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A transverse hydraulic jump in a model of the Faroe Bank Channel outflow

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Abstract

Realizations of transverse hydraulic jumps in rotating overflows have been limited to simple shallow-water models and to one laboratory experiment. Such jumps are marked by an abrupt increase in the width of the current. The following note identifies an apparent example of a transverse jump in a model of the Faroe Bank Channel overflow, one that contains realistic stratification and bathymetry and that reproduces other observed features of the Faroe Bank outflow plume. We discuss structural similarities between the model jump and idealized transverse jumps, including a new example of a shallow-water jump in a channel with a parabolic cross section. More significantly, we calculate the long-wave speeds of the Faroe Bank model plume and show that the requisite subcritical-to-supercritical transition takes place across the observed jump.

1. Introduction

Internal hydraulic jumps are potentially important sources of overturning, mixing and energy dissipation in ocean overflows and exchange flows. However, documentation of jumps in the ocean has thus far been limited to estuaries, inlets, the Strait of Gibraltar (Armi and Farmer, 1988), the Romanche fracture zone (Ferron et al., 1998), and other sites where the influence of the Earth's rotation is weak.[1] Examples of jumps in rotating channels exist but are largely confined to relatively idealized analytical, laboratory or numerical models [e.g. Pratt (1983, 1987), Nof (1986) and Pratt et al. (2000)]. In nearly all cases, the flow is confined to a single active layer and the channel has a rectangular cross section with vertical sidewalls. When rotation is strong enough to cause the fluid to separate from the left wall (facing downstream in the northern hemisphere), there is a dramatic alteration in the properties of the jump. The change in layer thickness, which is abrupt in a classical, nonrotating jump, becomes much weaker and more gradual. Instead there is a sudden expansion in the width

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[1] Hogg (1983) notes a possible exception in the Vema Channel overflow, which is strongly influenced by rotation. The jump is inferred from measurements of energy dissipation.

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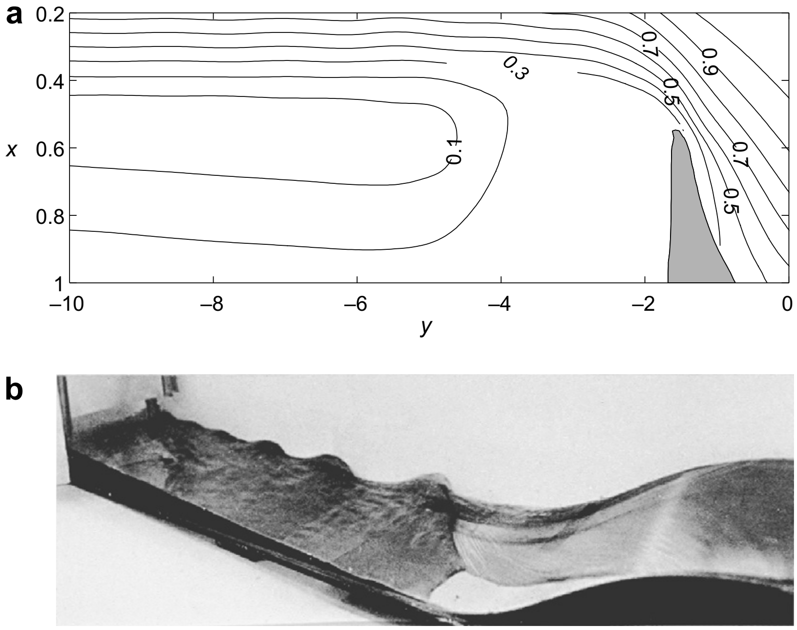

Fig. 1. (a) A transverse hydraulic jump in a rotating channel of rectangular cross section, generated by a shallow-water model (Pratt et al., 2000). The contours indicate interface elevation and approximate the streamlines. The flow is from right to left and the shaded, triangular region shows where the lower layer has separated from one wall, exposing the bottom. The separated flow is spilling down from a sill located at $y = 0$ and is supercritical. The jump lies at $y \cong -1.7$. (b) A transverse jump produced in a laboratory experiment (Pratt, 1987). The flow is from right to left and enters the picture where it spills down a slope. The triangular, silver region shows where the fluid has separated. The jump consists of the downstream termination of this region.

of the stream, with the upstream portion separated from the left wall and the downstream portion attached. Examples from a numerical simulation and from a laboratory experiment are shown in Fig. 1a and b. In each case, a detached supercritical stream is generated when a shallow layer of fluid spills across an obstacle. The hydraulic jump is marked by an abrupt[2] reattachment of the separated stream. Naturally occurring channels in the ocean and elsewhere have smoothly varying topography, so that the thickness of a dense flow within goes to zero at either side. A transverse jump under such conditions would presumably be manifested by a sudden widening of the stream.

