Nonlinear Waves: Woods Hole GFD Program 2009

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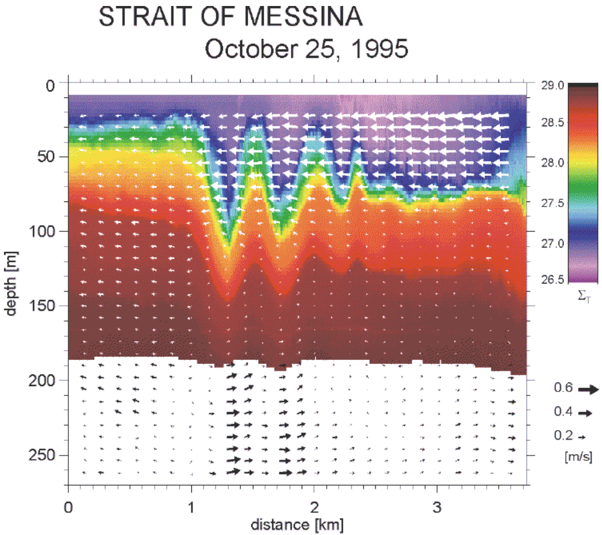
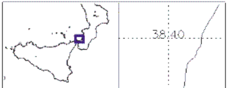
A tidal bore on the Dordogne river in France.

Internal Solitary Waves in the Atmosphere

An atmospheric undular bore in Northern Australia: the "Morning Glory".

Internal Solitary Waves in the Ocean





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position: 38.305 N 15.752 E

end time: 16:14 UTC

6.1: Shallow water flow over an obstacle

Consider one-dimensional shallow-water flow past topography. The flow variables are the **total local depth** $H = \zeta + h - F(x)$ and the **depth-averaged horizontal velocity** V. Here ζ is the surface elevation above the undisturbed depth h and the bottom is located at $z = -h + F(x)$ where $F(x)$ is the obstacle. Then the fully nonlinear shallow water equations for conservation of mass and momentum are

$$\zeta_t + (HV)_x = 0, \tag{1}$$

$$V_t + VV_x + g\zeta_x = -\frac{(H^2 D^2 H)_x}{3H} - \frac{(H^2 D^2 F)_x}{2H} - \frac{F_x D^2(\zeta + F)}{2}, \tag{2}$$

$$\text{where} \quad D = \frac{\partial}{\partial t} + V\frac{\partial}{\partial x}.$$

Equation (1) is exact, but equation (2) is a long-wave approximation; the terms on the right-hand side are the leading order effects of wave dispersion. They form the **Su-Gardner** equations (also known as the Green-Naghdi equations).

6.2: Linearized shallow water theory

If the Su-Gardner equations are linearized about the constant state U, h, that is $V = U + u, |u| << V$, $|\zeta| << h$, and also the dispersive terms are omitted, they reduce to the forced linear wave equation

$$D_l^2\zeta - c^2\zeta_{xx} = U^2 F_{xx}\,, \quad D_l = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\,, \quad c = \sqrt{gh}\,. \tag{3}$$

Here c is the linear long-wave speed, and a key parameter is the **Froude** number $\mathbf{Fr = U/c}$. Provided that $Fr \neq 1$, this initial-value problem is easily solved, and as $t \to \infty$, we get the steady solution

$$\zeta = \frac{U^2}{U^2 - c^2}F(x)\,, \tag{4}$$

describing a stationary depression (elevation) over the obstacle for subcritical (supercritical) flow, that is $Fr < (>)1$. But clearly this solution fails when $Fr \approx 1$ and then it is necessary to invoke **weak nonlinearity and weak dispersion**. The outcome is then the forced Korteweg-de Vries equation.

6.3: Forced Korteweg de Vries equation

For water waves this is, in nondimensional form based on a length scale h and a velocity scale c,

$$-\zeta_t - \Delta\zeta_x + \frac{3}{2}\zeta\zeta_x + \frac{1}{6}\zeta_{xxx} + \frac{1}{2}F_x = 0\,. \tag{5}$$

