Chapter 3. Stability theory for zonal flows :formulation

3.1 Introduction

Although flows in the atmosphere and ocean are never strictly zonal major currents are nearly so and the simplifications springing from such idealizations are helpful in analyzing the basic nature of the instability process. Much of the literature on instability theory relates to the stability properties of zonal flows, i.e. flows directed along latitude lines and whose properties are independent of the downstream direction.

The basic flow

Although any flow can be examined for its stability we usually focus on simpler flows that possess some symmetry or simplicity which is “lost” as a result of its inherent instability. Such a process is often termed “symmetry breaking”. For example, a zonal flow may become unstable to wavelike disturbances so that the new state is a function of the downstream (x) coordinate. Or a steady flow may become time dependent. To give the general idea consider the problem that arises when the initial state is time independent and let’s examine the formulation of the stability problem for that basic state.

Consider a flow, determined by its geostrophic streamfunction,

$$\psi = \Psi_o(x,y,z) \tag{3.1.1}$$

which is a solution of the quasi-geostrophic potential vorticity equation. If Ψ_o is a solution it must satisfy the steady version of the quasi-geostrophic potential vorticity equation (qgpve),

$$J\left(\Psi_o, \nabla^2\Psi_o + \beta y + \frac{1}{\rho_s}\frac{\partial}{\partial z}\rho_s\frac{f_o^{\,2}}{N^2}\frac{\partial\Psi_o}{\partial z}\right) = curl(\Im) + \frac{f_o}{\rho_s}\frac{\partial\rho_s \mathrm{H}}{\partial z} \equiv \Sigma \qquad (3.1.2)$$

In the above we use the notation for the Jacobian (with respect to x and y) introduced in Chapter 2). We will also use the Laplacian, as in (3.1.2) to refer to the two-dimensional Laplacian in the x-y plane. The sum of the dissipation and heating terms on the right hand side of the equation is Σ and may well be a function of the streamfunction.

In some cases, e.g. a purely zonal flow for which Ψ_o is independent of x, the left hand side of (3.1.2) will vanish identically and so Ψ_o will automatically satisfy the inviscid form of the qgpve. If not we imagine that the forcing has been chosen so that our selected Ψ_o satisfies (3.1.2).

Now we suppose Σ can be split into two parts. One part may depend on the streamfunction itself, for example the frictional diffusion of vorticity or the thermal diffusion of temperature. A second part we consider as an external forcing and is a given function of space but not explicitly of the streamfunction. That is,

$$\Sigma = \Sigma_i(\psi) + \Sigma_e(x,y,z) \qquad (3.1.3)$$

Then (3.1.2) can be simply written

$$J(\Psi_o, q_o) = \Sigma_i(\Psi_o) + \Sigma_e,$$

$$q_o = \nabla^2\Psi_o + \frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s\frac{f_o^{\,2}}{N^2}\frac{\partial\Psi_o}{\partial z}) + \beta y \qquad (3.1.4\ a,b)$$

We call the solution of this system the *basic flow*. Note that as we have formulated the problem it is independent of time. The solution Ψ_o is what it is and will not change as a result of an instability. The *mean zonal* flow may change and depart from the basic flow. The basic flow is defined as that flow which exists under specified external forcing *in the absence of perturbations* (or eddies).

Now the general qgpve can be written

$$\frac{\partial q}{\partial t} + J(\psi, q) = \Sigma_i(\psi) + \Sigma_e \qquad (3.1.5)$$

Now let's consider the perturbation problem. Let

$$\psi = \Psi_o + \psi'(x, y, z, t),$$
$$q = q_o + q'(x, y, z, t) \qquad (3.1.6\ \text{a,b})$$

Inserting this partition into (3.1.5) and subtracting from the result the equation for the basic state (3.1.4 a), we obtain,

$$\frac{\partial q'}{\partial t} + J(\Psi_o, q') + J(\psi', q_o) + J(\psi', q') = \delta\Sigma_i(\Psi_o, \psi')$$
$$\delta\Sigma_i(\Psi_o, \psi') \equiv \Sigma_i(\Psi_o + \psi') - \Sigma_i(\Psi_o) \qquad (3.1.7\ \text{a.b})$$

Note that (3.1.7 a) is non linear (in general). The second term on the left hand side is the advection of perturbation potential vorticity by the basic flow. The third term is the advection of the potential vorticity of the basic flow by the perturbation velocities (horizontal) while the fourth term on the left had side is the self-advection of the perturbation potential vorticity by the perturbation field itself. This is a quadratic nonlinearity. The right hand side is the dissipation term due to the perturbations. Note that it vanishes if the perturbation is zero. If for example,

$$\Sigma_i(\psi) = A\nabla^2\psi,$$

$$\delta\Sigma_i = A\nabla^2\psi'$$

That is, if the dissipation term is linear in the streamfunction the dissipation term in the perturbation qgpve will depend only on the perturbation.