The lack of observations of transverse jumps in ocean overflows may be due to several causes. The jump may occur over an along-stream distance that is small and may be missed when the flow is sampled at widely spaced cross sections. Or, the jump may be subsumed by small scale interfacial instabilities[3] that lead to entrainment of overlying fluid and that cause gradual changes along the stream. This behavior has been documented with nonrotating jumps in a two-layer system (Wilkinson and Wood, 1971). A further complication is that overflows such as that of the Denmark Strait are strongly time-dependent and their outflow plumes are marked by the presence of large horizontal eddies. The jumps may be intermittent or hidden by eddies and meanders, or the jump may actually be components of the eddies themselves.

Given the difficulties of observing a jump in the field, one might turn to numerical models that use realistic bottom topography and that are able to reproduce some of the characteristics of the overflow that they simulate. We will concentrate on a recent regional model of the Faroe Bank Channel outflow plume (Riemenschneider and Legg, 2007). This model was developed using the MITgcm (Massachusetts Institute of Technology global circulation model), a z-level model. It encompassed a region spanning the Faroe-Shetland Channel in the east to the Iceland Ridge in the west (Fig. 2). The model has been used to simulate the devel-

[2] In rotating hydraulic theory, a feature is considered ‘abrupt’, and may therefore qualify as a jump or shock, if the change occurs over an along-stream distance comparable to the Rossby radius of deformation. In the most of the examples considered herein, the transitional distance is, in fact, much shorter.

[3] These instabilities occur when the Richardson number becomes sufficiently low.