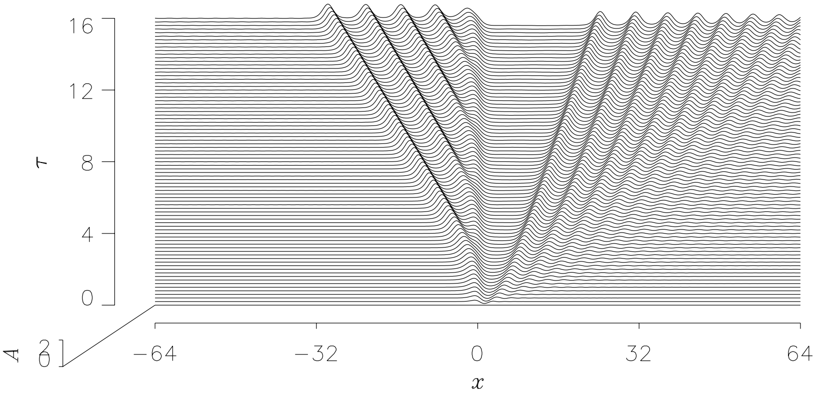
Here $\mathbf{\Delta = Fr - 1}$ measures the degree of criticality, **subcritical for $\Delta < 0$ or supercritical for $\Delta > 0$**. The equation describes the usual KdV balance between nonlinearity, dispersion and time evolution, supplemented here by forcing and criticality. Note that the scaling requires that the response ζ scales with $\sqrt{F}$ and with the detuning Δ, typical of **resonantly** forced systems. In canonical form, the fKdV equation is

$$-A_t - \Delta A_x + 6AA_x + A_{xxx} + F_x(x) = 0\,. \tag{6}$$

That is, $\zeta = 2A/3, \Delta = \tilde{\Delta}/6, t = 6\tilde{t}, F = 2\tilde{F}/9$, and then omit "tilde". The forcing function is localized at $x = 0$ with a maximum height of $F_M > 0$, and is typically of a "Gaussian" shape.

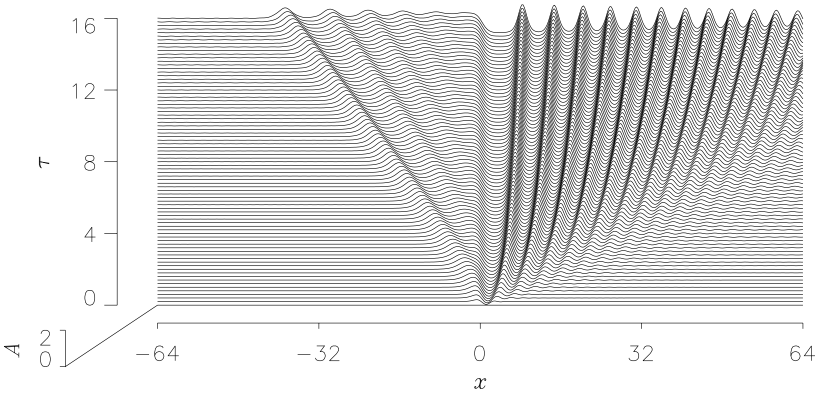
6.4: fKdV equation, localized forcing, critical case

A typical solution of the fKdV equation (6) in canonical form at exact criticality $\Delta = 0$. The forcing (not shown in the plot) is located at $x = 0$ and has a maximum height of $F_M = 1$.



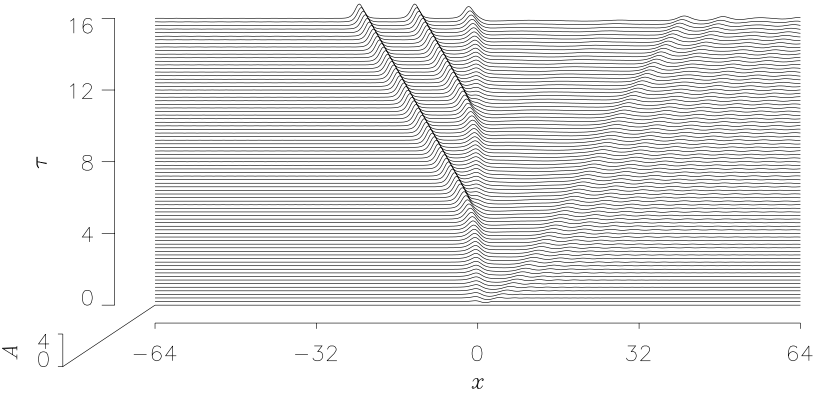
6.5: fKdV equation, localized forcing, subcritical case

A typical solution of the fKdV equation (6) in canonical form for a subcritical case, $\Delta = -1.5$. The forcing (not shown in the plot) is located at $x = 0$ and has a maximum height of $F_M = 1$.



6.6: fKdV equation, localized forcing, supercritical case

A typical solution of the fKdV equation (6) in canonical form for a supercritical case $\Delta = 1.5$. The forcing (not shown in the plot) is located at $x = 0$ and has a maximum height of $F_M = 1$.



