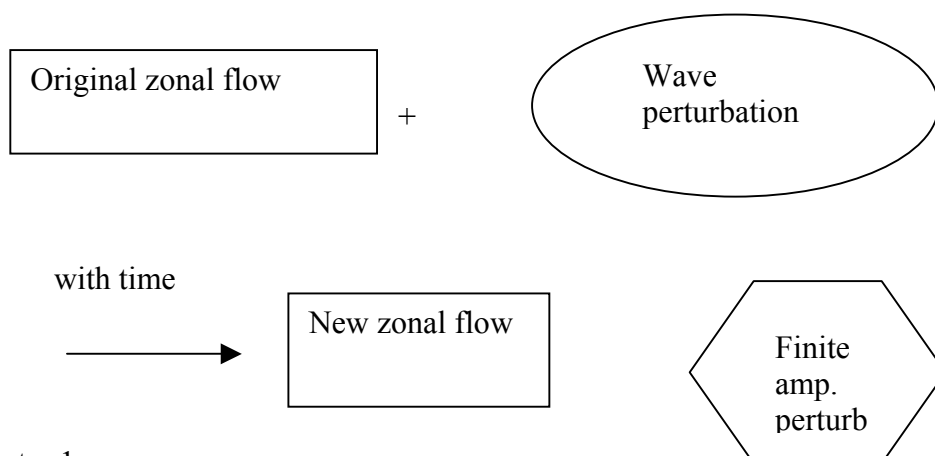


Chapter 1.

1.1 Introduction

This course aims to present an account of concepts and results in Geophysical Fluid Dynamics regarding the instability of large scale flows in the atmosphere and ocean. One can also think of it as a necessary conceptual preface to a study of the turbulent character of both geophysical systems. The course is primarily theoretical in character and will concentrate on developing an appreciation of both the physics of large scale *quasi-geostrophic* instabilities and the techniques required for that study. The techniques are general and are easily applied to the issue of instability of other physical systems.

Although, in principle, the goal of stability theory is to determine whether a particular flow is stable or unstable we are, in fact, not much concerned about *whether* a flow is unstable but rather we hope to use the favored (a term we must clarify) *mode* of instability to account for the observed rather complex, finite-amplitude disturbances in the atmosphere and oceans. These disturbances are at time and space scales impossible to relate directly to external forcing and we imagine the observed motions as springing from the instability of a simpler state of flow. The total motion then consists of the original flow (no longer observed) plus the perturbation (which usually includes a portion we can identify as a *change* in the basic flow). For example, if the basic flow were a zonal flow (a current directed in the east-west direction independent of longitude):



+ +

The new field of motion can be thought of as the original flow, plus a finite perturbation of that zonal flow which represents a new zonal flow, plus a finite amplitude perturbation with zero zonal average.

The new zonal flow is thus a response to the instability and it is not the original (or basic) flow. The new zonal flow is what we might imagine the *observed* zonal flow is, assuming we have the dynamics right and its structure is already shaped by the presence of the instability. It follows as a matter of simple logic therefore, that the observed zonal flow, perhaps constructed by averaging the observed turbulent field, is *not* the appropriate flow structure to test for instability. What we need to test is the basic flow, that is, the flow that would exist in the *absence* of perturbations, e.g. the pre-existing state to the perturbation. To explain the existence and structure of the observed perturbations we first should construct the structure of the flow that would exist if there were no instability. This is usually too hard to do.

This forces us to examine the stability of *classes* of flows to be sure that the derived instability is not special and delicately peculiar to the particular flow we have chosen to test since that flow may not be exactly the correct *basic* flow. This leads us naturally to a view of stability theory less tied to detailed simulations and more connected with general processes.

1.2 Model problems

Let's illustrate some of the aforementioned ideas with a model stability problem with which we can explicitly carry out all the steps implied in the above description in a simple way. The real stability problems are usually quite complicated. For example, determining the right basic state is often an difficult theoretical problem in itself. The model problem we discuss here has many of the features of the general stability problem in a simple form.