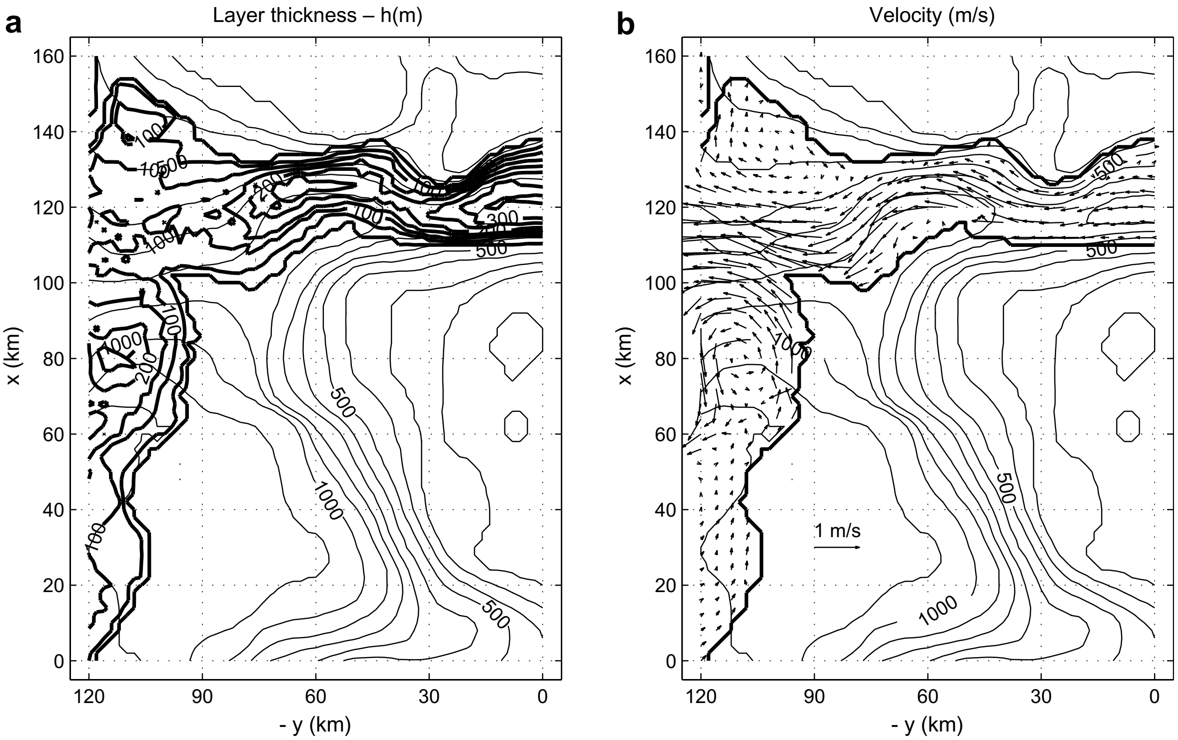

Fig. 2. An excerpt of the domain used in a regional model of the Faroe Bank Channel overflow by Riemenschneider and Legg (2007). (a) The thickness (m) of the outflow layer is plotted with thick contours, over the bathymetry in thin contours. The flow is from right to left and the sill is located at $-y = 30$ km. (b) Vector field of the horizontal velocity, averaged over the thickness of the plume. The thicker, solid curves approximate the edges of the plume, where its thickness goes to zero.

opment of the dense deep-water plume as it makes its way through the Faroe Bank Channel and onto the Faroe-Iceland slope. It has 2 km resolution in the horizontal and 64 levels in the vertical. Below 600 m, where most of the plume's activity takes place, the levels are 25 m thick. Many global models spanning this region resolve the channel and its outflow very poorly. The higher resolution regional model is able to reproduce many observed structural features (Mauritzen et al., 2005) of the plume, including the rate of entrainment and spreading, and its thickness and velocity. The model does not use any subgridscale parameterization of mixing, since the purpose of the work was to study the magnitude of the numerical mixing in a typical z-level model.

We have used the model output to investigate a sudden widening of the outflow that occurs downstream of the sill. In the instantaneous plume thickness and velocity fields shown in Fig. 2, the widening occurs near $y = -90$ km.[4] The left edge (again facing downstream, or negative y) crosses isobaths as it abruptly veers to the left, showing that the widening is not simply the result of topographic steering. Although this feature clearly resembles a transverse jump, a definitive judgment requires proof that the flow undergoes the supercritical-to-subcritical transition that characterizes all hydraulic jumps. In particular, one must calculate the long wave modes of the flow and show that the hydraulically relevant mode, a left-edge frontal wave, propagates in the upstream direction on the downstream of the jump, and *vice versa*. That is, a supercritical-to-subcritical transition with respect this wave takes place through the jump. The main finding, then, is the identification of a transverse jump in a flow with a realistic topography and stratification, and with other properties that lie beyond the scope of idealized models. The analysis will also show that the model outflow undergoes a subcritical-to-supercritical transition, and is therefore hydraulically controlled, just downstream of the sill. The

[4] In order to maintain the right-to-left flow convention between various figures, we require the $-y$ direction to be downstream. Negative wave speed will therefore indicate propagation in the downstream direction.

existence and location of this transition are in agreement with an analysis of direct observations due to Girton et al. (2006), lending further confidence in the model.

Since the method of analysis is somewhat new, and since past simulations of the jumps have utilized rectangular channels with vertical side walls, we will first present an example of the analysis, as applied to a shallow-water jump in a channel with smoothly varying cross section and no vertical walls. The corresponding discussion, which appears in Section 2, will help motivate the Section 3 discussion of the more involved Faroe Bank Channel jump.