6.7: Asymptotic analysis for localized forcing

The origin of the upstream and downstream wavetrains can be found in the structure of the **locally steady** solution over the obstacle. In the transcritical regime this local steady solution is characterised by a transition from a constant elevation $A_- > 0$ upstream $(x < 0)$ of the obstacle to a constant depression $A_+ < 0$ downstream $(x > 0)$ of the obstacle, independently of the details of the localized forcing term $F(x)$. Explicit determination of A_+ and A_- requires some knowledge of the forcing term $F(x)$. However, in the dispersionless, or "hydraulic", limit when the linear dispersive term in (6) can be neglected, it is readily shown that, for all localised $F(x)$, with a maximum height $F_M > 0$,

$$6A_{\pm} = \Delta \mp (12F_M)^{1/2} . \tag{7}$$

This expression also serves to define the transcritical regime, which is

$$|\Delta| < (12F_M)^{1/2} . \tag{8}$$

Thus upstream of the obstacle there is a transition from the zero state to A_-, while downstream the transition is from A_+ to 0; each transition is effectively generated at the obstacle, $x = 0$. Both transitions are resolved by **undular bores** .

6.8: Undular bore

A simple representation of an undular bore can be obtained from the solution of the KdV equation, in canonical form

$$A_t + 6AA_x + A_{xxx} = 0\,. \tag{9}$$

with the initial condition of a step, $\mathbf{A = A_0 H(-x)}$ with $A_0 > 0$; $H(x)$ is the Heaviside function. An asymptotic solution can be found using Whitham's modulation theory. The asymptotic solution of (9) with this initial condition is represented as the modulated periodic wave train

$$\begin{aligned} A &= a\{b(m) + \mathrm{cn}^2(\kappa(x - Vt); m)\} + d\,, && (10)\\ \text{where} \quad b(m) &= \frac{1-m}{m} - \frac{E(m)}{mK(m)}\,, \quad a = 2m\kappa^2\,, \\ \text{and} \quad V &= 6d + 2a\left\{\frac{2-m}{m} - \frac{3E(m)}{mK(m)}\right\}. && (11) \end{aligned}$$

Here $\mathrm{cn}(x;m)$ is the Jacobian elliptic function of modulus $m, 0 < m < 1$. As $m \to 1$, $\mathrm{cn}(x;m) \to \mathrm{sech}(x)$ and then (10) is a **solitary wave**, riding on a background level d. As $m \to 0, a \to 0$, $\mathrm{cn}(x;m) \to \cos x$ and so (10) collapses to a linear **sinusoidal wave**.

6.9: Undular bore

This family of solutions contains three free parameters, which are chosen from the set (a, κ, V, d, m). In **Whitham's modulation theory**, these parameters are all allowed to be slowly varying functions of x, t. The equations for these modulations are found by averaging conservation laws for the KdV equation (9). The outcome is a set of three nonlinear hyperbolic equations for three of the available free parameters, or rather better, from an appropriate combinations of them. The relevant asymptotic solution corresponding to the "step" initial condition is then constructed in terms of the similarity variable x/t, and is given by

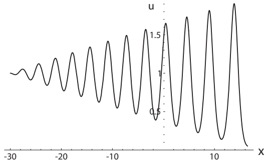
$$\frac{x}{A_0 t} = 2\left\{1 + m - \frac{2m(1-m)K(m)}{E(m) - (1-m)K(m)}\right\}, \quad -6 < \frac{x}{A_0 t} < 4 , \tag{12}$$

$$a = 2A_0 m , \quad d = A_0 \left\{m - 1 + \frac{2E(m)}{K(m)}\right\} . \tag{13}$$

Ahead of the wavetrain, $x/t > 4A_0$, $A = 0$, $m \to 1$, $a \to 2A_0$ and $d \to 0$; the leading wave is a **solitary wave** of amplitude $2A_0$. Behind the wavetrain $x/t < -6A_0$, $A = A_0$, $m \to 0$, $a \to 0$, and $d \to A_0$. On any individual crest in the wavetrain, $m \to 1$ as $t \to \infty$, and so, in this sense, the undular bore evolves into a **train of solitary waves**.

6.10: Undular bore

A plot of the undular bore given by (12, 13) for $A_0 = 1, t = 5$:



If $A_0 < 0$ in the initial condition (10), then an "undular bore" solution analogous to that described by (12, 13) does not exist. Instead, the asymptotic solution is a **rarefaction wave**,

$$\begin{aligned} A &= 0 \quad \text{for} \quad x > 0\,, \\ A &= \frac{x}{6t} \quad \text{for} \quad A_0 < \frac{x}{6t} < 0\,, \\ A &= A_0\,, \quad \text{for} \quad \frac{x}{6t} < A_0 (< 0)\,. \end{aligned} \tag{14}$$

Small oscillatory wavetrains are needed to smooth out the discontinuities in A_x at $x = 0$ and $x = -6A_0$.