It is very important to note that the external forcing:

1) Σ_e, *does not appear in the equation governing the nonlinear development of the perturbation* even when there is dissipation. Its physical presence is, instead, manifested by the appearance of the basic state which is maintained against dissipation by the external forcing.

2) If Σ_e represents an energy source and Σ_i a drain this does not imply that the perturbation field $\psi' \rightarrow 0$ for large t. The energy source is implicit in the prescribed and unchanging field Ψ_o.

3) Each term in (3.1.7) goes to zero as the perturbation → 0.

4) The problem for ψ' leaves Ψ_o unchanged in both the linear and nonlinear versions of the problem. In the nonlinear problem ψ' will, in general, have a non zero x-average which we can identify with the alteration of the zonal flow from the basic flow. That is $\overline{\psi'} \neq 0$ so that the total x-independent stream function $\Psi(y,z) = \Psi_o(y,z) + \overline{\psi'}$. This correction to the mean field is related to the potential vorticity fluxes in the perturbation field as described in Chapter 2 so that normally, the correction to the mean, zonal flow is of order (amplitude)^2. Note that this emphasizes again that it is Ψ_o and not Ψ that should be used to formulate the stability problem.

The boundary condition at the lower boundary, (2.2.3)

$$\frac{f_o}{N^2}\frac{d\psi_z}{dt} + \delta_E \nabla_h{}^2 \psi + J(\psi, h) = H, \tag{3.1.8}$$

can be partitioned in the same way and its perturbation form is,

$$\frac{\partial \psi'_z}{\partial t} + J(\Psi_o, \psi'_z) + J(\psi', \Psi_{o_z}) + J(\psi', \psi'_z) + \frac{N^2}{f_o} J(\psi', h) + \delta_E \nabla^2 \psi' = \frac{N^2}{f_o}\delta H(\Psi_o, \psi') \tag{3.1.9}$$

where the right hand side is the perturbation of the heating at the level z=0 due to the perturbation.

3.2 Stability problem for zonal flows

If the basic flow is a zonal flow♦ so that

$$\Psi_o = \Psi_o(y,z) \tag{3.2.1}$$

then

$$\begin{aligned} U_o &= -\Psi_{o_y} \\ q_o &= \beta y + \Psi_{o_{yy}} + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s f_o^2}{N^2}\Psi_{o_z}\right) \end{aligned} \tag{3.2.2}$$

Using the fact that the basic state is independent of x, (3.1.7) becomes

$$\frac{\partial q'}{\partial t} - \Psi_{o_y} q'_x + \psi'_x q_{o_y} + J(\psi', q') = \delta\Sigma_i \tag{3.2.3}$$

or,

$$\left\{\frac{\partial}{\partial t} + U_o\frac{\partial}{\partial x}\right\} q' + \frac{\partial \psi'}{\partial x}\frac{\partial q_o}{\partial y} + J(\psi', q') = \delta\Sigma_i \tag{3.2.6}$$

The properties of the basic flow that enter the stability problem are the basic velocity itself and the meridional gradient of the basic state potential vorticity. The bracketed term on the left hand side is the linearized form of the advective derivative of the basic flow while the next term is the advection by the perturbation's meridional

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♦ Note that for the flow to consistently be independent of x the bottom boundary condition requires either that *h* be independent of x or that the zonal flow be zero at the bottom. We shall choose the former otherwise the perturbation problem becomes quite complex. Henceforth then, *h=h(y)*.

velocity in the meridional gradient of the ambient or background potential vorticity of the basic state. This, as we shall see is a crucial property of the flow that helps determine its stability. In particular,

$$\frac{\partial q_o}{\partial y} = \beta - U_{o_{yy}} - \frac{1}{\rho_s}\left[\rho_s \frac{f_o^{\,2}}{N^2} U_{o_z}\right]_z \tag{3.2.7}$$

so that the potential vorticity gradient depends on the planetary vorticity gradient, β, the horizontal *curvature* of the basic flow and the vertical shear and its derivative. (If the background density can be considered constant and if N is constant then only the vertical curvature of the basic flow profile enters). For weak flows the potential vorticity gradient is dominated by the beta term and we might imagine that since stable Rossby waves are the solution for a constant basic flow the flow for weak shear would be stable. We shall be surprised!

