Nonlinear Waves: Woods Hole GFD Program 2009

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Lecture 7: Solitary Waves

The basic paradigms for nonlinear waves, at least for small amplitudes, is the central role played by model equations, such as the KdV equation and the NLS equation . In turn, the solution of these equations are often controlled by their **soliton** solutions. This suggests that even in the full system from which these model equations are derived, **solitary waves** will be of great importance. Here we will indicate how small-amplitude solitary waves can be found by an asymptotic perturbation procedure directly from the full system, rather than from a model equation.

Because solitary waves are required to decay in their tail regions, some information about their possible existence or otherwise can often be obtained by an examination of these tail regions, where, except in certain exceptional cases, a linearized analysis is applicable. One-dimensional steady solitary waves, propagating in the x-direction with speed c are functions of $\xi = x - ct$, together with a set of other spatial transverse variables which define the modal structure. For instance, for surface or internal solitary waves, the dependence is on ξ and z (x is horizontal and z is vertical), and there is no dependence on the remaining horizontal variable y.

7.1 Linear spectrum

In the tail region, where we assume that a linearized analysis holds, we seek solutions proportional to

$$Re\left\{\exp\left(ik(x-ct)\right)\right\}. \tag{1}$$

The linearized equations will then yield the linear dispersion relation

$$c = c(k)\,, \tag{2}$$

written here for the phase speed c rather than the frequency $\omega = ck$. Whereas usually this dispersion relation (which may have several branches) is considered as an equation for c given a real wavenumber k, for solitary wave tails it needs to be considered as an equation for a complex-valued k given a real speed c. Indeed, it is immediately clear that if there exist real-valued solutions of (2) for the given value of c, then it is unlikely that the solitary wave can decay to zero in its tail region. Instead, it will probably be accompanied by a non-decaying co-propagating oscillatory wave field. This consideration leads to the notion that solitary waves generally can only exist in the **gaps** in the linear spectrum.

7.2 Linear spectrum: water waves

For instance, for water waves, the dispersion relation is

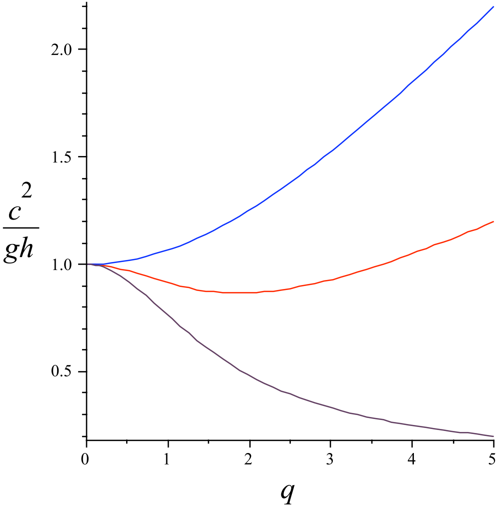
$$\frac{c^2}{gh} = \frac{(1+Bq^2)}{q} \tanh q \,, \qquad q = kh \,, \tag{3}$$

where the Bond number $B = \Sigma/gh^2$ and $\rho\Sigma$ is the coefficient of surface tension (ρ is density), and has a value of 74 $dynes/cm$ at $20^o C$.

It may then be shown that solitary waves of the **KdV** type can exist only when either $B = 0, c^2 > gh$ or when $B > 1/3, c^2 < gh$ with a bifurcation from wavenumber zero ($k = 0$) and $c^2 = gh$ in both cases.

Otherwise when $0 < B < 1/3$ solitary waves can exist for $|c| < c_m$ where c_m^2 is the minimum value that c^2 can take in (3) as q takes all real values (in deep water, $|q| \to \infty$, $c_m^2 = 2(g\Sigma)^{1/2}$ and occurs at $|k| = k_m = (g\Sigma)^{1/2}$). These solitary waves bifurcate at a finite wavenumber k_m and from the speed c_m, and have decaying oscillations in their tail regions. They are **envelope solitary waves**, of a quite different kind from the afore-mentioned KdV-type solitary waves, and closely related to the NLS equation.