6.11: Asymptotic analysis for localized forcing, upstream

We now return to the asymptotic solution of the fKdV equation, and resolve the upstream and downstream transitions by these “undular-bore” asymptotic solutions. That in $x < 0$ occupies the zone

$$\Delta - 4A_- < \frac{x}{t} < \max\{0,\ \Delta + 6A_-\}\,. \tag{15}$$

Note that this upstream wavetrain is constrained to lie in $x < 0$, and hence is only fully realised if $\Delta < -6A_-$. Combining this criterion with (7) and (8) defines the regime

$$-(12F_M)^{1/2} < \Delta < -\frac{1}{2}(12F_M)^{1/2}\,, \tag{16}$$

where a **fully developed undular bore solution can develop upstream**. On the other hand, the regime $\Delta > -6A_-$ or

$$-\frac{1}{2}(12F_M)^{1/2} < \Delta < (12F_M)^{1/2}\,, \tag{17}$$

is where the **upstream undular bore is only partially formed, and is attached to the obstacle**. In this case the modulus m varies from 1 at the leading edge (a solitary wave) to a value m_- (< 1) at the obstacle, where m_- can be found from (12) by replacing x with Δt, A_0 with A_-.

6.12: Asymptotic analysis for localized forcing, downstream

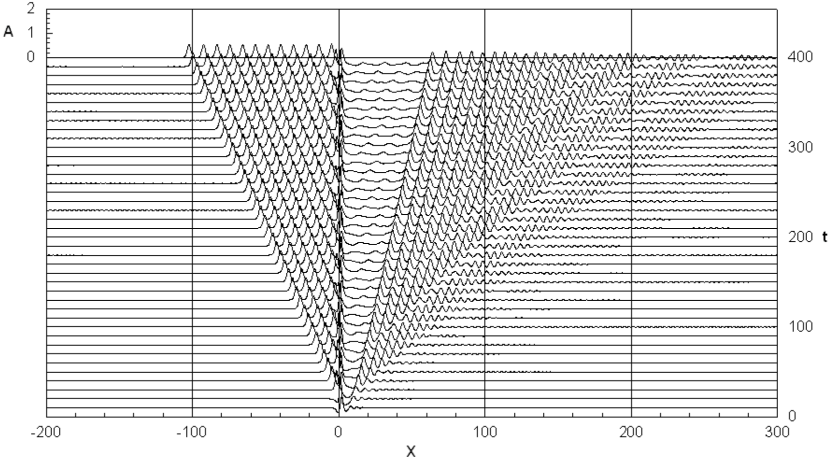
The "undular-bore" aysmptotic solution in $x > 0$ occupies the zone

$$\max\{0,\ \Delta - 2A_+\} < \frac{x}{t} < \Delta - 12A_+ \,. \tag{18}$$

The downstream wavetrain is constrained to lie in $x > 0$, and hence is only fully realised if $\Delta > 2A_+$. Combining this criterion with (7) and (8) defines the regime (17), and so **a fully detached downstream undular bore coincides with the case when the upstream undular bore is attached to the obstacle**. On the other hand, in the regime (16), when **the upstream undular bore is detached from the obstacle, the downstream undular bore is attached to the obstacle**, with a modulus $m_+(< 1)$ at the obstacle, where m_+ can be founding from (12) by replacing x with $\Delta - 6A_+$, A_0 with $-A_+$. Indeed, now a stationary lee wavetrain develops just behind the obstacle.
For the case when the obstacle has **negative polarity** (that is $F(x)$ is negative, and non-zero only in the vicinity of $x = 0$), the upstream and downstream solutions are qualitatively similar. However, the solution in the vicinity of the obstacle remains transient, and this causes a modulation of the "undular bore" solutions.

6.13: Localized negative forcing

Numerical simulation of the fKdV equation (5) (water wave case) for **localized negative** forcing, $F_M = -0.1$ and $\Delta = 0.0$.

