Nonlinear Waves: Woods Hole GFD Program 2009

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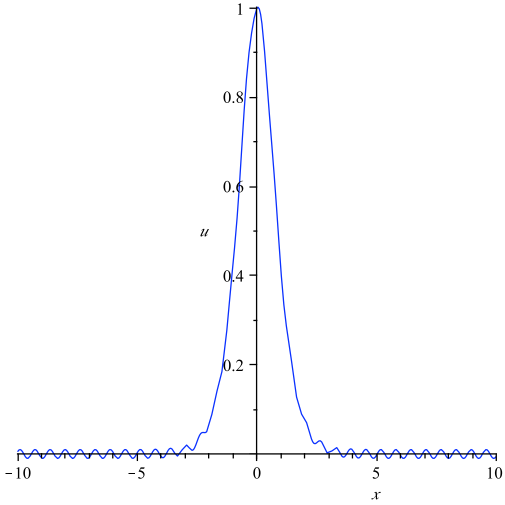
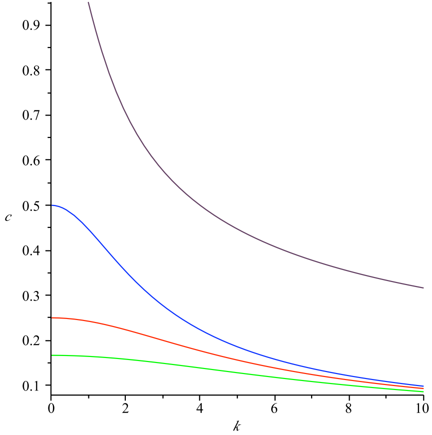
Lecture 8: Generalized Solitary Waves



8.1 Generalized Solitary Waves

We have seen that solitary waves, either of the KdV-type with a "pulse"-like profile, or as the envelope of a wave packet, play a key role in nonlinear wave dynamics. However, there are physical situations when such KdV-type waves may not be genuinely localized. Instead they are accompanied by **co-propagating** small oscillations which spread out to infinity **without decay**. These are **generalized solitary waves** and occur for water waves with surface tension for Bond numbers less than $1/3$, for interfacial waves when there is a free surface, and for all internal waves with mode numbers $n \geq 2$. The underlying reason is that there is a resonance between a long wave with wavenumber $k \approx 0$ and a short wave with a finite wavenumber. When the amplitude of the central core is small, $O(\epsilon^2)$, the amplitude of the oscillations is **exponentially small**, typically $O(\exp(-C/\epsilon))$ where C is a positive constant. Hence they cannot usually be found by conventional asymptotic expansions, and need **exponential asymptotics**.

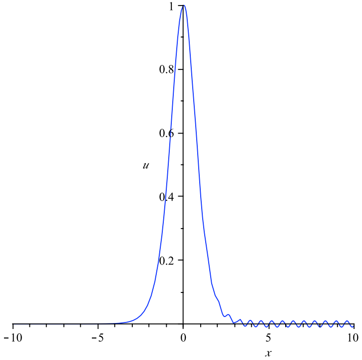
8.2 Generalized Solitary Waves



Plot of a schematic set dispersion curves for internal waves: mode 1 (blue), mode 2 (red), mode 3 (green) and the surface mode (violet).

8.3 Generalized Solitary Waves

Steady generalized solitary waves are necessarily symmetric. But this means they cannot be realized physically as then the group velocity of the small oscillations is the same at both ends, which implies that energy sources and sinks are needed. In practice, they are generated with a core and small oscillations only on **one side**, determined by the group velocity. Consequently, they are unsteady and slowly **decay** due to this radiation.



8.4 Generalized Solitary Wave

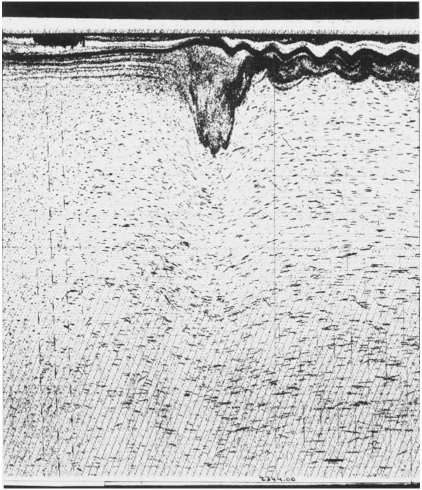


FIGURE 2. Acoustical image of internal-wave disturbances generated by stratified flow past a sill in the field experiments of Farmer & Smith (1980). The streamlines indicate that the main disturbance is a mode-2 solitary-like wave and is followed by a train of smaller-amplitude mode-1 short waves.