Consider a basic flow u which is exposed to a perturbation ϕ and suppose these fields are governed by the following equations:

$$\begin{aligned}\phi_{tt} + \nu\phi_t - (u - u_c)\phi + N\phi^3 &= 0 \\ u_t + \eta u &= Q - \eta\delta\phi^2\end{aligned}\tag{1.2.1 a,b}$$

In the above equations the parameters ν, η are coefficients of frictional dissipation (note that they might be different for the basic flow and the perturbation. The *parameter* u_c is a value of u that will yield a threshold for instability; it is a *critical* value for the basic flow. The last term in (1.2.1 a) is a nonlinear term measuring the self-interaction of the wave and we take it to be a cubic nonlinearity as is often the case. Note too, that there is another nonlinearity in (1.2.1a) which is the interaction of the perturbation with the evolving mean flow u . The mean flow $u(t)$ evolves with time. It is forced by some external forcing Q and is also altered by a *quadratic* nonlinearity due to the perturbation. The coefficient δ is a measure of how the nonlinear effects in the perturbation can alter the mean flow. Subscripts denote differentiation and the basic simplicity of the above model is due to the fact that the system is just a coupled system of ordinary differential equations. The equations of fluid mechanics and the resulting stability problems are, of course, partial differential equations. As we shall see, the system is nevertheless rich enough to illustrate many of the points we discussed in the introduction.

A. Perturbation – free state.

Let's first calculate the state of motion in the absence of perturbations. If $\phi=0$ and Q is independent of time it is an easy matter to calculate the resulting “flow” u ,

$$u \equiv u_o = Q/\eta\tag{1.2.2}$$

Note that we could, instead, choose u_o and calculate Q . This would be much easier to do if η were a complicated operator. Any basic state can consistently correspond to

some forcing. Of course, we have to decide whether the required forcing is too bizarre to be considered acceptable.

B. Small perturbation instability of u_o

Let us suppose that the parameter ε is introduced as a measure of the relative amplitude of the perturbation to the size of the basic flow u_o .

Then we imagine we can construct the following asymptotic series :

$$\begin{aligned}\phi &= \varepsilon \phi_o + \varepsilon^2 \phi_1 + \dots, \\ u &= u_o + \varepsilon u_1 + \varepsilon^2 u_2 + \dots\end{aligned}\tag{1.2.3 a,b}$$

Examining (1.21 b) it seems obvious that we will find u_1 to be zero and the first correction to u to be order ε^2 .

If we insert the series (1.2.3 a,b) into (1.2.1 a,b) and equate like powers of the parameter ε we obtain the following *linear* problem for the small amplitude instability:

$$\phi_{ott} + \nu \phi_{ot} - (u_o - u_c) \phi_o = 0\tag{1.2.4}$$

It is important to note that the forcing term Q , which you can think of as the ultimate energy supply, is absent from the linear stability problem in its explicit form. It is present indirectly. The basic flow u_o is a manifestation of the forcing and is *fixed*. Note that the stability problem depends on the difference between the basic flow and its critical value.

Linear solutions of (1.2.4) can be found in the form:

$$\phi = A e^{\sigma t}\tag{1.2.5}$$

where the *growth rate* σ satisfies the algebraic equation,

$$\sigma^2 + \nu \sigma - (u_o - u_c) = 0,$$

$$\Rightarrow \quad (1.2.6 \text{ a,b})$$

$$\sigma = -\frac{\nu}{2} \pm \left\{ \frac{\nu^2}{4} + (u_o - u_c) \right\}^{1/2}$$

If $u_o - u_c > 0$ then the root with the + sign will be positive and the perturbation will exponentially grow. The other root will lead to decaying perturbations.

Neutral stability.

The real part of the growth rate will vanish for the most unstable mode when $u_o = u_c$ at which point :

$$\sigma_+ = 0, \quad (1.2.7 \text{ a,b})$$

$$\sigma_- = -\nu.$$

If there were no friction, i.e. if ν were zero both roots would coalesce at the critical value of u_o .

Suppose $u_o = u_c + \Delta$, where Δ is the supercriticality of the flow, that is, the excess of the basic flow above its critical value u_c .