2. A transverse jump in a shallow layer in a rotating, parabolic channel

The transverse jump shown in Fig. 3a and b takes place in a rotating channel with a parabolic cross section. The channel narrows and shoals near $y = 0$, forming an isolated sill. The smooth geometry implies that the

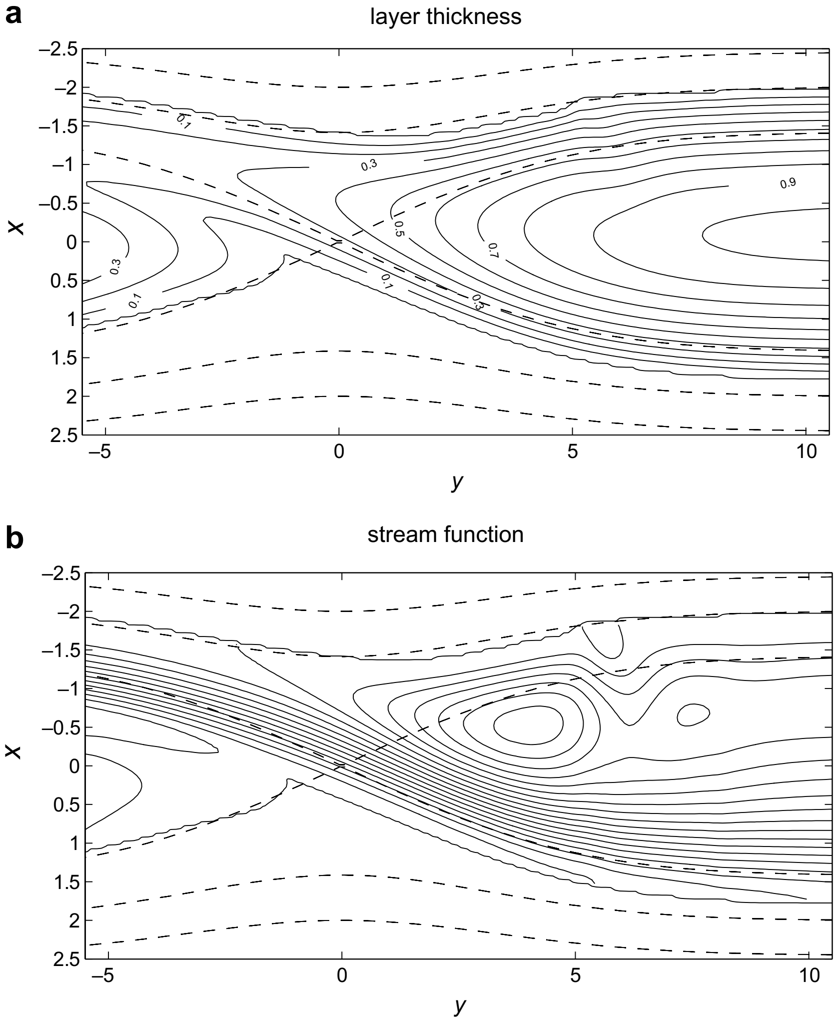

Fig. 3. (a) A transverse jump in a channel with a parabolic cross section, as produced in a shallow-water model. The dashed curves give the bathymetry while the solid curves are contours of the layer thickness. The outermost solid curves represent the free edges of the stream. The flow is from right to left and passes through a region where the channel narrows and shoals. The shallowest, narrowest section is at $y = 0$. The jump begins near $y = -1$. (b) Streamlines for the above flow. (c) The phase speeds c_L and c_R of the long, frontal waves trapped to the left and right edges of the flow. Positive values of $-c_R$ or $-c_L$ indicate downstream propagation. Larger dots indicate that the phase speed lies in the range of the fluid velocity at the section in question. (d) The stream width w. The vertical dashed line marks the minimum width, just upstream of the jump. All quantities are nondimensional.