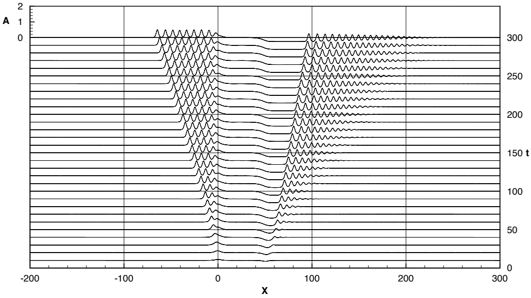


6.14: fKdV equation, step forcing, $\Delta = 0.0$

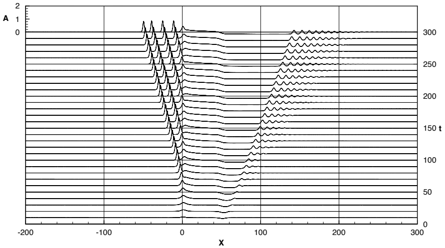
Next the localized obstacle is replaced by a **step** (Zhang & Chwang 2001, Grimshaw, Zhang & Chow 2007, 2009),

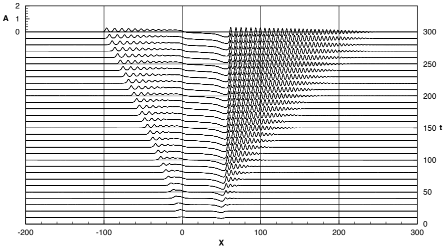
$$F(x) = \frac{F_M}{2}(\tanh \gamma x - \tanh \gamma (x - L)), \tag{19}$$

that is, a **step up** $F_M > 0$ at $x = 0$ and a **step down** at $x = L >> 1$. Numerical simulation of the fKdV equation (5) (water wave case) at exact criticality $\Delta = 0$ for $F_M = 0.1$, $\gamma = 0.25$, $L = 50$,

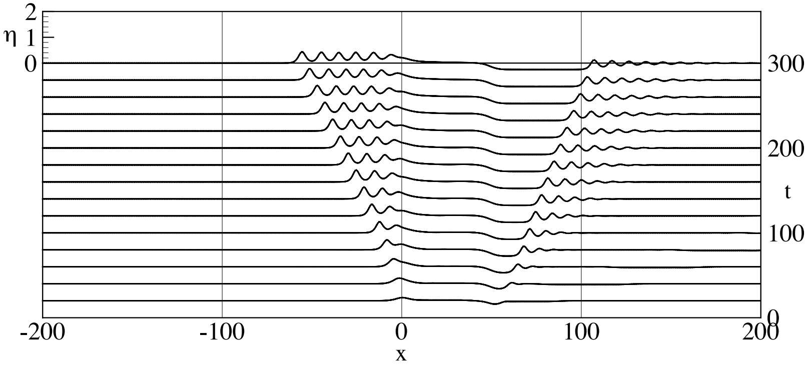


6.15: fKdV equation, step forcing, $\Delta = 0.2$, $\Delta = -0.2$

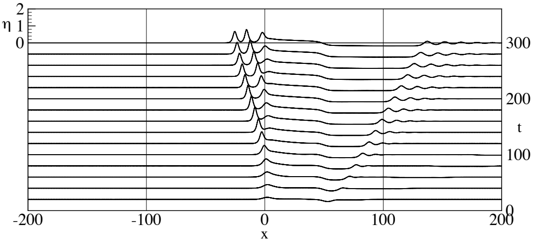


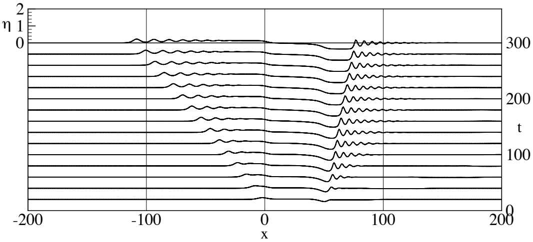


6.16: Euler equations, step forcing $Fr = 1, \Delta = 0$



6.17: Euler equations, step forcing, $\Delta = 0.2$, $\Delta = -0.2$





6.18: fKdV equation, asymptotic analysis for step forcing

Consider **positive and negative steps**, where

$$\begin{aligned} F(x) &= 0\,, \quad \text{for} \quad x < 0\,, \\ F(x) &= F_M\,, \quad \text{for} \quad x > W\,, \end{aligned} \tag{20}$$

and $F(x)$ varies monotonically in $0 < x < W$. A positive (negative) step has $F_M > 0 (< 0)$. Strictly $F(x)$ should return to zero for some $L >> W$. For this analysis we ignore this, and in effect assume that $L \to \infty$. It means that the asymptotic solutions constructed below are only valid for some limited time, determined by how long it takes for a disturbance to travel the distance L.