8.5 Coupled KdV equations

The technique we use to find the tail oscillations is based on extending the usual asymptotic expansion into the complex plane, and using **Borel summation**. It is similar to the techniques used by Pomeau et al (1988) and Kruskal and Segur (1991). To exhibit it here, we use the model system of two coupled KdV equations, which can be shown to describe the interaction between two weakly nonlinear long internal waves whose linear long wave speeds are nearly equal.

$$u_t + 6uu_x + u_{xxx} + (pv_{xx} + quv + \frac{1}{2}rv^2)_x = 0\,, \tag{1}$$

$$v_t + \Delta v_x + 6vv_x + v_{xxx} + \lambda(pu_{xx} + ruv + \frac{1}{2}qu^2)_x = 0\,. \tag{2}$$

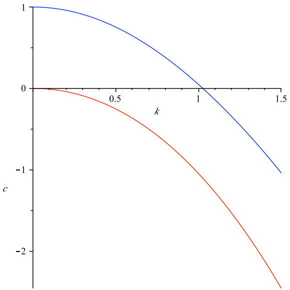
This system is Hamiltonian, and conserves the "mass" u, v, the "momentum" $\lambda u^2 + v^2$, and the Hamiltonian. For stability we choose the coupling parameter $\lambda > 0$. Δ is the detuning parameter, proportional to the difference between the two linear long wave speeds; without loss of generality we take $\Delta > 0$.

8.6 Resonance between long waves and short waves

First examine the linear spectrum, for waves of wavenumber k and phase speed c,

$$c = \frac{\Delta}{2} - k^2 \pm \{\lambda p^2 k^4 + \frac{\Delta^2}{4}\}^{1/2} \tag{3}$$

If we let the coupling parameter $\lambda \to 0$ these linear modes uncouple into a u-mode with spectrum $c = -k^2$ and a v-mode with spectrum $c = \Delta - k^2$. This situation persists for $\lambda > 0$, and there is a **resonance** between the long wave (u-mode) and a short wave (v-mode), with a resonant wavenumber $k_0 = (\Delta/1 - \lambda p^2)^{1/2}$ provided that $\lambda p^2 < 1$. A typical plot is for $\Delta = 1, p = 0.5, \lambda = 0.2$



8.6 Steady travelling waves

We seek solutions of the form

$$u = u(x - ct)\,, \quad v = v(x - ct)\,, \tag{4}$$

so that the coupled KdV system (1, 2) becomes

$$-cu + 3u^2 + u_{xx} + pv_{xx} + quv + \frac{1}{2}rv^2 = 0\,, \tag{5}$$

$$-cv + \Delta v + 3v^2 + v_{xx} + \lambda(pu_{xx} + ruv + \frac{1}{2}qu^2) = 0\,. \tag{6}$$

The two constants of integration have been set to zero, essentially by imposing solitary wave boundary conditions, or better, by translating u, v by constants. Equations (5, 6) form a fourth order ODE system. We shall show that they have symmetric generalized solitary wave solutions, with co-propagating oscillatory tails of small amplitude. This amplitude will be found using either exponential asymptotics, or more directly by expanding in λ.

8.7 Asymptotic expansion for small waves

First, expand around $k = 0$ for the long (u-mode) wave. Thus, we introduce a small parameter $\epsilon << 1$, and seek a solution as an asymptotic expansion,

$$u_s(\epsilon x) = \sum_{n=1}^{\infty} \epsilon^{2n} u_n \,, \quad v_s(\epsilon x) = \sum_{n=1}^{\infty} \epsilon^{2n} v_n \,, \quad c = \sum_{n=1}^{\infty} \epsilon^{2n} c_n \,. \tag{7}$$

Substitution into (5, 6) yields

$$u_1 = 2\gamma^2 \mathrm{sech}^2(\epsilon\gamma x) \,, \quad v_1 = 0 \,, \quad c_1 = 4\gamma^2 \,, \tag{8}$$

$$u_2 = \frac{\lambda}{\Delta}\{(20p^2 + q^2 - 8pq)c_1 u_1 - (q - 6p)(q - 10p)u_1^2\} \,, \tag{9}$$

$$v_2 = -\frac{\lambda}{\Delta}\{pc_1 u_1 + \frac{1}{2}(q - 6p)u_1^2\} \,, \tag{10}$$

$$c_2 = -\frac{\lambda}{\Delta} p^2 c_1^2 \,. \tag{11}$$

This expansion can be continued to all orders in ϵ^2 without any oscillatory tail being detected.