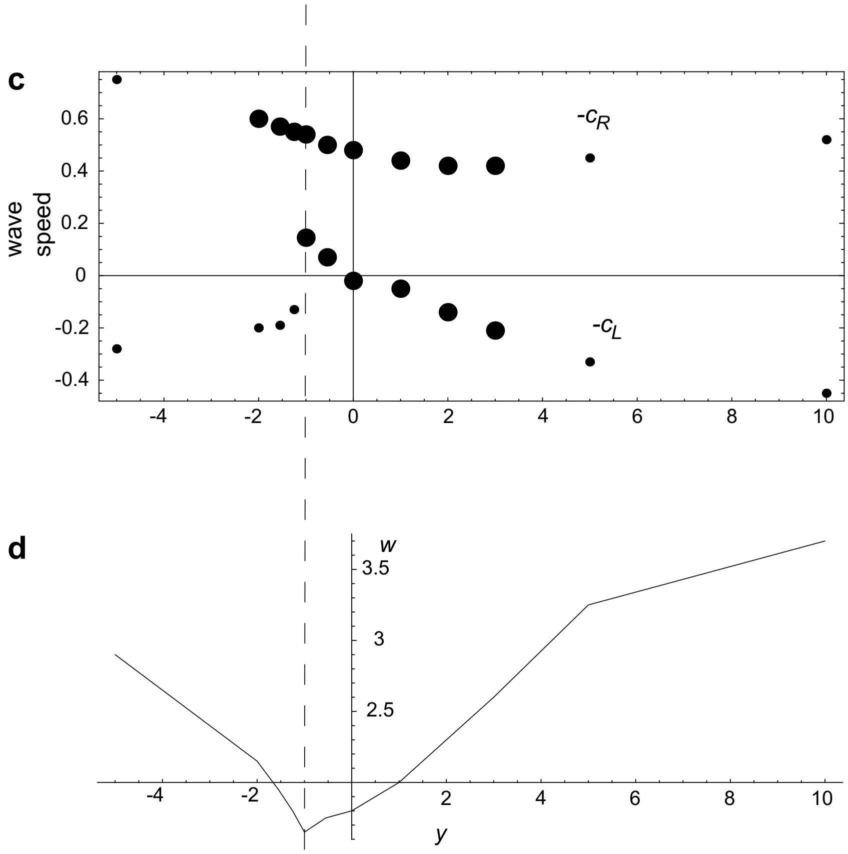

Fig 3. (*continued*)

depth of the single, active layer goes to zero at both edges, an important departure from previous models. The solution is calculated using a shock-capturing, shallow-water code. (Details of the code may be found in Helfrich et al., 1999). The current accelerates and veers to its right as it spills across the sill at $y = 0$. Farther downstream, near $y = -1$, the flow starts to widen. Although the widening is abrupt, it is not discontinuous as in the rectangular channel examples (Fig. 1a and b). A sluggish, cyclonic recirculation cell exists downstream of the widening and can partially be seen in the streamfunction map (Fig. 3b near $y = -5$ and $x > -1$).

The terms 'subcritical' and 'supercritical' generally mean that the single layer flow does or does not allow upstream propagation of long waves. This usage must be qualified in the present case since a variety of long waves are possible for the rotating flows in question. The linear wave analysis described in Appendix A shows that the flow admits several classes of waves. Of primary interest here are two 'frontal' waves, one trapped to each edge of the flow. These are the continuation of the two Kelvin waves that would exist if the channel had vertical sidewalls. In classical models (e.g. Gill, 1977) of rotating hydraulic behavior under gravity, sub- or supercriticality is defined according to the ability of the left-wall Kelvin wave to propagate upstream.[5] In the present situation, the object is to determine whether the left-edge frontal wave can or cannot propagate upstream, and the flow is classified as subcritical or supercritical accordingly. It is also possible to find potential vorticity (Rossby-like) waves that may or may not propagate upstream, but these are deemed less important for a flow that is driven primarily by gravity.

[5] Hydraulic behavior with respect to potential vorticity waves is also possible. The reviewer is referred to Johnson and Clarke (2001) or Pratt and Whitehead (2007), Chapter 6. In the present context of overflows driven by gravity, Kelvin waves are generally regarded as most relevant.

In the present model, and especially in the more involved model considered below, identification of the long Kelvin or frontal modes can be a challenge. The correct eigenfunctions generally decay monotonically inward from the edge along which the wave propagates, though there are some exceptions. Appendix A describes the method of computation and gives some guidance concerning identification of the desired eigenfunctions. The phase speeds c_{L} and c_{R} of the left- and right-edge modes have been calculated at 12 sections (Fig. 3c), starting from upstream of the sill and proceeding downstream (towards negative y) across the sill and through the region of abrupt widening. As shown, the wave trapped to the right edge of the current always moves in the downstream direction ($-c_R > 0$). The wave trapped to the left edge may propagate either upstream or downstream, indicating subcritical or supercritical conditions, respectively.