We shall sketch how the asymptotic solution for the localized forcing described above becomes modified for a step. Thus, in the forcing region, $A = A(x), 0 < x < W$, while otherwise

$$A = A_- \quad \text{for} \quad x \to -\infty\,, \tag{21}$$

$$A = A_+ \quad \text{for} \quad x \to \infty\,. \tag{22}$$

6.19: fKdV equation, asymptotic analysis for step forcing

Omitting the dispersive term in (6) it is readily found that

$$-\Delta A + 3A^2 + F = C \,, \tag{23}$$

$$\text{or} \quad 6A = \Delta \pm (\Delta^2 + 12C - 12F)^{1/2} \tag{24}$$

There are thus two branches. Application of the limits (21, 22) yields

$$C = -\Delta A_- + 3A_-^2 = -\Delta A_+ + 3A_+^2 + F_M \,, \tag{25}$$

The system is closed by determining the constant C from the long-time limit of the unsteady hydraulic solution. That is, we omit the linear dispersive term in (6) and write the resulting nonlinear hyperbolic equation in the characteristic form

$$\frac{dx}{dt} = \Delta - 6A \,, \quad \frac{dA}{dt} = F_x(x) \,. \tag{26}$$

with the initial condition that $A = 0$ at $t = 0$. It follows that all characteristics have an initial slope Δ which then decreases. The key issue is whether the characteristics reach a turning point, where $\Delta = 6A$.

6.20: fKdV equation, positive step $F_M > 0$

For $\Delta \leq 0$, all characteristics have a negative slope, and there are no turning points; in this case, clearly $A_+ = 0, C = F_M$ and the upper branch must be chosen in (24). Similarly, for $\Delta > (12F_M)^{1/2}$, there are no turning points, and all characteristics have a positive slope; in this case $A_- = 0, C = 0$ and the lower branch is chosen. But, for $0 < \Delta < (12F_M)^{1/2}$, characteristics emerging from the step at $F = F_0, 0 < 12F_0 < 12F_M - \Delta^2$ have a turning point and then go upstream into $x < 0$, while those with $12F_M - \Delta^2 < F_0 < 12F_M$ pass over the step and go downstream into $x > W$; it follows that $12C = 12F_M - \Delta^2$, and that $6A_+ = \Delta$, while A_- is then obtained from the upper branch of (24). In summary, the outcome is,

$$\Delta \leq 0: \qquad 6A_- = \Delta + (\Delta^2 + 12F_M)^{1/2}\,, 6A_+ = 0\,, \tag{27}$$

$$0 < \Delta < (12F_M)^{1/2}: \qquad 6A_- = \Delta + (12F_M)^{1/2}\,, 6A_+ = \Delta\,, \tag{28}$$

$$\Delta > (12F_M)^{1/2}: \qquad 6A_- = 0\,, 6A_+ = \Delta - (\Delta^2 - 12F_M)^{1/2}\,. \tag{29}$$

Upstream $A_- > 0$ and an **undular bore** forms. **Downstream** $A_+ > 0$ (in (27) $A_+ = 0$), and an undular bore is not needed. Instead the solution is terminated by a **rarefraction wave** .

6.21: fKdV equation, positive step $F_M > 0$

In all cases, the upstream solution $A_- > 0$ is a "jump" in the hydraulic limit (in (29) it has zero strength), which needs to be resolved by an **undular bore**, given by

$$\Delta - \frac{x}{t} = 2A_- \left\{1 + m - \frac{2m(1-m)K(m)}{E(m) - (1-m)K(m)}\right\}, \qquad (30)$$

$$\text{for} \qquad \Delta - 4A_- < \frac{x}{t} < \max\{0, \Delta + 6A_-\}, \qquad (31)$$

$$a = 2A_- m, \quad d = A_- \left\{m - 1 + \frac{2E(m)}{K(m)}\right\}. \qquad (32)$$

For a **fully detached undular bore**, $\Delta + 6A_- < 0$, and combining this criterion with (27, 28, 29), we get the regime

$$\Delta < -2(F_M)^{1/2} < 0. \qquad (33)$$

On the other hand the regime where $\Delta + 6A_- > 0$ but $\Delta - 4A_- < 0$, or

$$-2(F_M)^{1/2} < \Delta < (12F_M)^{1/2}, \qquad (34)$$

is where the upstream undular bore is **only partially formed and is attached to the obstacle**.

In this case, the analogous procedure yields $C = 0, -\Delta^2/12, F_M$ respectively, and so

$$\Delta \geq 0 : \quad 6A_- = 0\,, 6A_+ = \Delta - (\Delta^2 - 12F_M)^{1/2}, \tag{35}$$
$$-(|12F_M|)^{1/2} < \Delta < 0 : \quad 6A_- = \Delta\,, 6A_+ = \Delta - (|12F_M\ )^{1/2}\,, \tag{36}$$
$$\Delta < -([12F_M])^{1/2} : \quad 6A_- = \Delta + (\Delta^2 + 12F_M)^{1/2}\,, 6A_+ = 0 \tag{37}$$