7.3 Linear spectrum: water waves



Plot of the water wave dispersion relation (3) for $B = 0$ (violet), $B = 0.2$ (red), $B = 0.4$ (blue).

7.4 Reformulation as a dynamical system

This approach has recently been developed into the basis of a rigorous approach to finding solitary waves, often called the "dynamical-systems" method . Since here we are considering only solitary waves which occur in **conservative** systems, which is the common and traditional scenario for solitary waves, we shall suppose that the underlying physical system is **Hamiltonian** (that is, energy-conserving) and **reversible** (that is, there is a symmetry under the transformation $\xi \to -\xi$). In this case it can be shown the the solutions k of the dispersion relation (2) for each real value of c have the property that $-k$ and k^* (complex conjugate) are also solutions. It follows that generically the solutions form a **quartet** $(k, k^*, -k, -k^*)$, with an associated four-dimensional subspace for the corresponding wave mode. For solitary waves we require solutions with $Im(k) > 0 (< 0)$ when $\xi \to \infty (\to -\infty)$, in order to ensure that the solution decays to zero in its tail region. In the general case when $Im(k) \neq 0$ we see that there are generically two such roots available as $\xi \to \infty$ and, due to the reversible symmetry, two other roots available as $\xi \to -\infty$. Thus, for the corresponding wave mode, as $\xi \to \infty$ two boundary conditions are needed at each of $\pm\infty$. This count is consistent with the existence of a solitary wave solution, which from this dynamical systems point of view, is a **homoclinic orbit**.

7.5 Reformulation as a dynamical system

Next, consider how this quartet structure may change as some system parameter is varied. Bifurcations arise when two solutions for k coalesce, for which the necessary condition is that $\partial c/\partial k = 0$ simultaneously with the dispersion relation (2). When this occurs at a real value of k, it is equivalent to the condition that $\mathbf{c = c_g}$ where

$$c_g = d\omega/dk = c + k\,dc/dk$$

is the group velocity. Generically, there are four possibilities:

(1) $(0, 0, i\gamma, -i\gamma)$ where $\gamma > 0$ is real-valued .

(2) $(0, 0, \beta, -\beta)$ where $\beta > 0$ is real-valued .

(3) $(\beta, \beta, -\beta, -\beta)$ where $\beta > 0$ is real-valued .

(4) $(i\gamma, i\gamma, -i\gamma, -i\gamma)$ where $\gamma > 0$ is real-valued .

Case (1) corresponds to a KdV-type solitary wave, and case (3) corresponds to an envelope (NLS) solitary wave. Case (2) corresponds to a so-called generalized solitary wave, which does not decay at infinity, but instead is accompanied there be small-amplitude co-propagating oscillations. Case (4) has only rarely been studied and corresponds to a transition from a KdV-type solitary wave to an envelope solitary wave.

7.6 Reformulation as a dynamical system

The full system is now projected onto the appropriate four-dimensional subspace, and the resulting bifurcation analyzed within the framework of this subspace. Of course, rigorous results require a delicate and sophisticated justification of this process. Here we shall instead briefly describe the structure of the subspace, which we suppose is represented by the 4-vector $\mathbf{W}(\xi)$. It satisfies an equation of the form

$$\mathbf{W}_\xi = L(\mathbf{W};\epsilon) + N(\mathbf{W})\,. \tag{4}$$

Here $L(\mathbf{W};\epsilon)$ is a linear operator and $N(\mathbf{W})$ contains all the nonlinear terms. The bifurcation parameter is ϵ, and is such that the spectrum of L at $\epsilon = 0$ reproduces one of the cases (1) to (4) describe above. That is, the eigenvalues $\lambda = ik$ of the linear operator $L(\mathbf{W};0)$ are respectively :

$$\begin{aligned}
&(1) \quad (0,0,-\gamma,\gamma)\,.\\
&(2) \quad (0,0,i\beta,-i\beta)\,.\\
&(3) \quad (i\beta,i\beta,-i\beta,-i\beta)\\
&(4) \quad (-\gamma,-\gamma,\gamma,\gamma)\,.
\end{aligned}$$