Then,

$$\sigma = -\frac{\nu}{2} \pm \left\{ \frac{\nu^2}{4} + \Delta \right\}^{1/2} \quad (1.2.8)$$

If Δ were small, i.e. in the limit as $\Delta \rightarrow 0$, the resulting form for σ depends on the relative size of the supercriticality to the friction.

If

$$\nu = O(1), \Delta \ll \nu,$$

$$\begin{aligned}\sigma_+ &= \frac{\Delta}{\nu} \\ \sigma_- &= -\nu\end{aligned}\tag{1.2.9 a,b,}$$

In this limit one root is slowly growing and the other root is rapidly decaying.
On the other hand if the perturbation dynamics is basically inviscid, i.e. if

$$\nu \ll \Delta^{1/2} \ll 1$$

then to lowest order,

$$\sigma = \frac{-\nu}{2} \pm \Delta^{1/2}$$

so that in the limit as $\nu \rightarrow 0$ the two roots have the same magnitude, one representing a growing mode and the other a decaying mode with the rates proportional to the square root of the supercriticality, i.e. there is a branch point for σ as Δ passes through zero.

We sketch in below the growth rate curve for the two cases. The form is the same but the off-set with respect to the zero line is crucial.

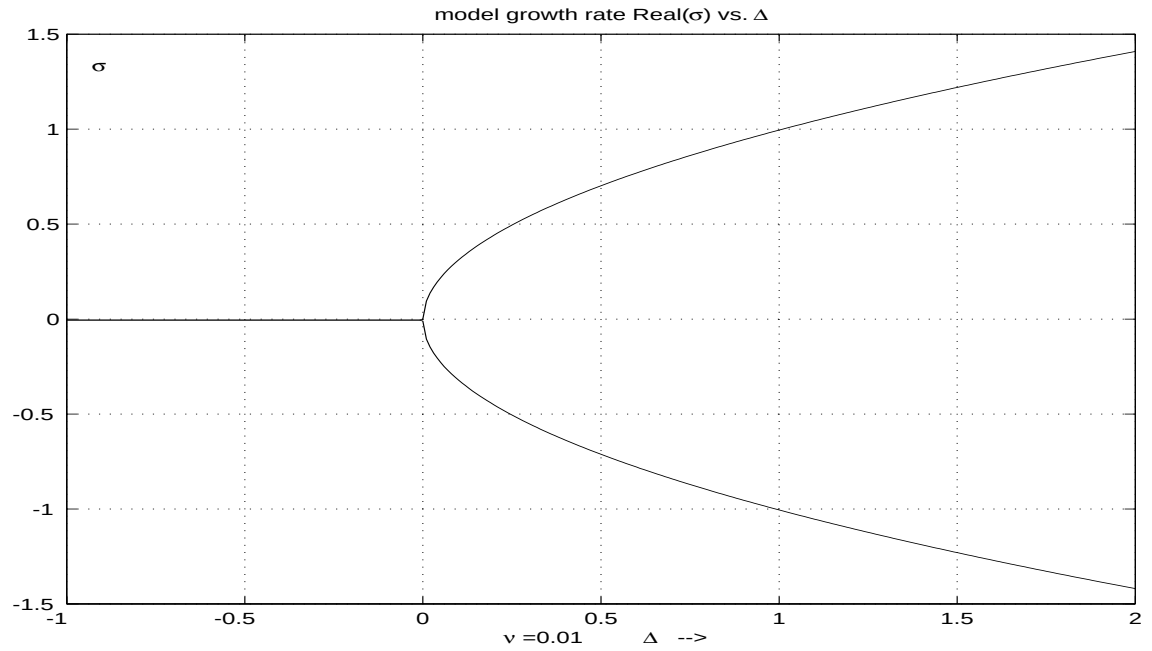


Figure 1.2.1 a. The growth rate curve for very small friction. Note that the growth rate σ has a branch point behavior near $\sigma = 0$.