In all cases the downstream solution $A_+ < 0$ is a jump, and needs to be resolved by an **undular bore**, occupying the zone

$$\max\{0,\ \Delta - 2A_+\} < \frac{x - W}{t} < \Delta - 12A_+\,. \tag{38}$$

where A_+ is given by (35, 36, 37). For a **fully detached (partially attached) undular bore**, $\Delta - 2A_+ > 0 (< 0)$, and combining with the criteria (35, 36, 37) we get the regimes

$$\Delta > -(-3F_M)^{1/2}\,, \text{ or } -(-12F_M)^{1/2} < \Delta < -(-3F_M)^{1/2} < 0\,. \tag{39}$$

For $\Delta < -(-12F_M)^{1/2}$ a stationary lee-wave train forms downstream. For $\Delta < 0$ the **upstream** solution, $A_- < 0$, is terminated by a **rarefaction wave**. For $\Delta > 0$ the upstream solution is zero.

6.23: Comparison between Euler and fKdV equations

A quantitative comparison of the results from the fKdV equation (5) and the Euler equations. $A_{-}(A_{+})$ is the elevation just upstream (downstream) of the positive (negative) step at $x = 0(50)$ respectively, and $A_{w-}(A_{w+})$ is the amplitude of the leading wave in the corresponding undular bore

Δ	fKdV				Theory			
	A_{w-}	A_{-}	A_{w+}	A_{+}	A_{w-}	A_{-}	A_{w+}	A_{+}
0.2	0.83	0.44	0.31	-0.16	0.80	0.40	0.32	-0.16
0.1	0.66	0.38	0.39	-0.20	0.66	0.33	0.40	-0.20
0.0	0.50	0.30	0.51	-0.26	0.52	0.26	0.52	-0.26
-0.1	0.39	0.22	0.64	-0.33	0.40	0.20	0.66	-0.33
-0.2	0.30	0.16	0.84	-0.40	0.32	0.16	0.80	-0.40
-0.3	0.24	0.13	0.64	-0.38	0.26	0.13	0.92	-0.46
-0.4	0.19	0.10	0.00	0.00	0.20	0.10	0.00	0.00

Δ	fKdV				Euler			
	A_{w-}	A_{-}	A_{w+}	A_{+}	A_{w-}	A_{-}	A_{w+}	A_{+}
0.2	0.83	0.44	0.31	-0.16	0.75	0.40	0.28	-0.18
0.1	0.66	0.38	0.39	-0.20	0.57	0.36	0.32	-0.21
0.0	0.50	0.30	0.51	-0.26	0.44	0.33	0.37	-0.25
-0.1	0.39	0.22	0.64	-0.33	0.32	0.20	0.43	-0.30
-0.2	0.30	0.16	0.84	-0.40	0.23	0.13	0.53	-0.36
-0.3	0.24	0.13	0.64	-0.38	0.16	0.08	0.57	-0.38
-0.4	0.19	0.10	0.00	0.00	0.10	0.01	0.00	0.00

6.24: fKdV equation, long-time limit for step forcing

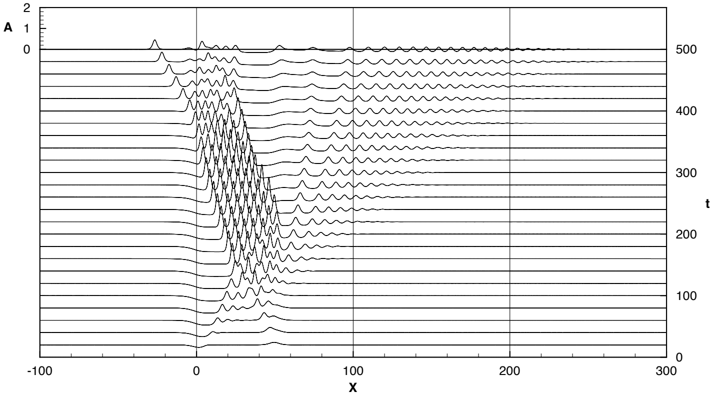
The long-time solution for flow over a step of **finite length** L (that is, (19) for instance with $F_M > 0, \gamma L >> 1$) will be that predicted by Grimshaw and Smyth (1986) in the framework of the fKdV equation for flow over a localised obstacle. Indeed, at exact criticality $\Delta = 0$, the wavetrains generated by the elongated step are in fact exactly the same as those predicted for flow over a localised obstacle. Otherwise, for $\Delta \neq 0$, the upstream and downstream undular bores initially generated by the positive and negative steps have (slightly) different amplitudes to those generated by a localized obstacle, but for sufficiently long times ($t > L/|\Delta|$) there is communication between the two steps by a rarefraction wave, followed by an adjustment to precisely the same solution predicted for localized forcing.

