-k_o{}^2)}{(V_1-V_2)^2} \qquad (11.3.3\text{a })$$

and spatial instability for real frequencies if

$$\omega^2 < \frac{4(-k_o{}^2)(V_1V_2)^2}{(V_1-V_2)^2} \qquad (11.3.3\text{b})$$

To check whether the apparent spatial instability is physically meaningful add the damping term and so shift ω according to the rule, $\omega \to \omega + i\sigma$. For very large σ the two roots of (11.3.2b) become,

$$k_+ = \frac{i\sigma}{V_2},\ \ k_- = \frac{i\sigma}{V_1} \qquad (11.3.4\ a,b)$$

so that both roots start in the upper half plane. As $\sigma \rightarrow 0$ the plus root becomes,

$$k_+ = \frac{\omega(V_1 + V_2)}{2V_1V_2} + \frac{i}{2}\left(4\left|k_o\right|^2 - \frac{\omega^2(V_1 - V_2)^2}{V_1{}^2V_2{}^2}\right)^{1/2} \qquad (11.3.5a)$$

so that this root remains in the upper half plane. The other root crosses the real axis and is,

$$k_- = \frac{\omega(V_1 + V_2)}{2V_1V_2} - \frac{i}{2}\left(4\left|k_o\right|^2 - \frac{\omega^2(V_1 - V_2)^2}{V_1{}^2V_2{}^2}\right)^{1/2} \qquad (11.3.5b)$$

and so ends in the lower half plane. This root therefore represents spatially growing solutions for x >0. It is important to note that k_+ represents a decaying solution in x >0 and does not represent a spatial instability to negative x.

D) $V_1 > 0,\ V_2 < 0,\ k_o{}^2 > 0$

In this case temporal instability is possible and with the oppositely directed velocities we might wonder whether absolute instability occurs. Spatial instability is clearly not possible for real frequencies. To check for absolute instability we calculate $\partial\omega/\partial k$ and see if it vanishes. It is easier to take the *k* derivative of (11.3.1) and set $\partial\omega/\partial k$ =0, to obtain,

$$\omega = \frac{2V_1V_2}{(V_1 + V_2)}k,\ \ \Leftrightarrow k = \frac{(V_1 + V_2)\omega}{2V_1V_2} \quad \text{at } \omega_k = 0 \qquad (11.3.6\ a,b)$$

Further, using (11.3.4 a,b) we see that for large σ the roots start in opposite half planes, one above and one below the real axis. As $\sigma \rightarrow 0$ they coalesce at the point obtained by using (11.3.6b) in (11.3.1) to obtain,

$$\omega^2 = -4{k_o}^2 {V_1}^2 {V_2}^2 / (V_1 - V_2)^2,$$

(11.3.7 a,b)

$$\omega = 2ik_o \frac{|V_1 V_2|}{(V_1 - V_2)}$$

so that absolute instability occurs at the complex wavenumber, (from (11.3.6 b)

$$k = 2ik_o \frac{|V_1 V_2|}{(V_1 V_2)} \frac{(V_1 + V_2)}{(V_1 - V_2)} \qquad (11.3.7b)$$

so that locally, near x=0 where the absolute instability occurs, the amplitude also grows spatially.