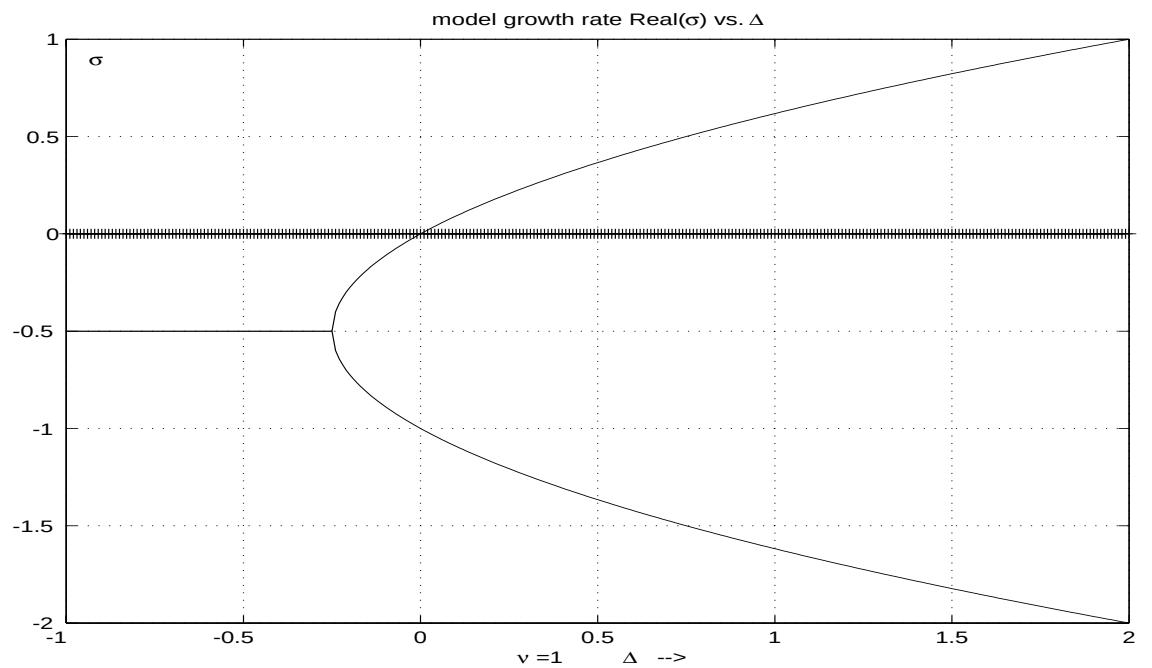


Figure 1.2.1.b Here the cross hatch has been place along the zero growth rate axis to emphasize that in this case that as Δ increases through zero, one root grows linearly and becomes positive as given by (1.2.9 a) while the other root is negative and $O(1)$.

C. Finite amplitude steady state.

Although a solution with $\phi = 0$ is always a possible solution, we can also look for steady solutions of (1.2.1 a,b). For the solution with ϕ different from zero, (1.2.1a) implies:

$$\phi^2 = \frac{u - u_c}{N} \quad (1.2.10)$$

Thus, if $u > u_c$ a real solution for ϕ is possible if N is positive. Such solutions are called *supercritical*. If instead, $N < 0$, real solutions for ϕ require $u < u_c$ and are termed *subcritical*.

From (1.2.1b) it follows that for steady solutions,

$$u = \frac{Q}{\eta} - \delta\phi^2 = u_o - \delta\phi^2 \quad (1.2.11)$$

or,

$$\phi^2 = \frac{u_o - \delta\phi^2 - u_c}{N} \quad (1.2.12)$$

so that,

$$\phi = \left(\frac{\Delta}{N}\right)^{1/2} \frac{1}{\left[1 + \frac{\delta}{N}\right]^{1/2}} \quad (1.2.13)$$

In the figure below the *bifurcation* diagram shows how the amplitude, in the case of supercritical bifurcation, springs from the zero supercriticality point in the forward direction with a square root branch point.