7.7 Case (1)

Let us first consider case (1). At the bifurcation point ($\epsilon = 0$) the linearized system (4) has a double-zero eigenvalue, and generically there is a corresponding single eigenvector $\mathbf{V}_0$, and a single generalized eigenvector $\mathbf{V}_1$. Small-amplitude solutions are then sought in the form

$$\mathbf{W} = A(\xi)\mathbf{V}_0 + B(\xi)\mathbf{V}_1 + \mathbf{W}^{(2)} . \tag{5}$$

Here A, B are real variables of $O(\alpha), \alpha << 1$, where α measures wave amplitude. The leading terms form a two-dimensional subspace (A, B), while $\mathbf{W}^{(2)}$ is a small error term of $O(\alpha^2, \alpha\epsilon)$, where $\epsilon, \alpha << 1$ are both small parameters. Note that the two remaining eigenvalues $\mp\gamma$ play no role at the leading order here, since they correspond to strong exponential decay at infinity, and their effects are included in the small error term $\mathbf{W}^{(2)}$.

7.8 Case (1)

Projection onto the two-dimensional subspace and a normal form analysis then reveals that (A, B) satisfy the system

$$\begin{aligned} A_\xi &= B\,, \\ B_\xi &= \epsilon A + \mu A^2 + \cdots, \end{aligned} \tag{6}$$

where μ is a real-valued coefficient, specific to the system being considered, and the omitted terms are $O(\alpha\epsilon^2, \alpha^2\epsilon, \alpha^3)$. The coefficient ϵ yields the perturbed eigenvalues $\pm\epsilon^{1/2}$ for $\epsilon > 0$, and $\pm i|\epsilon|^{1/2}$ for $\epsilon < 0$; the former case yields the solitary wave solution. Comparison with the dispersion relation (2) leads to the identification of ϵ as

$$\epsilon = -\frac{2(c - c(0))}{c_{kk}(0)}\,. \tag{7}$$

It follows that for solitary waves, $c > (<) c(0)$ according as $c_{kk}(0) < (>) 0$, as expected. When the error terms in (6) are omitted, the resulting system can be recognised as the **steady-state KdV equation**, and has the well-known **"sech^2" solution**. It is then a delicate and intricate task to establish that this solitary wave solution persists when the error terms are restored.

7.9 Case (2)

Next consider case (2). At the bifurcation point ($\epsilon = 0$) the linearized system (4) again has a double-zero eigenvalue, with a corresponding single eigenvector $\mathbf{V}_0$, and a single generalized eigenvector $\mathbf{V}_1$. However, account must now be taken of the other two eigenvalues $\pm i\beta$, with their associated eigenvectors $\mathbf{V}_2, \mathbf{V}_2^*$, since they do not now lead to decaying solutions at infinity. Small-amplitude solutions are sought in the form

$$\mathbf{W} = A(\xi)\mathbf{V}_0 + B(\xi)\mathbf{V}_1 + C(\xi)\mathbf{V}_2 + C^*(\xi)\mathbf{V}_2^* + \mathbf{W}^{(2)} . \tag{8}$$

Here C is a complex-valued variable, and the leading terms form a four-dimensional subspace (A, B, C), while $\mathbf{W}^{(2)}$ is again a small error term. Projection onto this four-dimensional subspace, and a normal form analysis reveals that (A, B, C) satisfy the system