To find the tail oscillations, we observe that u_n, v_n are singular in the complex plane at $x = \pm i\pi/2\epsilon\gamma, \pm 3i\pi/2\epsilon\gamma, \cdots$. This motivates us to examine this singularity by the change of variables

$$x = \frac{i\pi}{2\epsilon\gamma} + z\,, \tag{12}$$

Then as $\epsilon z \to 0$, $\text{sech}^2(\epsilon\gamma x) \sim -1/\epsilon^2\gamma^2 z^2$, and so

$$u_s \sim -\frac{2}{z^2} - \frac{\lambda}{2\Delta z^4}(q-6p)(q-10p) + \cdots + O(\epsilon^2)\,, \tag{13}$$

$$v_s \sim -\frac{2\lambda}{\Delta z^4}(q-6p) + \cdots + O(\epsilon^2)\,. \tag{14}$$

Next, we consider the **inner problem** in which we seek solutions of (5, 6) in the form $u = u(z), v = v(z)$, for which the expressions (13, 14) form an **outer boundary condition**. The outcome is just the same system (5, 6) with x replaced by z. Note that $c = O(\epsilon^2)$, and can be omitted at the leading order.

8.9 Borel summation

We seek a solution as a Laplace transform

$$[u, v] = \int_\Gamma \exp(-zs)[U(s), V(s)]\, ds\,, \tag{15}$$

where the contour Γ runs from 0 to ∞ in the half-plane $\mathrm{Re}(sz) > 0$. We seek a power series solution

$$[U(s), V(s)] = \sum_{n=1}^{\infty} [a_n, b_n] s^{2n-1}\,, \tag{16}$$

where $a_1 = -2\,, b_1 = 0$, $a_2 = -\lambda(q - 6p)(q - 10p)/12\Delta$, $b_2 = -\lambda(q - 6p)/3\Delta$. In general, substitution of (16) into the Laplace transform (15) generates the asymptotic series

$$[u, v] \sim \sum_{n=1}^{\infty} [\alpha_n, \beta_n] z^{-2n}\,, \quad [\alpha_n, \beta_n] = (2n-1)! [a_n, b_n]\,. \tag{17}$$

This agrees with the asymptotic series (13, 14), and in effect the Laplace transform is a **Borel summation** of the asymptotic series.

8.10 Recurrence relation

Substitution of the Laplace transform (15) and the series (16) into the differential equation system (5, 6) yields a recurrence relation for $[a_n, b_n]$. Putting $\Delta[A_n, B_n)] = (-k_0^2)^n[a_n, b_n]$, we get

$$\frac{(n+1)(2n+5)}{(n-1)(2n-1)}A_{n-1} + \{p - \frac{q}{(n-1)(2n-1)}\}B_{n-1} = F_n \,, \tag{18}$$

$$(1-\lambda p^2)B_n - B_{n-1} - \lambda p A_{n-1} + \lambda \frac{rB_{n-1} + qA_{n-1}}{(n-1)(2n-1)} = G_n \,, \tag{19}$$

where F_n, G_n are quadratic convolution expressions in $A_2, \cdots A_{n-2}$, $B_2 \cdots, B_{n-2}$. As $n \to \infty$, these nonlinear terms can be neglected, and we find that

$$[A_n, B_n] \to [-p, 1]K \quad \text{as} \quad n \to \infty \,, \tag{20}$$

where K is a constant whose value depends on λ, p, q, r. It now follows that the series (16) **converges for $|s| < k_0$**, $k_0^2 = \Delta/(1-\lambda p^2)$. The result (20) shows that as $|s| \to k_0$ there is a pole singularity given by