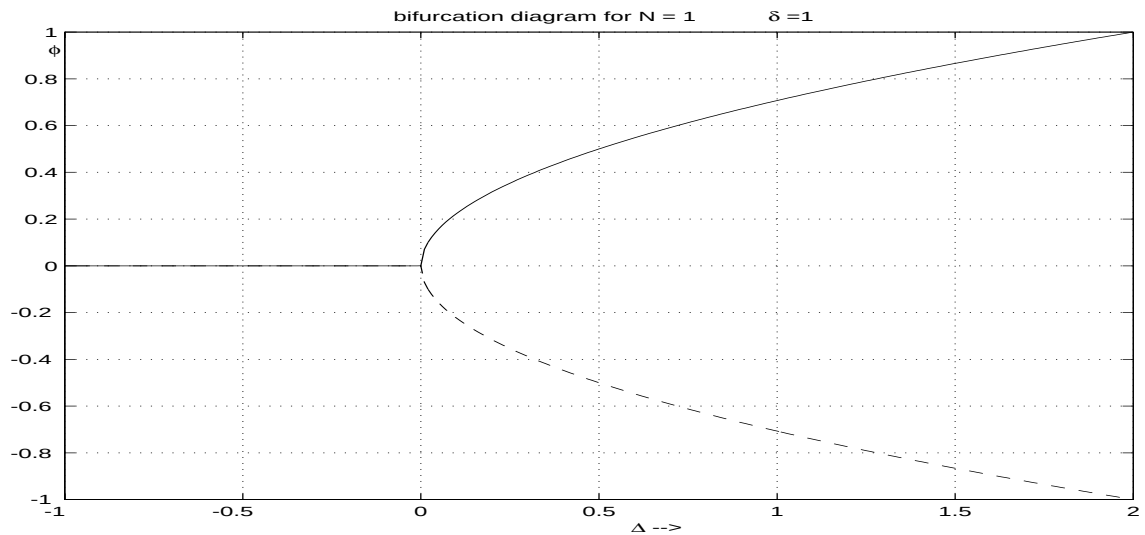


Figure 1.2.2 a. Supercritical bifurcation ($N > 0$).. Steady finite amplitude solutions exist only where the basic flow is linearly unstable ($\Delta > 0$).

On the other hand, if $N < 0$, steady solutions occur where the basic flow is *linearly stable*.

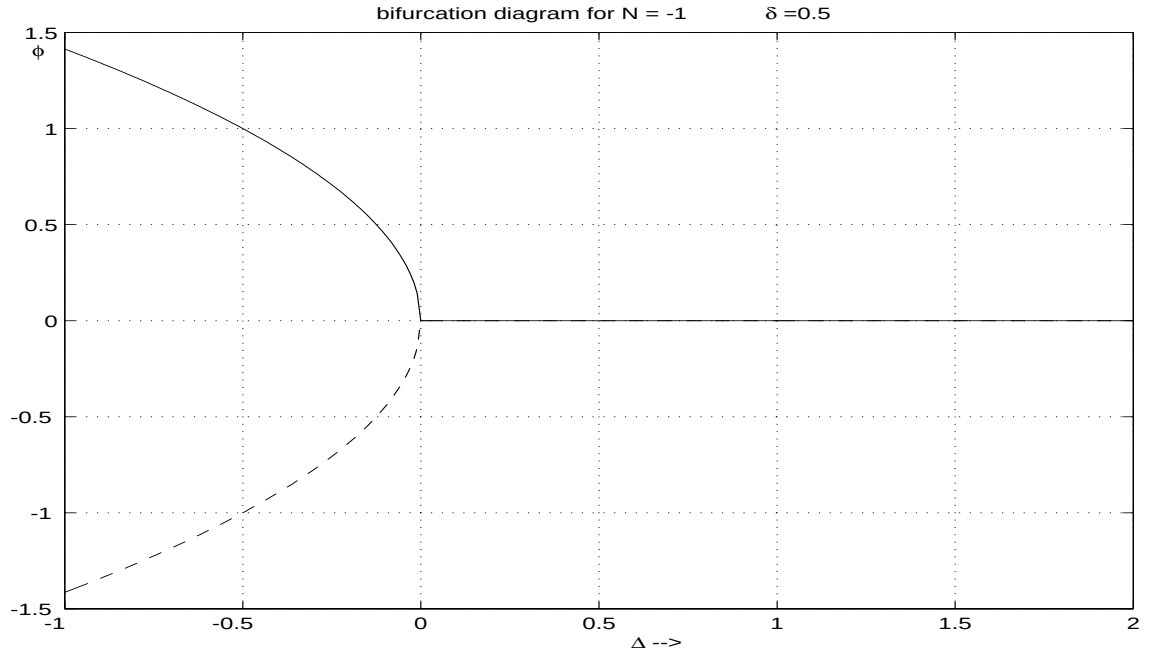


Figure 1.2.2 b. The bifurcation diagram for steady solutions when $N < 0$.