Upstream of the sill, the $-c_L$ values are negative, indicating subcritical conditions. Upon passing the sill ($y = 0$), $-c_L$ become positive and the flow therefore becomes hydraulically supercritical. The flow narrows and becomes increasingly supercritical until $y \cong 1$, where the sudden widening begins (Fig. 3d). The flow abruptly becomes subcritical ($-c_L < 0$) at the point of sudden widening and this would appear to confirm the judgment that the transition is a transverse jump.

3. Transverse jump in the Faroe Bank plume model

The method described above has been used in an attempt to evaluate, albeit in an approximate manner, the hydraulic criticality of the simulated Faroe Bank outflow plume (Fig. 2). The dense overflow waters are identified using a passive tracer concentration. Pure, undiluted overflow water has a tracer concentration of $\tau = 1$. Across the interface separating the dense water from the ambient water masses the tracer concentration decreases from $\tau = 1$ to 0. We define the cut-off between ambient and dense waters to be $\tau = 0.1$.[6] All waters with tracer concentration $\tau > 0.1$ are defined to be overflow (or 'plume') waters. For the purposes of wave mode calculation, this water mass is treated as a deep, homogeneous layer. The horizontal velocity at a given location is taken as the average over the plume thickness at that point. A reduced gravity at each point is then defined based on the difference between the average densities of the plume water and overlying water. The wave speed calculation ignores the motion of any overlying water, which is reasonable in the region of the supposed jump.

The primary calculation is based on the instantaneous realization of the plume shown in Fig. 2. We calculate the two frontal wave speeds at 27 sections, beginning at a point 30 km upstream of the Faroe Bank Channel sill and ending 90 km downstream of the sill. (In the coordinate system of Fig. 2, the sill lies near $-y = 30$ km.) It is possible at each location to identify the left and right frontal waves and it is again found that the latter always propagates in the downstream direction. We therefore focus on the left frontal wave, the phase speed of which is plotted in Fig. 4a. Beginning at the point farthest upstream, the $-c_{\mathrm{L}}$ values are negative, indicating subcritical flow. A transition to supercritical flow occurs at 40 km (10 km downstream of the sill).[7] [The channel bottom undergoes a sudden increase in slope and the proximity of critical flow to this point agrees with what was found by Girton et al. (2006) in their analysis of the *in situ* data.] Between the transitional point and the beginning of the abrupt widening of the current, roughly 40–95 km, $-c_{\mathrm{L}}$ ranges slightly above and below zero. These weak transitions between subcritical and supercritical flow are not present in the time mean (see below) and we attribute them to transients in the flow field. Also, there are several instances in which two eigenfunctions posses the expected characteristics of a left frontal wave. In such cases both of the c_{L} values are plotted at the position in question, and it can be seen that the sign of each value is always the same.

Just downstream of the 80 km mark, the flow accelerates and becomes substantially supercritical, as shown by the relatively large, positive $-c_{\mathrm{L}}$ values. Then between 90 and 100 km, the $-c_{\mathrm{L}}$ values plunge and become negative. Comparison with the plot of plume width (Fig. 4b) shows that supercritical-to-subcritical transition

[6] The tracer cut-off of $\tau = 0.1$ was chosen because it best illustrated the wider structure of the plume. Using lower cut-off values included some regions of the plume which were felt to be artifacts of the model rather than realistic representations of the plume.

[7] It is common for bottom drag or interfacial friction to displace the critical section downstream relative to the sill. See Gerdes et al. (2002) for example.