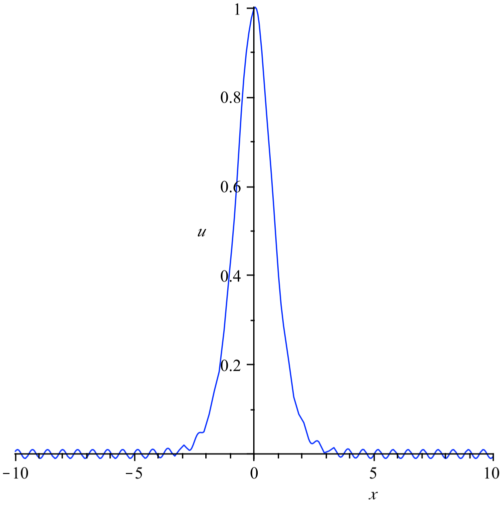
$$\begin{aligned} A_\xi &= B\,, \\ B_\xi &= \epsilon A + \mu A^2 + \nu |C|^2 \cdots , \\ C_\xi &= i\gamma(1 + \delta A)C + \cdots . \end{aligned} \tag{9}$$

Here μ, ν, δ are real-valued coefficients specific to the system being considered, and the omitted terms are small error terms as above.

7.10 Case (2)

When the error terms are omitted the system is integrable. Indeed in that limit, $|C| = C_0$ is a constant, and after a change of origin, the system reduces to the same form as (6) in case (1). Thus, for the case $\epsilon > 0$ (when case (1) is a KdV-type solitary wave), the solution is a one-parameter family of homoclinic-to-periodic solutions, with $|C| = C_0$ constant and $(A, B) \to (A_0,)$ as $\xi \to \pm\infty$ where A_0 is a real constant, given by $\epsilon A_0 + \mu A_0^2 + \nu C_0^2 = 0$. The solution is a **generalized solitary wave** which typically has a **"sech^2" core, and decays at infinity to non-zero oscillations of constant amplitude C_0 and wavenumber γ**. A delicate analysis of the full system (4) with the the small error terms shows that at least two of these solutions persist; the minimal amplitude C_0 being **exponentially small**, that is $O(\exp(-K/|\epsilon|^{1/2}))$ where K is a positive real constant. Although such waves are permissible as solutions of the steady-state equations, they have infinite energy and their associated group velocity is inevitably inward at one end and outward at the other end. Hence, they cannot be realised in a physical system from any localized initial condition. Instead localized initial conditions will typically generate a one-sided generalized solitary wave, whose central core is accompanied by small-amplitude outgoing waves on one side only. Such waves cannot be steady, and instead will slowly decay with time.

7.11 Generalized solitary wave



7.12 Case (3)

Finally we consider case (3), when there is a double eigenvalue $\lambda = i\beta$ with generically a corresponding single eigenvector $\mathbf{V}_0$, and a single generalized eigenvector $\mathbf{V}_1$, while the complex conjugate double eigenvalue $\lambda = -i\beta$ has corresponding complex conjugate eigenvectors. Small-amplitude solutions are now sought in the form

$$\mathbf{W} = A(\xi)\mathbf{V}_0 + B(\xi)\mathbf{V}_1 + A^*(\xi)\mathbf{V}_0^* + B^*(\xi)\mathbf{V}_1^* + \mathbf{W}^{(2)} . \qquad (10)$$

Here A, B are complex-valued variables, forming a four-dimensional subspace while $\mathbf{W}^{(2)}$ is again a small error term. Projection onto this subspace and a normal form analysis reveals that

$$A_\xi = i\beta A + B + iAP(\epsilon, |A|^2, K) + \cdots ,$$
$$B_\xi = i\beta B + iBP(\epsilon, |A|^2, K) + AQ(\epsilon, |A|^2, K) + \cdots . \qquad (11)$$
$$\text{where} \qquad K = i(AB^* - A^*B) , \qquad (12)$$

Here P, Q are real-valued polynomials of degree 1, that is we may write

$$P(\epsilon, |A|^2, K) = \epsilon + \nu_1|A|^2 + \nu_2 K ,$$
$$Q(\epsilon, |A|^2, K) = 2\epsilon\beta + \mu_1|A|^2 + \mu_2 K \qquad (13)$$

where all coefficients are real-valued.