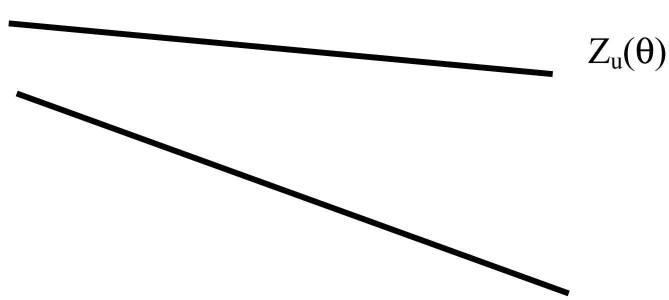
Using the thermal wind relation,

$$f_o U_{o_z} = -g \frac{\vartheta_{o_y}}{\vartheta_s} \tag{3.2.8}$$

and the definition of N^2, the last term in (3.2.7) can be written

$$\frac{\partial}{\partial z}\left(\frac{f_o \vartheta_{o_y}}{\vartheta_{s_z}}\right) = -f_o \frac{\partial}{\partial z}\left(\frac{\partial z}{\partial y}\right)_{\vartheta} \tag{3.2.9}$$

so that it represents the narrowing of the slope of isopycnal layers northward.



Z_L (θ)

Fig. 3.2.1 The interpretation of the vertical shear contribution to the potential vorticity gradient of the basic flow.

In the case shown the slope of the isentropic surfaces (or isopycnals for the ocean) decreases upward so the term in the potential vorticity gradient is positive. It is interesting to note that in this sense the squeezing together of the isentropic surfaces acts like a beta effect in precise analogy to the topographic beta effect in a homogeneous fluid (note the similar factor of f_O)

To start our discussion of the stability of the zonal flow it is useful to return to the equations for the evolution of the mean zonal flow. Using the identity we proved before between the perturbation fluxes (we will often call them the eddy fluxes) of the departures from the zonal average and the departures from the basic flow,

$$\frac{\partial \overline{u}}{\partial t} = -(\overline{u'v'})_y + f_o \overline{v}_a$$

$$\frac{\partial}{\partial t}\left(g\frac{\overline{\vartheta}}{\vartheta_s}\right) = -\left(\overline{v' g\vartheta'/\vartheta_s}\right)_y - \overline{w}N^2 \qquad \text{(3.2.10 a,b)}$$

We have multiplied the equation for potential temperature by the factor g/ϑ_s so that (3.2.10b) is written in terms of the buoyancy $\overline{b} = g\overline{\vartheta}/\vartheta_s$ of the mean zonal flow thus naturally introducing N^2. Note that b (in this case the zonal average, but we are speaking in general) has the dimensions of an acceleration.

We form the energy equation for the zonal mean flow by multiplying (3.2.10) by $\rho_s\overline{u}$ and (3.2.10b) by $\rho_s b/N^2$ and then adding to obtain,

$$\frac{\partial}{\partial t}\rho_s\left[\frac{\bar{u}^2}{2}+\frac{\bar{b}^2}{2N^2}\right]=-\rho_s\bar{u}(\overline{u'v'})_y-\rho_s\bar{b}(\overline{v'b'})_y/N^2$$

$$+\rho_s f_o\bar{u}\bar{v}_a-\rho_s\bar{b}\,\bar{w} \tag{3.2.11}$$

We have ignored dissipation for the present purpose. It is left to the student to fill in the minor changes required when dissipation of momentum and buoyancy are considered.

The quantity in the square bracket on the left hand side is the sum of the kinetic energy of the zonal flow and its available potential energy. The latter term should be familiar from your studies of internal gravity waves in 12.802. We will denote the total of the left hand side after multiplication by the density as $E(\bar{\psi})$.

To interpret the right hand side let us start with the last two terms. With the aid of the geostrophic and hydrostatic relations, $\rho_s f_o\bar{u}=-\bar{p}_y, \quad \bar{b}=(\bar{p}/\rho_s)_z$

$$\rho_s f_o\bar{u}\bar{v}_a-\rho_s\bar{b}\,\bar{w}=-\bar{v}_a\frac{\partial\bar{p}}{\partial y}-\rho_s\bar{w}\frac{\partial}{\partial z}\left(\frac{\bar{p}}{\rho_s}\right)$$

$$=-(\bar{p}\bar{v}_a)_y-(\bar{p}\bar{w})_z \tag{3.2.12}$$

These terms we recognize as the energy flux vector of the mean flow or equivalently, the rate of pressure work of the mean flow. If we were to integrate over a closed domain, again assuming that h is independent of x so that $\bar{w}$ is zero at the surface, the total energy flux will integrate to zero so that these terms would yield no net transfer of energy to or from the zonal mean flow. Let us use a bracket to denote a further integration over y and z. Then,

$$\frac{\partial}{\partial t}\langle E(\bar{\psi})\rangle=-\iint\rho_s dydz\left\{\bar{u}(\overline{u'v'})_y+\bar{b}(\overline{v'b'})_y/N^2\right\} \tag{3.2.13}$$