$$[U(s), V(s)] \approx \Delta \frac{[p, -1]K}{2(s - ik_0)} \,. \tag{21}$$

8.11 Singularity

We have now established that the solution in the z-variable is given by

$$[u, v] = \int_\Gamma \exp(-zs)[U(s), V(s)]\, ds\,,$$

where $[U(s), V(s)]$ has a pole singularity at $s = ik_0$, also at the complex conjugate point $s = -ik_0$ and at all their harmonics $s = \pm ink_0\,, n = 2, 3$ etc. Hence the contour Γ should be chosen to avoid the imaginary s-axis, and to be explicit we choose it to lie in $\mathrm{Re}\, s > 0$. But we seek a **symmetric solution**, which in the z-variable requires that $\mathrm{Im}\,[u.v] = 0$ when $\mathrm{Re}\, z = 0$. But the presence of the pole prevents (15) from satisfying this condition, and so we must correct it by adding a subdominant term

$$[u, v] = \int_\Gamma \exp(-zs)[U(s), V(s)]\, ds + \frac{ib}{2}[p, -1]\exp(-ik_0 z + i\delta). \quad (22)$$

Here b, δ are real constants, and note that $|\exp(-ik_0 z)|$ is smaller than any power of $|z|^{-1}$ as $z \to \infty$ in $\mathrm{Re}\;\; z > 0\,, \mathrm{Im}\, z < 0$, recalling that $x = (i\pi/2\epsilon\gamma) + z$. The symmetry condition is now applied by bringing the contour Γ onto $\mathrm{Re}\, s = 0$ and deforming around the pole at $s = ik_0$.

8.12 Singularity

The outcome is

$$b \cos \delta = \pi K \, . \tag{23}$$

which is substituted into

$$[u, v] = \int_{\Gamma} \exp(-zs)[U(s), V(s)]\, ds + \frac{ib}{2}[p, -1] \exp(-ik_0 z + i\delta).$$

The final step is to bring this solution (22) back to the real axis, using $x = (i\pi/2\epsilon\gamma) + z$. Taking account of the corresponding singularity at $s = -ik_0$, we finally get that

$$[u, v] \sim [u_s, v_s] + b\Delta[-p, -1]) \exp(-\pi k_0/2\epsilon\gamma) \sin(k_0|x| - \delta)\, . \tag{24}$$

This is a two-parameter family, the parameters being $\gamma, \delta, 0 < \delta < \pi/2$. The minimum tail amplitude occurs at $\delta = 0$. Note that the constant in the exponential term is determined by the location of the singularity, but the amplitude needs the exponential asymptotics.

8.13 Embedded solitons

The constant K is determined by the recurrence relations (18, 19). It is a function of the system parameters λ, p, q, r and in general is found numerically. But $K = 0$ for $q = 6p$ (see (8, 9)), and in general we found many parameter combinations where $K = 0$. In particular

$$K \approx \frac{\lambda(6p - q)}{3\Delta} \quad \text{as} \quad \lambda \to 0\,. \tag{25}$$

These special values imply that the solitary wave decays to zero at infinity, even although its speed lies inside the linear spectrum, at least in this asymptotic limit. These are called **embedded solitons**. They are usually not stable, but are then **metastable**, or are said to exhibit **semi-stability**. Nevertheless they are found useful in several applications, such as nonlinear optics and solid state physics. For water waves with surface tension, generalized solitary waves exist for Bond numbers $0 < B < 1/3$, but from numerical simulations it seems there are no embedded solitons.

8.14 One-sided generalized solitary waves

These symmetric solitary waves cannot be realized in practice, since they require an energy source and sink at infinity. Instead, they are replaced by **solitary waves with radiating tails on one side only**, determined by the group velocity. That is, in $x > 0$ for $c_g > c$, or in $x < 0$ for $c_g < c$, where c_g is the group velocity at the resonant wavenumber. For the present case, the linear dispersion relation is (3) and so for the relevant u-mode, $c_g = \Delta - 3k^2 < c = \Delta - k^2$. Hence there are no oscillations in $x > 0$, but they will appear in $x < 0$.

Hence in $x > 0$, or more generally in Re $z > 0$, the solution is completely defined by the Laplace transform integral (15), with the contour Γ lying in $\mathrm{Re}\, s > 0$. Then for $x < 0$, or $\mathrm{Re}\, z < 0$, the contour Γ must be moved to Re $z < 0$ across the axis $\mathrm{Re}\, s = 0$. In this process the solution collects a contribution from the pole at $s = ik_0$, which generates the tail oscillation. The final outcome is that (24) is replaced by

$$[u, v] \sim [u_s, v_s] - H(-x) 2\pi K \Delta[-p, -1]) \exp(-\pi k_0 / 2\epsilon\gamma) \sin(k_0 x) \quad (26)$$

where $H(\cdot)$ is the Heaviside function. That is, in effect the phase shift $\delta = 0$, there are no oscillations in $x > 0$ and **the amplitude in $x < 0$ is exactly twice the amplitude of the symmetric solution**.