7.13 Case (3)

$$\begin{aligned} A_\xi &= i\beta A + B + iAP(\epsilon, |A|^2, K)\,, \\ B_\xi &= i\beta B + iBP(\epsilon, |A|^2, K) + AQ(\epsilon, |A|^2, K)\,. \\ \text{where} \qquad & K = i(AB^* - A^* B)\,, \end{aligned}$$

This truncated system, which has the error terms omitted, is integrable. There are two integrals, K, H both constants, where

$$H = |B|^2 - (2\epsilon\beta|A|^2 + \frac{\mu_1}{2}|A|^4 + \mu_2 K|A|^2)\,. \tag{14}$$

For a solitary wave solution we must have $K = H = 0$ and it then follows that

$$|A|_\xi^2 = 2\epsilon\beta|A|^2 + \frac{\mu_1}{2}|A|^4\,. \tag{15}$$

Thus solitary wave solutions exist provided that $\epsilon > 0$, and that the nonlinear coefficient $\mu_1 < 0$. The condition $\epsilon > 0$ implies that the perturbed eigenvalues, $\lambda \approx i\beta + (2\epsilon\beta)^{1/2}$ have split off the imaginary axis, and so provide the conditions needed for exponential decay at infinity; the condition $\mu_1 < 0$ depends on the particular physical system being considered.

7.14 case (3)

The solution of the truncated system is

$$A = a \exp\left(i[\beta + \epsilon]\xi\right)\mathrm{sech}(\gamma\xi)\,, \text{ where } \gamma = (2\epsilon\beta)^{1/2}\,,\ |a|^2 = -\frac{4\epsilon\beta}{\mu_1}\,. \quad (16)$$

This solution describes an **envelope solitary wave** , with a carrier wavenumber $\beta + \epsilon$ and an envelope described by the "sech"-function. These solitary waves can also be obtained from the **soliton solutions of the NLS equation** , for that special case when the phase velocity equals the group velocity, $c = c_g$, or more precisely when $c + \Omega/K = c_g + V$, where V is the soliton speed and Ω, K are the frequency and wavenumber corrections. Note that the solution (16) contains an arbitrary phase in the complex amplitude a, meaning that the location of the crests of the carrier wave *vis-a-vis* the maximum of the envelope (here located at $\xi = 0$) is arbitrary. However, restoration of the error terms leads to the result that **only two of these solutions persist**, namely, those for which a carrier wave crest or trough is placed exactly at $\xi = 0$, so that the resulting solitary wave is either one of **elevation or depression**. This result requires very delicate analysis, but could be anticipated by noting that these are the only two solutions which persist under the symmetry transformation $\xi \to -\xi$.

7.15 Water Waves

For water waves, for which the dispersion relation is (3), these two cases (1) and (2) imply that pure solitary waves of elevation exist for $B = 0$, and of depression for $B > 1/3$, while generalized solitary waves arise whenever $0 < B < 1/3$.

For the case of generalized solitary waves, there is always the possibility that the amplitude of the oscillations is zero, and the solution then reduces to a pure solitary, called an **“embedded”** solitary wave. There are now many examples of such embedded solitary waves arising in various physical systems, notably for internal waves, but from various numerical and analytical studies, it would seem that they do not arise in the water wave context. This “dynamical-systems” approach to finding solitary waves has also been applied to interfacial waves, where again the linear dispersion relation holds the key to where solitary waves can be found.

Concerning case (3) the conditions are met for capillary -gravity waves with $0 < B < 1/3$, where it can be shown that the coefficient $\mu_1 < 0$ as required. Hence we find envelope solitary waves.

Lecture 7: References

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