The rate of change of the energy of the mean flow can be changed by either (or both) of perturbation flux terms. The first is the gradient (in y) of eddy Reynolds stress, or momentum transfer by the perturbations, times the mean momentum itself, the second term is due to the meridional buoyancy flux times the mean buoyancy. Note that in each case it is the meridional gradient of the fluxes that enters the energy equation and this is not surprising since as we saw earlier it is the meridional gradient of the potential vorticity flux that alters the mean flow and hence its energy.

Assuming that the geostrophic meridional velocity vanishes at the boundaries of the domain in y (they may be at infinity in some models), we can integrate the right hand side of (3.2.13) by parts to obtain,

$$\frac{\partial}{\partial t}\langle E(\overline{\psi})\rangle = \iint \rho_s dydz \left\{ \overline{u}_y(\overline{u'v'}) + \overline{b}_y(\overline{v'b'})/N^2 \right\} \tag{3.2.14}$$

Thus if the momentum flux of the perturbations is negative correlated with the mean meridional shear of the basic state so that on average $\overline{u}_y\overline{u'v'} < 0$, i.e. that the eddy momentum flux is *down* the mean gradient, transferring momentum from where it is high in the mean flow to where it is low, and if the same is true of the buoyancy flux (i.e. if the heat flux is *down* the mean temperature gradient or the density is *down* the mean density gradient) then the mean flow will be losing energy as a consequence of the perturbations. Where does that energy go? We anticipate that the energy must be transformed from the mean into the energy in the perturbation field. Thus a condition for instability is that the perturbations be configured such that the right hand side of (3.2.14) is less than zero. It is important to realize that either term could be positive or negative. Only the sum must be negative to decrease the mean energy.

Returning to the equation for the mean zonal momentum itself (3.2.10a) it follows that *at each y* an integral over the depth of the fluid yields,

$$\frac{\partial}{\partial t}\int \rho_s \overline{u}\,dz = -\int \rho_s(\overline{u'v'})_y + \underbrace{\int \rho_s f_o \overline{v}_a dz}_{=0} \tag{3.2.15}$$

The last term is zero by an vertical integral of the continuity equation which shows that $\partial/\partial y \int \rho_s \overline{v}_a dz = 0$, applying the condition that the ageostrophic meridional velocity vanish on at least one of the zonal boundaries implies it the integral is zero for all y (as much mass must be going northward and going southward when integrated in x and z).

Thus, the mean momentum must decrease if there is a divergence of perturbation momentum flux in y (more momentum leaving a latitude band than entering it). Decreasing the momentum where the momentum is positive decreases the kinetic energy of the mean and this is consistent with the first term in (3.2.13) although the interpretation is not strictly accurate since we need to form the square of $\bar{u}$ before vertically integrating. Nonetheless it show the tendency to remove mean energy in a region of divergent Reynolds fluxes.

Returning to the right hand side of (3.2.14) and using the geostrophic stream function to evaluate the eddy fluxes,

$$\bar{u}_y \overline{u'v'} = -\bar{u}_y \overline{\psi'_x \psi'_y} = -\bar{u}_y \overline{(\psi'_y)^2 \frac{\psi'_x}{\psi'_y}} = \bar{u}_y \overline{(u')^2 \left(\frac{\partial y}{\partial x}\right)_{\psi'}} \tag{3.2.16}$$

The degradation of the mean energy and therefore the delivery of energy to the perturbations by the action of the Reynolds momentum stress is favored by perturbations that *lean* against the shear, i.e. perturbations whose isolines of constant ψ' are sloped against the shear, as shown in the figure below,

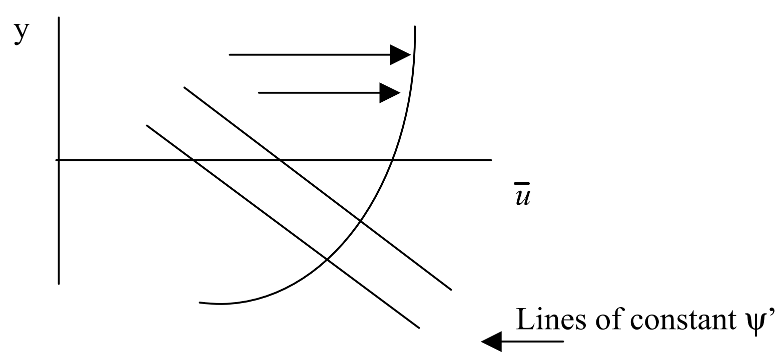


Figure 3.2.1 The shear of the mean flow is positive here. The lines of constant ψ' are leaning against the shear and extracting energy.