8.14 Weak coupling

$$-cu + 3u^2 + u_{xx} + pv_{xx} + quv + \frac{1}{2}rv^2 = 0\,,$$
$$-cv + \Delta v + 3v^2 + v_{xx} + \lambda(pu_{xx} + ruv + \frac{1}{2}qu^2) = 0\,.$$

Let us suppose that $0 << \lambda < 1$ and expand,

$$[u, v] \sim \sum_{n=0}^{\infty} \lambda^n [u_n, v_n]\,, \quad c \sim \sum_{n=0}^{\infty} \lambda^n c_n\,. \tag{27}$$

$$u_0 = 2\beta^2 \mathrm{sech}^2(\beta x)\,, \quad v_0 = 0\,, \quad c_0 = 4\beta^2\,. \tag{28}$$

This leading term is a u-mode solitary wave. Note that in comparison with the previous expansion (7) $\beta = \epsilon\gamma$, but now the amplitude can be order unity. At the next order

$$-c_0 u_1 + 6u_0 u_1 + u_{1xx} + pv_{1xx} + qu_0 v_1 - c_1 u_0 = 0\,, \tag{29}$$

$$(\Delta - c_0)v_1 + v_{1xx} + pu_{0xx} + \frac{q}{2}u_0^2 = 0\,. \tag{30}$$

8.15 Weak coupling

$$(\Delta - c_0)v_1 + v_{1xx} = f(x) = -pc_0u_0 + (6p - q)\frac{u_0^2}{2}\,. \qquad (31)$$

Note that in this limit $\lambda \to 0$, the resonant wavenumber $k_0 \approx (\Delta - c_0)^{1/2}$ and takes account of the **finite speed** of the wave. We must now take $c_0 < \Delta$ to get tail oscillations, and for $c_0 > \Delta$ the expansion yields a genuine solitary wave. The general solution of (31) is

$$v_1 = A\sin k_0x + B\cos k_0x + \frac{1}{2k_0}\int_{-\infty}^{\infty} f(x')\sin(k_0|x - x'|)dx'\,. \qquad (32)$$

To determine the constants A, B we impose a symmetry condition on v_1, so that $A = 0$, and then

$$v_1 \sim b_1\sin(k_0|x| - \delta) \quad \text{as} \quad |x| \to \infty\,, \qquad (33)$$

$$b_1\cos\delta = L = \frac{1}{2k_0}\int_{-\infty}^{\infty} f(x)\cos(k_0x)dx\,. \qquad (34)$$

With v_1 known, we can find u_1 from (29), and

$$u_1 \sim -p\frac{(\Delta - c_0)}{c_0}b_1\sin(k_0|x| - \delta)\,, \quad \text{as} \quad |x| \to \infty\,, \qquad (35)$$

8.16 Weak coupling

$$[u_1, v_1] \sim [-p\frac{(\Delta - c_0)}{c_0}, 1]\, b_1 \sin(k_0|x| - \delta)\,, \quad \text{as} \quad |x| \to \infty\,,$$

$$b_1 \cos\delta = L = \frac{1}{2k_0}\int_{-\infty}^{\infty} f(x)\cos(k_0 x)dx\,,$$

$$f(x) = -pc_0 u_0 + (6p - q)\frac{u_0^2}{2}\,.$$

We find that

$$L = -\frac{6k_0}{\beta^2}\{k_0^2(q - 6p) + 4\beta^2 q\}\int_{-\infty}^{\infty} \text{sech}^2(\beta x)\cos(k_0 x)dx\,. \tag{36}$$

Then as $\beta = \epsilon\gamma \to 0$, this reduces to

$$L \sim \frac{\pi k_0^2}{3}(6p - q)\exp(-\pi k_0/2\epsilon\gamma)\,, \tag{37}$$

which agrees with the previous result (25) from the exponential asymptotics, since $L = \pi K$. The one-sided solutions are obtained by setting $\delta = 0$, and replacing b_1 in (33, 35) by $0, 2b_1$ for $x > 0, x < 0$.

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