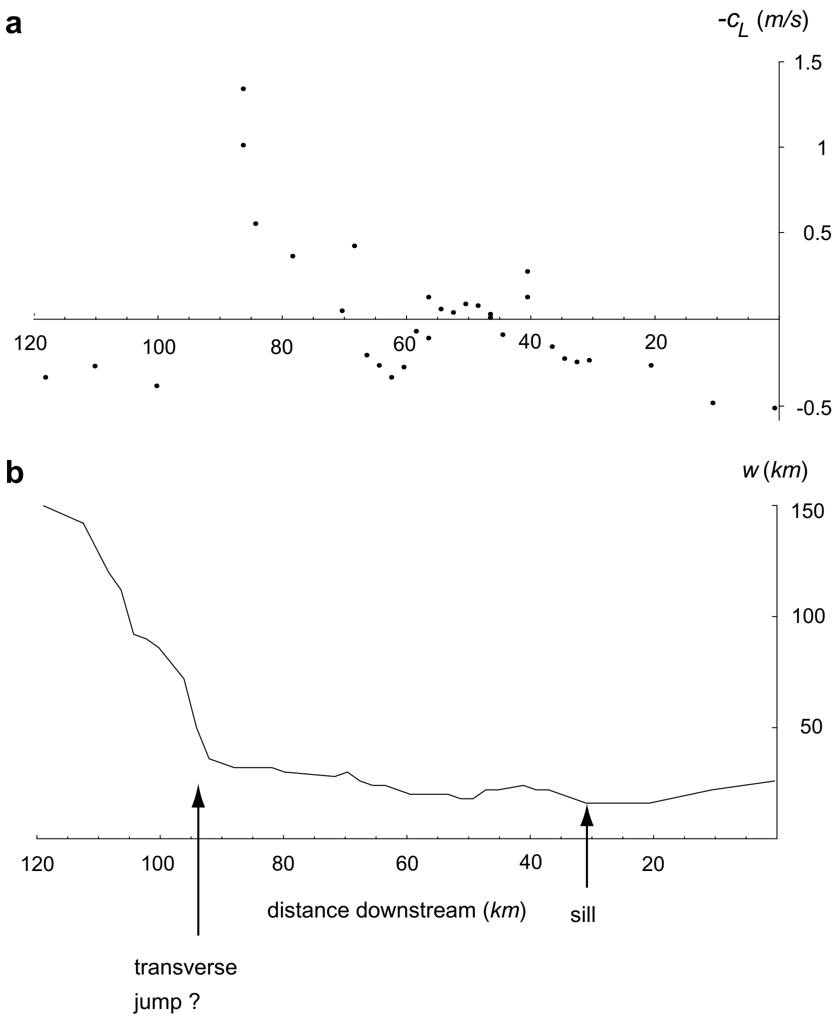

Fig. 4. (a) Phase speed values (c_L) for the left frontal wave, as calculated for the Faroe Bank plume simulation (Fig. 2). (b) The stream width w as a function of distance along the channel.

corresponds to the abrupt widening of the stream. The existence and location of the widening are quite stable throughout the simulation. The transition itself ranges from being nearly discontinuous to less abrupt, but always occurs over the scale comparable to the Rossby radius of deformation (about 20 km). The widening is followed slightly downstream by a prominent cyclonic gyre (centered at $-y = 105$ and $x = 80$ in Fig. 2b). Similar features are also found in the two single-layer simulations (Fig. 1a and Fig. 3a and b), and the laboratory experiment (Fig. 1b).

A similar calculation (not shown) has been performed with a time average realization of the plume, taken over a period of 20 days. The meandering and eddying of the plume lead to a time average flow that is wider and that varies more gradually in the downstream direction than the instantaneous realization. The widening associated with the transverse jump is present but is more gradual. An analysis of the wave modes of the average state show that a subcritical-to-supercritical transition takes place slightly downstream of the sill, as before, and that the stream remains supercritical as it moves downstream until a point just upstream of where the widening begins. The flow downstream of this point is subcritical. The weak, transient transitions found in the instantaneous flow are absent but the major widening and transition from supercritical to subcritical flow is present.

4. Discussion

We have argued that the major widening of the simulated Faroe Bank overflow plume is a transverse hydraulic jump. This finding is based on its structural similarity to transverse jumps in idealized systems, its transcriticality, its persistence (as suggested by its presence in the mean) and by the presence of a characteristic downstream recirculation region. It would appear, then, that transverse hydraulic jumps can occur in models of overflows with realistic stratification and bathymetry. The possibility of such a jump may effect future interpretations of observations or model data of overflow plumes. What is not understood at this time are the implications for mixing and entrainment of plume waters. Does the jump primarily cause lateral stirring within the plume, or can ambient water masses be pulled into plume laterally near the point of sudden widening?