Of course we have no guarantee that the final state reached after a large time will be steady in finite amplitude.

Note that in the steady state,

$$\begin{aligned}
 u &= u_s = u_o - \delta \frac{\Delta}{N \left(1 + \frac{\delta}{N} \right)} \\
 &= u_o - \frac{\delta [u_o - u_c]}{N \left(1 + \frac{\delta}{N} \right)}
 \end{aligned}
 \tag{1.2.14}$$

from which it follows,

$$u_s - u_c = \frac{N}{N + \delta}(u_o - u_c), \quad (1.2.15 \text{ a,b})$$

$$\phi_s^2 = \frac{(u_o - u_c)}{N + \delta}$$

where the subscript s refers to the steady finite amplitude solution.

Note that as $N \rightarrow 0$ for fixed supercriticality, $u_o - u_c$, u actually approaches u_c (i.e. flow is driven towards the stability threshold), but the disturbance ϕ reaches a maximum as a function of N . In this limit the entire supercriticality is eaten up by the perturbation and the flow is driven down to its critical threshold but the perturbation field is large. It would clearly be misleading to use the finite amplitude value of u to do a stability analysis in that case. We would find the flow nearly stable (a little friction would damp the disturbances) and yet the nonlinear theory shows that this is a case where large amplitude perturbations exist. In fact, it is just those perturbations that drive the basic flow towards the critical threshold and shows the peril of using $u - u_c$ rather than $u_o - u_c$ to discuss the instability.

D. Stability of the steady solutions.

Now that we have found possible steady, finite amplitude solutions we can ask whether they are stable. If they are not stable they will never be reached by initial conditions that initiate far from those states.

We can take the solutions given by (1.2.15) and perturb them (this is, as you can imagine an extremely difficult problem in the fluid mechanical case).

We write

$$\begin{aligned} \phi &= \phi_s + \phi', \\ u &= u_s + u' \end{aligned}$$

The prime variables are the perturbations to the steady solution. Assuming, first that they are small perturbations, inserting the above forms into (1.2.1 a, b) and keeping only the linear terms in the primed variables leads to,

$$\phi'_{tt} + \nu \phi'_t - (u_s - u_c) \phi' - u' \phi_s + 3N \phi_s^2 \phi' = 0, \quad (1.2.16 \text{ a,b})$$

$$u'_t + \eta u' = -2\eta \delta \phi_s \phi'$$

Using the explicit form of the steady solution the above equations reduce to,

$$\phi'_{tt} + \nu \phi'_t + \frac{2N}{N + \delta} [u_o - u_c] \phi' - u' \phi_s = 0, \quad (1.2.17 \text{ a,b})$$

$$u'_t + \eta u' = -2\eta \delta \phi_s \phi'$$

This is just a linear system with constant coefficient so that solutions can be found in the form,

$$\phi' = A e^{st}, \quad u' = U e^{st} \quad (1.2.18 \text{ a,b})$$

leading to,

$$A \left\{ s^2 + \nu s + \frac{2N}{N + \delta} \Delta \right\} = \phi_s U, \quad (1.2.19, \text{a,b})$$

$$U \{s + \eta\} = -2\eta \delta \phi_s A$$

This in turn yields the dispersion relation for the growth rate s ,

$$\left\{ s^2 + \nu s + \frac{2N}{N + \delta} \right\} = -2 \frac{\eta}{s + \eta} \delta \phi_s^2$$

$$= -2 \frac{\eta}{s + \eta} \delta \frac{N}{(N + \delta)} \quad (1.2.20)$$

which is a cubic for s ,

$$(s + \eta) \left[s^2 + \nu s + \frac{2N\Delta}{N + \delta} \right] = -2\eta \frac{\delta\Delta}{N + \delta} \quad (1.2.21)$$

Instead of solving the cubic in general (which is so complex as to be not illuminating) it is better instead to look at some important special cases.