Note that this is precisely opposite to what we would expect of a passive tracer painted on the shear that would be tilted in the forward direction by the shear. To release energy the perturbations are clearly not acting like *passive* tracers. Of course, such a direction for the isolines of the disturbance need not have this relation to the shear at every location to release energy. It is only necessary that this must be satisfied on average when integrated over the domain of the fluid. Such transformations which could occur even in a homogeneous, barotropic fluid are called barotropic conversion terms even if the fluid is stratified.

Similarly, the second term in (3.2.14) can be written,

$$\overline{b}_y \overline{v'b'}/N^2 = -\frac{f_o^2}{N^2}\overline{u}_z \overline{\psi'_x \psi'_z} = \frac{f_o^2}{N^2}\overline{u}_z \overline{(\psi'_z{}^2)\left(\frac{\partial z}{\partial x}\right)_{\psi'}} \tag{3.2.17}$$

which has the same form as (3.2.16) with y replaced with z (aside from the factors of f_O and N). So that energy (this time potential energy associated with the smoothing out of the mean buoyancy gradient) will be released if the perturbations lean against the *vertical* shear of the mean current.

Returning to (3.2.14) and once more using the thermal wind relation,

$$\frac{\partial}{\partial t}\langle E(\overline{\psi})\rangle = \iint \rho_s dydz\left[\overline{u'v'}\overline{u}_y - \overline{v'b'}f_o\overline{u}_z/N^2\right] \tag{3.2.18}$$

Integrating by parts yields,

$$\begin{aligned}\frac{\partial}{\partial t}\langle E(\overline{\psi})\rangle &= -\iint dydz\left[\rho_s\overline{(u'v')}_y\,\overline{u} - \overline{(\rho_s v'b')}_z f_o\overline{u}/N^2\right] \\ &\quad -\int \rho_s dy\left\{\left(\frac{f_o}{N^2}\overline{u}\overline{v'b'}\Big|\right)_{z=z_t} - \left(\frac{f_o}{N^2}\overline{u}\overline{v'b'}\Big|\right)_{z=0}\right\}\end{aligned} \tag{3.2.19}$$

where z_t is the coordinate of the upper boundary.

Referring to (2.5.9) this becomes,

$$\frac{\partial}{\partial t}\langle E(\overline{\psi})\rangle = \iint \rho_s dydz\left[\overline{u}\,\overline{v'q'}\right]$$

$$-\int \rho_s dy\left\{\left(\frac{f_o}{N^2}\overline{u}\overline{v'b'}\right)_{z=z_t} - \left(\frac{f_o}{N^2}\overline{u}\,\overline{v'b'}\right)_{z=0}\right\} \tag{3.2.20}$$

Thus the change of the energy of the mean depends on the correlation of the perturbation potential vorticity flux with the mean current within the body of the fluid and the correlation of the buoyancy fluxes on the horizontal boundaries with the zonal mean current at those depths. What we shall see frequently, as we do here, is the essential equivalence for the stability problem of buoyancy fluxes on the boundary and potential vorticity fluxes in the interior. Note that for the first term in (3.2.20) it will be negative, leading to a decrease of the mean flow energy if the pv flux is negatively correlated with the basic current, a result that will have considerable significance when we examine the necessary conditions for the growth of perturbations on the basic flow.

Finally, returning to the equation for the zonal mean of the potential vorticity, (2.5.11)

$$\frac{\partial}{\partial t}\overline{q} = -\left(\overline{v'q'}\right)_y \tag{3.2.21}$$

an integration by parts after multiplication by $\overline{q}$ yields,

$$\frac{\partial\langle\overline{q}^2/2\rangle}{\partial t} = \left\langle(\overline{v'q'})\frac{\partial\overline{q}}{\partial y}\right\rangle \tag{3.2.22}$$

If the *enstrophy* $\overline{q}^2/2$ of the mean field is to decrease as a consequence of the perturbations the eddy transport of potential vorticity must be *down* the gradient of the zonal mean of the potential vorticity.