6.25: fKdV equation, negative step forcing, $\Delta = 0.0$

Numerical simulation of the fKdV equation (5) for the **step** forcing

$$F(x) = \frac{F_M}{2}(\tanh \gamma x - \tanh \gamma(x - L)), \tag{40}$$

where $F_M = -0.1$, $\gamma = 0.25$, $L = 50 >> 1$ and $\Delta = 0$.

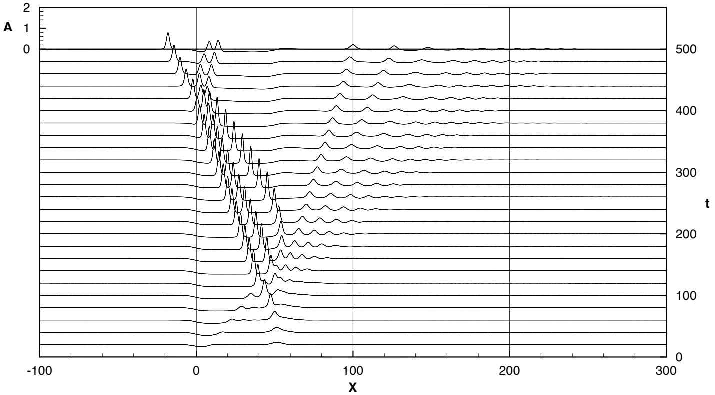


6.26: fKdV equation, negative step forcing, $\Delta = 0.2$

Numerical simulation of the fKdV equation (5) for the **step** forcing

$$F(x) = \frac{F_M}{2}(\tanh \gamma x - \tanh \gamma (x - L)),$$

where $F_M = -0.1$, $\gamma = 0.25$, $L = 50 >> 1$ and $\Delta = 0.2$.

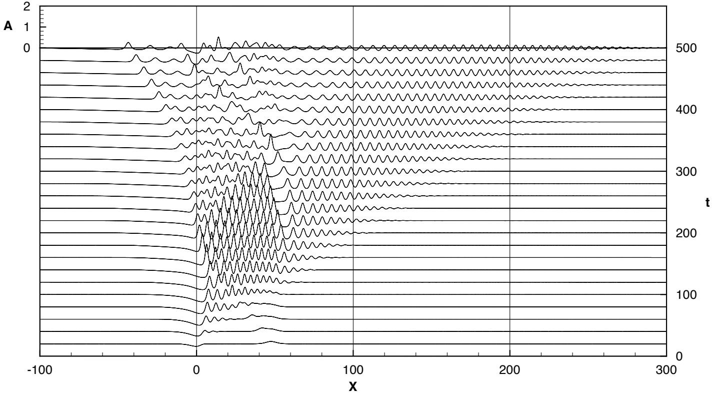


6.27: fKdV equation, negative step forcing, $\Delta = -0.2$

Numerical simulation of the fKdV equation (5) for the **step** forcing

$$F(x) = \frac{F_M}{2}(\tanh \gamma x - \tanh \gamma (x - L)),$$

where $F_M = -0.1$, $\gamma = 0.25$, $L = 50 >> 1$ and $\Delta = -0.2$.

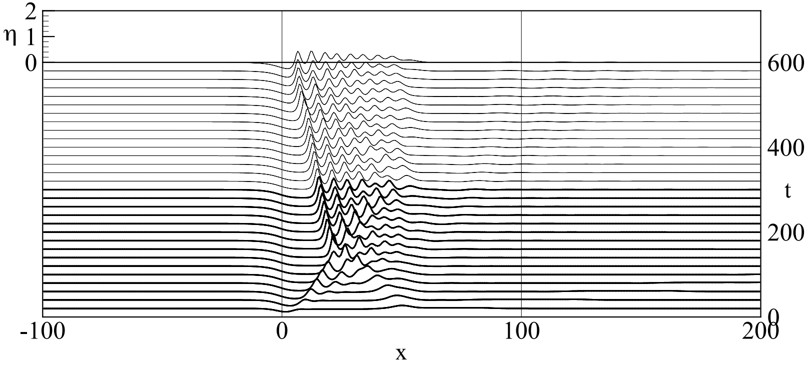


6.28: Euler equations, negative step forcing, $\Delta = 0.0$

Numerical simulation of the full Euler equations for the **step** forcing

$$F(x) = \frac{F_M}{2}(\tanh \gamma x - \tanh \gamma (x - L)),$$

where $F_M = -0.1$, $\gamma = 0.25$, $L = 50 >> 1$ and $Fr = 1.0, \Delta = 0.0$.



Lecture 6: References

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