I) Suppose $\eta = O(\nu) \gg 1$.

It follows pretty easily that two approximate roots are $s = -\eta$, $s = -\nu$ while the third root for s is small (order η^{-1}) and satisfies,

$$\left[s \nu + \frac{2N\Delta}{N + \delta} \right] \eta = -2\eta \frac{\delta\Delta}{N + \delta} \quad (1.2.22)$$

$$\Rightarrow s = -2 \frac{\Delta}{\nu}$$

so that all three roots are negative in this limit and the steady solution is *linearly* stable.

II) Suppose instead that $\eta = O(\nu) \ll 1$

Then it is necessary, as we shall see to expand s in the series,

$$s = s_o + \eta s_1 + \dots \quad (1.2.23)$$

Keeping terms only to order $\eta = O(\nu) \ll 1$ we obtain from (1.2.21)

$$\left(s_o^2 + 2s_1 s_o \eta + \nu s_o + \frac{2N\Delta}{N + \delta} \right) (s_o + \eta s_1 + \eta) = -2\eta \frac{\delta\Delta}{N + \delta} \quad (1.2.24)$$

At order unity either s_o is zero or the nonzero root satisfies,

$$s_o^2 = -2N \frac{\Delta}{N + \delta} \quad (1.2.25)$$

If N is positive and Δ is positive (otherwise we don't have a steady solution for positive supercriticality) the lowest order solution for the growth rate is imaginary representing an oscillation with frequency $\omega = \left(\frac{2N}{N+\delta} \Delta \right)^{1/2}$. To decide whether there is growth it is necessary to go to the next order, At order η a little algebra shows that,

$$2s_1 s_o^2 \eta = -v s_o^2 - 2 \frac{\eta \Delta \delta}{N + \delta} \quad (1.2.26)$$

or

$$s_1 = -\frac{v}{2\eta} + \frac{\delta}{2N} = \frac{\delta}{2N} \left[1 - \frac{Nv}{\delta\eta} \right] \quad (1.2.27)$$

Thus if

$$N < \frac{\delta\eta}{v} \quad (1.2.28)$$

the steady solution will be unstable.

The third root, which to lowest order was zero can be shown to be, at order η

$$s = -\eta \left[1 + \frac{\delta}{N} \right] \quad (1.2.29)$$

and is therefore always stable.

Even when the steady solutions are linearly stable it does not mean they will be realized as the result of an initial value problem. Since the initial conditions exist in solution space a finite distance from the steady solutions they represent a finite amplitude perturbation to the steady solution. Indeed, we will find that systems of the form (1.2.1) can have a very rich finite amplitude behavior even when the steady solutions are linearly

stable. The possibility exists in those cases for chaotic behavior in time. Indeed, the system (1.2.1a,b) is a very close cousin of the famous Lorenz equations which describe a variant of deterministic chaotic dynamics.

Thus, even the very simple model problem we have posed for ourselves shows very complex behavior. We must be prepared for at least this much complexity when we approach the fluid mechanical problem. As we proceed to the quasi-geostrophic problem let's keep in mind the overall pattern of the problem. It consists of the following elements:

- a) Find the basic flow or specify it and find the consistent forcing.
- b) Calculate the linear stability of the *basic flow* (the flow satisfying the forced problem with no eddies).
- c) Find the possible steady states.
- d) Find the stability properties of the steady states.
- e) Discuss the time dependent finite amplitude states.

This sequence of problems is ordinarily so difficult that we will rarely be able to proceed past (b). We will take up the nonlinear problem in a special case. However, we have to keep in mind that another important part of our investigation that we have utterly neglected in the model problem is the *spatial* behavior of the instability and its effect on the mean flow and that will be an important consideration for us to investigate.