Regional plume models that aim to provide a boundary condition for a course-resolution general circulation model ought to be able to reproduce a transverse jump, at least where a jump can reasonably be expected. 'Streamtube' models, including Smith's (1975) original and its descendents (e.g. Price and Baringer, 1994), filter out gravity wave dynamics and therefore are unable to produce a hydraulic jump. The neglect of gravity waves allows the models to be integrated downstream from some source point without the need to consider downstream conditions. The position and structure of a jump, on the other hand, depends on the downstream conditions. It might be possible to alter a streamtube model so as to provide sensitivity to downstream conditions, though one would lose the convenience of being able to integrate in a single direction. The alternative is a more general model, such as the one used to generate the plume in Fig. 2.

Appendix A. Wave mode calculation

Suppose that the layer thickness and along-channel velocity of the flow at a position y are denoted $V(x)$ and depth $D(x)$. We wish to find the linear wave modes of a hypothetical parallel (y-independent) flow having this cross-stream structure. These modes can be found by perturbing the basic state with small amplitude, wave-like disturbance of the form

$$\begin{pmatrix} u \\ v \\ d \end{pmatrix} = Re\left[\begin{pmatrix} \hat{u}(x) \\ \hat{v}(x) \\ \hat{d}(x) \end{pmatrix} \mathrm{e}^{\mathrm{i}l(y-ct)}\right].$$

Here u and v are the disturbance x-and y-velocity components, d is the disturbance thickness, l is the wave number and c is the phase speed. All fields have been made dimensionless using a typical depth scale H and with $(g'H)^{1/2}$ as a velocity scale, $(g'H)^{1/2}/f$ as a length scale, and f^{-1} as a time scale.

Substitution into the shallow water equations and linearization then leads to

$$[\mathrm{i}l(V-c)\hat{u}] - \hat{v} = -\frac{\mathrm{d}}{\mathrm{d}x}\hat{d} \tag{A.1}$$

$$\mathrm{i}l(V-c)\hat{v} + \left(1+\frac{\partial V}{\partial x}\right)\hat{u} = -\mathrm{i}l\hat{d} \tag{A.2}$$

and

$$\mathrm{i}l(V-c)\hat{d} + \frac{\partial}{\partial x}(D\hat{u}) + \mathrm{i}lD\hat{v} = 0. \tag{A.3}$$

The boundary conditions are

$$D\hat{u} = 0 \text{ (at free edges).} \tag{A.4}$$

Since D vanishes at the edges, one merely requires that $\hat{u}$ remain bounded there.

The eigenvalue problem ((A.1)–(A.4)) has been solved numerically using a finite difference scheme based on second order approximations to the derivatives. Since hydraulic behavior is concerned only with the dynamics of long waves, the wavenumber l is set to zero. The number of grid cells N typically ranges from 16 to 64, resulting in $3N$ eigenvalues (wave speeds) c_n and accompanying eigenfunctions. It is the structure of the eigenfunctions that allows one to identify the wave mode. For a basic flow in which V remains positive and both V and D vary gradually and simply over the cross section, there are often just two modes that decay monotonically inward from the edges, one for each edge. These modes are also characterized by having relatively weak $\hat{u}$ and strong $\hat{v}$ and $\hat{d}$, also typical of a standard Kelvin wave. There are additional modes that are smooth but have a finite number of zero crossings. These modes have substantial $\hat{u}$ and are identified with potential vorticity (Rossby-type) waves. Finally, there is generally a group of singular modes that have c_n lying in the range of $V(x)$ and for which the wave fields and/or their first derivatives are discontinuous where $c_n = V(x)$. These solutions are regarded as members of the continuous spectrum and are disregarded in the consideration of the hydraulic properties of the flow.

When the structure of V and D becomes more complicated, the frontal modes may no longer decay monotonically from the edges, and therefore may be less easy to identify. This situation can arise, for example, where $V(x)$ becomes negative near one or both edges of the flow. (Side bands of counterflow are not unusual in the Faroe Bank plume model.) In such cases it is helpful to temporarily simplify the basic flow, either by smoothing it, or by eliminating the thin edges of counterflow. The frontal modes of the simplified flow usually decay monotonically and are easy to identify. Once this is done, the investigator can gradually modify the basic flow back to its original form, tracing the desired root in the processes.

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